A Complete Classification of Finitely Generated Indecomposable Modules Over the Kronecker Algebra

Tore A. Forbregd

Abstract

We give a characterization of the indecomposable modules for the Kronecker algebra. Since the Kronecker algebra is of infinte representation type, this is not obtained in a straightforward way. We do this by means of the functors $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$. Also the Coxeter transformation plays a central role in the discussion.

1 Introduction

For any ring R, the indecomosable modules in the module category over R are always of great interest. As with the prime numbers over the integers, the indecomosable modules play the role of the building blocks in the category of R-modules. If our ring is a path algebra over a quiver, then the indecomposable modules will play an additional role. In the finitely generated case, we have a one to one correspondace between finitely generated representations over the quiver and finitely generated modules over the path algebra. Then for a representation over a quiver we may be interested in finding suitable basis for the vector spaces such that the linear transformations look 'nice'. One approach to this problem is to find the indecomposable representations, which, in a sense, fundamental representations, and also yield the linear transformations that cannot be broken down into easier ones.

Given two finite dimensional k-vector spaces, V_1, V_2 and two linear transformations T_α and T_β with $T_\alpha, T_\beta : V_1 \longrightarrow V_2$, we may then be intereseted in finding a basis for V_1 and V_2 such that the linear trasformations look nice. This is what happens if we are in the case of the Kronecker algebra.

The Kronecker algebra is of infinite representation type, meaning that the indecomposable modules over the Kronecker algebra constitute an infinte set. This poses, in general, a problem in describing the indecomposable objects, however, in the case of the Kronecker algebra we are able to give a full description of this set and its elements.

We do this by means of the Auslander-Reiten translation and the Coxeter transformation.

2 Preliminaries

During this section we go through some rudimentary facts and definitions. These facts are assumed to be well known to the reader, so proofs will be omitted. First things first, all rings will be unital and with $1 \neq 0$, for a ring R, we denote by $\mathrm{Mod}\,R$ the category of left R-modules. If M is a right R-module, then it is a left R^{op}-module in a natural way. Hence denote by $\mathrm{Mod}\,R^{\mathrm{op}}$ the category of right modules. And by $\mathrm{mod}\,R$ denote the full subcategory of $\mathrm{Mod}\,R$ which consists of finitely generated left R-modules, and $\mathrm{mod}\,R^{\mathrm{op}}$ the finitely generated right R-modules. For brevity, the word module will itself mean left module. We define the **projective dimension**, $\operatorname{pd} M$, of a module M to be the smallest number $n \in \mathbb{N}$ such that

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact and P_i projective. Otherwise, if no such n exists we say $\operatorname{pd} M = \infty$. Dually, the **injective dimension**, $\operatorname{id} M$, is the smallest $m \in \mathbb{N}$ such that

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_m \longrightarrow 0$$

is exact with the I_j's injective. We say that $\operatorname{id} M = \infty$ if there is no such long exact sequence. Obviously, $\operatorname{pd} M = 0$ if and only if M is projective, and $\operatorname{id} M = 0$ if and only if M is injective. Furthermore, for a ring R define the **left global dimension** of R as follows $\operatorname{gl.dim} R = \sup\{\operatorname{pd} M \mid M \text{ is a left } R\text{-module (not necessarily finitely generated)}\}$.

We call a ring, R, **left hereditary** if all left ideals are projective. R is left hereditary if and only if $\operatorname{gl.dim} R \leq 1$, this then gives a charachterization of hereditary rings in the sense of the global dimension. We shall call a left artin ring **hereditary** if it is left hereditary.

Let us turn our attention to exact sequences. The following fact is well known in homological algebra, and we need it here. It describes how one can continue short exact sequences when passed to Hom by Ext.

Proposition 2.1. *Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence of R-modules, and let M be a R-module, R ring. Then there exists long exact sequences:*

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(M, X) \longrightarrow \operatorname{Hom}_R(M, Y) \longrightarrow \operatorname{Hom}_R(M, Z) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_R(M, X) \longrightarrow \operatorname{Ext}^1_R(M, Y) \longrightarrow \operatorname{Ext}^1_R(M, Z) \longrightarrow \\ \longrightarrow \operatorname{Ext}^2_R(M, X) \longrightarrow \operatorname{Ext}^2_R(M, Y) \longrightarrow \operatorname{Ext}^2_R(M, Z) \longrightarrow \cdots \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(Z, M) \longrightarrow \operatorname{Hom}_R(Y, M) \longrightarrow \operatorname{Hom}_R(X, M) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_R(Z, M) \longrightarrow \operatorname{Ext}^1_R(Y, M) \longrightarrow \operatorname{Ext}^1_R(X, M) \longrightarrow \\ \longrightarrow \operatorname{Ext}^2_R(Z, M) \longrightarrow \operatorname{Ext}^2_R(Y, M) \longrightarrow \operatorname{Ext}^2_R(X, M) \longrightarrow \cdots \end{aligned}$$

□

Closely related with the vanishing of $\operatorname{Ext}_R(Y, X)$ is whether Y is projective and X injective.

Proposition 2.2. *R is a ring. For R-modules X and Y we have the following.*

(i) X is projective if and only if $\operatorname{Ext}^1_R(X, Y) = 0$ for all Y if and only if $\operatorname{Ext}^n_R(X, Y) = 0$ for all Y and $n \geq 1$.

(ii) Y is injective if and only if $\operatorname{Ext}^1_R(X, Y) = 0$ for all X if and only if $\operatorname{Ext}^n_R(X, Y) = 0$ for all X and $n \geq 1$. □

We also state an important duality of projectives over an artin algebra. We introduce some notation before we present the important fact. Given an artin algebra Λ, then by $(\ \ )^*$ we mean $\operatorname{Hom}_\Lambda(\ \ , \Lambda)$. Also, given $f : P \longrightarrow Q$, define $f^* : Q^* \longrightarrow P^*$ by $g \longmapsto gf$ for all $g \in \operatorname{Hom}_\Lambda(Q, \Lambda)$.

Proposition 2.3. *Let Λ be an artin algebra. Then the functor $(\ \ )^* : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda^{op})$ is a duality.* □

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a oriented multigraph, with Γ_0 the set of vertices and Γ_1 the set of edges (arrows). We call Γ a **quiver**, furthermore, if Γ_0 and Γ_1 are both finite, then we say that Γ is a finite quiver. Also, for each $i \in \Gamma_0$ define e_i as the trivial path from vertex i to i. Then given $\alpha \in \Gamma_1 \cup \{e_i\}_{i\in\Gamma_0}$, an arrow, we denote by $s(\alpha) = i$ and $e(\alpha) = j$ as the initial vertex and terminal vertex, respectively, of the arrow α. A **path** in the quiver Γ is a composition of arrows in Γ, $\alpha_r \cdots \alpha_1$, with $s(\alpha_t) = e(\alpha_{t-1})$ for $1 \leq t \leq r$, or the trivial paths e_i. And by $l(\rho)$ denote the length of a path ρ as the number of non-trivial arrows in the composition of ρ. We can then construct an k-algebra, k a field, in the following fashion; for α, β arrows, define the product $\beta\alpha$ as the following

$$\beta\alpha = \begin{cases} 0 & \text{, if } e(\alpha) \neq s(\beta). \\ \beta\alpha & \text{, if } e(\alpha) = s(\beta). \end{cases}$$

and by expanding this defenition inductively to paths we get a multiplication rule for paths. By letting the set of all paths be a basis for this algebra, and a general element of the algebra to be a k-linear combination of paths, we then get the desired algebra structure. We call this algebra the **path algebra** of Γ, and usually denote it by the symbol $k\Gamma$. We see that if Γ is a finite quiver, then $k\Gamma$ is a finite dimensional algebra if and only if Γ has no oriented cycles.

A **representation** of a quiver Γ is pair (V, T) such that V is a set of vector spaces $V = \{V_i\}$ for $i \in \Gamma_0$ and T is set of linear transformations $\{T_\alpha\}$ for $\alpha \in \Gamma_1$ such that $T_\alpha : V_i \longrightarrow V_j$ if there is an arrow from vertex i to j. Denote the category of such (finite dimensional) representations by Rep Γ. Now, if we are given a representation of a quiver Γ, we are able to construct a $k\Gamma$-module in the following manner. Let $M = \bigoplus V_i$ and for $m \in M$ and $T_\alpha : V_i \longrightarrow V_j$, we define $\alpha \cdot m$ as $(0, \ldots, T(v_i), 0, \ldots, 0)$, i.e. all zeros except at the j'th position, where $m = (v_i)_{i\in\Gamma_0}$. This then gives a module structure. Conversely, if we are given a $k\Gamma$-module, M, we then construct the representation (V, T) by $V_i = Me_i$, where e_i is the trivial path at vertex i, and $T_\alpha = \alpha$ for all arrows $\alpha \in \Gamma_1$. We have an equivalence of categories.

Proposition 2.4. *Γ a quiver with no oriented cycles and k a field, then there is an equivalence between* mod $k\Gamma$ *and* Rep Γ*.* □

Another important notion concerning modules over a path algebra, is the **dimension vector** of a module. Given a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ and k a field, denote by $\dim M = (\dim_k Me_1, \dim_k Me_2, \ldots, \dim_k Me_n)$ for a $k\Gamma$-module M, with $\Gamma_0 = \{1, 2, \ldots, n\}$.

Throughout the remaining of the paper, Λ will denote a hereditary artin algebra. If not stated otherwise, all modules are assumed to be finitely generated.

3 The Kronecker Algebra

Let Γ be the quiver

$$\Gamma : \ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

if k is a field, then we denote $\Lambda = k\Gamma$ to be the path algebra defined by the quiver Γ. This is algebra is called the Kronecker algebra. Λ is then a four dimensional algebra over k.

Now the simple Λ-modules are given by the representations $k \rightrightarrows 0$ and $0 \rightrightarrows k$. Taking the injective envelope and the projective cover we get the injective and projective indecomposable Λ-modules. They are given by the representations

$$k \rightrightarrows 0 \;\; , \;\; k^2 \overset{(1,0)}{\underset{(0,1)}{\rightrightarrows}} k \;\; , \;\; k \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\rightrightarrows}} k^2 \;\; , \;\; 0 \rightrightarrows k$$

with respective dimension vectors: $(1,0)$, $(2,1)$, $(1,2)$, $(0,1)$. We will denote these modules by I_1, I_2, P_1, P_2.

4 The Transpose

This section is devoted to the transpose and the dual of the transpose, we will introduce these notions here, and go through some elementary and some non-trivial properties of them. In this section, Λ will be an artinian algebra over a commutative artin ring k, that is we have a ring homomorphism $\varphi : k \longrightarrow \Lambda$ with $\operatorname{Im} \varphi \subseteq Z(\Lambda)$ and Λ is finitely generated as a R-module.

Given a minimal projective presentation $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \longrightarrow 0$ of X in $\operatorname{mod} \Lambda$, i.e. $p : P_0 \longrightarrow X$ and $f : P_1 \longrightarrow \operatorname{Ker} p$ are projective covers, we define the transpose of X, $\operatorname{Tr} X = \operatorname{Coker} f^*$. This means we have the exact sequence $P_0^* \xrightarrow{f^*} P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$. We state some immediate consequences of this definition.

Proposition 4.1. *All modules are in* $\operatorname{mod} \Lambda$

(i) $P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{q} \operatorname{Tr} X \longrightarrow 0$ *is a minimal projective presentation of* $\operatorname{Tr} X$ *whenever* $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \longrightarrow 0$ *is a minimal projective presentation of an indecomposable nonprojective module* X*.*

(ii) $\operatorname{Tr} (\bigoplus_{i=1}^n X_i) \simeq \bigoplus_{i=1}^n \operatorname{Tr} X_i$ *, n finite.*

(iii) $\operatorname{Tr} X = 0$ *if and only if* X *is projective.*

(iv) $\operatorname{Tr} \operatorname{Tr} X \simeq X_{\mathscr{P}}$ *for all* X*, where* $X_{\mathscr{P}}$ *are the non-projective direct summands of* X*.*

(v) If X *and* Y *have no nonzero projective direct summands, then* $\operatorname{Tr} X \simeq \operatorname{Tr} Y$ *if and only if* $X \simeq Y$*.*

Proof. (iv) and (v) will follow from the other parts of the proposition.

(i) Assume the contrary, since $(\ \ )^*$ is an equivalence of finitely generated projectives, it sends a projective Λ-module to a projective Λ^{op}-module, thus we are then left with two options either 1) $P_1^* \xrightarrow{q} \operatorname{Tr} X$ is not a projective cover or 2) $P_0^* \xrightarrow{f^*} \operatorname{Ker} q$ is not a projective cover. Assume 2), and let $R \xrightarrow{h} \operatorname{Ker} q$ be a projective cover of $\operatorname{Ker} q$. Since P_0^* is projective and f^* is epimorphic on the kernel of q, we then get the following commutative diagram

$$\begin{array}{ccccc}
R & \xrightarrow{h} & \operatorname{Ker} q & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \| & & \\
P_0^* & \xrightarrow{f^*} & \operatorname{Ker} q & \longrightarrow & 0 \\
\downarrow{\scriptstyle v} & & \| & & \\
R & \xrightarrow{h} & \operatorname{Ker} q & \longrightarrow & 0
\end{array}$$

Now let $g = vu$ and consider the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{h} & \operatorname{Ker} q \\ {\scriptstyle g}\big\downarrow & & \big\| \\ R & \xrightarrow{h} & \operatorname{Ker} q \end{array}$$

We now claim that g is an isomorphism. g is epimorphic since $h = hg$ is epimorphic and h is an essential epimorphism. From the diagram above we assert that $\operatorname{Ker} g \subseteq \operatorname{Ker} h$. Now $\operatorname{Ker} g$ is a direct summand of R since g is a map onto a projective, that is $R = \operatorname{Ker} g \oplus N$ with N in $\operatorname{mod} \Lambda$. But the commutativity of the above diagram gives that the composition $N \longrightarrow R \xrightarrow{h} K$ is epi, which by the essentiality of h gives $0 \longrightarrow N \longrightarrow R$ epi. Thus $\operatorname{Ker} g = 0$ and we get as desired that g is an isomorphism. This then means that u is split mono and v is split epi. We can then view u as an inclusion, and if we let $\operatorname{Ker} v = S$ we get that $P_0^* \simeq R \oplus S$, we also get that $f^*|_S = hv|_S = 0$. By applying the contravariant additive functor $(\ \ )^*$ and identifiyng P_0^{**} with P_0 we arrive at $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$, where $P_0 \simeq R^* \oplus S^*$. Since $f^*|_S = f^* \iota_S = 0$, where ι_S is the natural embedding of S into $P_0^* \simeq R \oplus S$, we get that $0 = (f^* \iota_S)^* = \pi_{S^*} f$ where f is identified with f^{**} and π_{S*} is the natural projection onto S^*. Thus $\operatorname{Im} f \subseteq R^*$. Since $X = \operatorname{Coker} f \simeq P_0/\operatorname{Im} f \simeq S^* \oplus (R^*/\operatorname{Im} f)$. This is impossible, since X is assumed to be indecomposable nonprojective.

Now assume that $P_1^* \xrightarrow{q} \operatorname{Tr} X$ is not a projective cover. For the same reason as above we get a non-trivial decomposition of $P_1^* \simeq S \oplus T$ with $S \longrightarrow \operatorname{Tr} X$ the projective cover and $q|_T = 0$, in other words $T \subseteq \operatorname{Ker} q$. This yields $\operatorname{Ker} q \simeq M \oplus T$ with $M = (S \cap \operatorname{Ker} q)$. Since T is a direct summand in a projective, T is itself projective. Now $P_0^* \longrightarrow \operatorname{Ker} q$ is the projective cover we get the following commutative diagram:

$$\begin{array}{ccc} & & T \\ & {\scriptstyle h}\swarrow & \big\| \\ P_0^* & \xrightarrow{f'} & T \end{array}$$

where $f' = p_T f^*$ and p_T is the canonical projection $p_T : M \oplus T \longrightarrow T$. This then implies that $P_0^* = \operatorname{Im} h \oplus \operatorname{Ker} f' \simeq T \oplus \operatorname{Ker} f'$ since $t = f'h(t) = 0$ for $t \in \operatorname{Ker} h$. Let $K = \operatorname{Ker} f'$. If we now take the projective covers of M and T, we get the following diagram

$$\begin{array}{ccc} K \oplus T & \simeq & P_M \oplus P_T \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} \pi_M & 0 \\ 0 & \pi_T \end{pmatrix}} \\ M \oplus T & = & M \oplus T \end{array}$$

with $\pi_M : P_M \longrightarrow M$ and $\pi_T : P_T \longrightarrow T$ projective covers. This then yields a decomposition of f^* into $f_K \oplus f_T$ where f_T is an isomorphism onto T. If we now apply $(\ \ )^*$ and identify P_i^{**} with P_i for $i = 0, 1$ we get that P_0 and P_1 both have a projective direct summand isomorphic to T^* and f decomposes to $f_{K^*} \oplus f_{T^*}$ where f_{T^*} is an isomorphism of T^*. Recall that $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ is exact, which means that $T^* \subseteq \operatorname{Ker} p$. This then contradicts the assumption that $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ is a minimal projective presentation, since the composition $K^* \longrightarrow K^* \oplus T^* \longrightarrow X$ is epimorphic with the inclusion map on the left, however the inclusion is not epi, hence violating the condition of essensiality.

(ii) It follows from the fact that $\operatorname{Hom}_\Lambda(\ \ , \Lambda) : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda^{op})$ is a duality and the universal property of Coker .

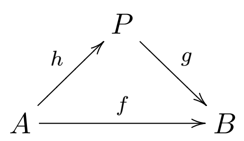
(iii) If X is projective indecomposable, then $0 \longrightarrow 0 \longrightarrow X \longrightarrow X \longrightarrow 0$ is a minimal projective presentation. Then $X^* \longrightarrow 0 \longrightarrow \operatorname{Tr} X \longrightarrow 0$ is exact and we get $\operatorname{Tr} X = 0$. If $\operatorname{Tr} X = 0$ and X indecomposable then $P_0^* \longrightarrow P_1^* \longrightarrow 0 \longrightarrow 0$ is not a minimal projective presentation, thus by (i) we get that X is projective. Now by (ii) we get the case when X is decomposable. □

Tr will not usually define a functor between module categories in general, in order for it to be a functor we need to move to **stable categories** modulo projectives. We will denote by $\mathscr{P}(A, B)$ the R-submodule of $\operatorname{Hom}_\Lambda(A, B)$ which consist of all morphisms $f : A \longrightarrow B$ which factor through a projective, i.e. there is a projective,P, in $\operatorname{mod} \Lambda$ such that $f = hg$ for some morphisms $g : A \longrightarrow P$ and $h : P \longrightarrow B$. We then define $\underline{\operatorname{Hom}}_\Lambda(A, B) = \operatorname{Hom}_\Lambda(A, B)/\mathscr{P}(A, B)$. Furthermore, we will denote the stable category of finitely generated Λ-modules modulo projectives by $\underline{\operatorname{mod}}\, \Lambda$, which objects are exactly the objects of $\operatorname{mod} \Lambda$ and morphisms are the factors $\underline{\operatorname{Hom}}_\Lambda(A, B)$. We state the next proposition without proof, for proof see [ARS97, IV.1].

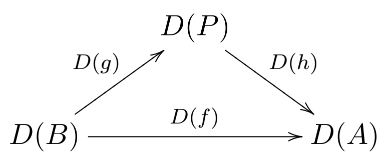
Proposition 4.2. *The functor* $\operatorname{Tr} : \underline{\operatorname{mod}}\, \Lambda \longrightarrow \underline{\operatorname{mod}}\, \Lambda^{\mathrm{op}}$ *is an equivalence of categories.* □

Since R is artin, we only have finitely many non-isomorphic simple R-modules, say $S_1, \ldots, S_n$. Let $I = \bigoplus I(S_i)$, where $S_i \longrightarrow I(S_i)$ is the injective envelope. Then the contravariant functor $\operatorname{Hom}_R(\ \ , I) : \operatorname{mod} R \longrightarrow \operatorname{mod} R$ is a duality, and this duality then induces a duality $D = \operatorname{Hom}_R(\ \ , I) : \operatorname{mod} \Lambda \longrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$. For a path algebra over a field k, this duality reduces to $D = \operatorname{Hom}_k(\ \ , k)$. If (V, T) is a representation of a quiver Γ, then $D(V, T)$ is the representation of the opposite quiver, Γ^{op}, with $(D(V))_i = D(V_i) = V_i^*$ the usual dual space of a vector space, and for $T_\alpha : V_i \longrightarrow V_j$ in (V, T) we have $D(T_\alpha) : D(V_j) \longrightarrow D(V_i)$ given by $D(T_\alpha)(g)(v) = g(T_\alpha v)$ for $g \in D(V_j)$ and $v \in V_i$. If we are in the case of the Kronecker algebra, and given the representation $k \overset{1}{\underset{t}{\rightrightarrows}} k$, then $D(V_1) = D(V_2) = k$ and $D(1) = 1$ and $D(t) = t$ by the correct change of basis. Thus the dual representation is then $k \overset{1}{\underset{t}{\leftleftarrows}} k$.

We might now be intereseted in knowing what happens on $\underline{\operatorname{mod}}\, \Lambda$ under the action of D. If $f \in \mathscr{P}(A, B)$, that is



is commutative for some projective P. Since D is a duality we get the following commutative diagram



with $D(P)$ injective in $\operatorname{mod} \Lambda^{\mathrm{op}}$. Thus if $f : A \longrightarrow B$ factors through a projective, $D(f) : D(B) \longrightarrow D(A)$ factors through an injective. We are then tempted to introduce the stable category modulo injectives. Let A and B be in $\operatorname{mod} \Lambda$ and let $\mathscr{I}(A, B) \subseteq \operatorname{Hom}_\Lambda(A, B)$ be the R-submodule consisting of all morphism which factor through an injective, that is all morphisms $f : A \longrightarrow B$ which for some $g : A \longrightarrow I$ and $h : I \longrightarrow B$ and I injective in $\operatorname{mod} \Lambda$ are such that $f = hg$. We will usually denote the factor module

$\mathrm{Hom}_\Lambda(A,B)/\mathscr{I}(A,B)$ by $\overline{\mathrm{Hom}}_\Lambda(A,B)$. We will then write $\overline{\mathrm{mod}}\,\Lambda$ when refering to the stable cateory modulo injectives, that is the category consisting of the same objects as $\mathrm{mod}\,\Lambda$ but the hom-sets are the factor modules $\overline{\mathrm{Hom}}_\Lambda(A,B)$ for Λ-modules A and B. From the observation above we see that the duality $D : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda$ induces a duality $D : \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$. Combining this fact with Proposition 4.2 results in the following proposition.

Proposition 4.3. *The compositions* $D\,\mathrm{Tr} : \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$ *and* $\mathrm{Tr}\,D : \overline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda$ *are inverse equivalences of categories.* □

We now give some basic properties of $D\,\mathrm{Tr}$ following Proposition 4.1.

Proposition 4.4.

(i) *If* $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$ *is a minimal projective presentation of an indecomposable non-projective* Λ*-module* X*, then* $0 \longrightarrow D\,\mathrm{Tr}\,X \xrightarrow{g} D(P_0^*) \xrightarrow{D(f^*)} D(P_1^*)$ *is a minimal injective copresentation, that is* $g : D\,\mathrm{Tr}\,X \longrightarrow D(P_0^*)$ *and the induced morphism* $h : \mathrm{Coker}\,g \longrightarrow D(P_1^*)$ *are injective envelopes.*

(ii) $(DI_0)^* \longrightarrow (DI_1)^* \longrightarrow \mathrm{Tr}\,DX \longrightarrow 0$ *is a minimal projective presentation of* $\mathrm{Tr}\,DX$ *whenever* $0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$ *is a minimal injective copresentation of a noninjective module* X*.*

(iii) $D\,\mathrm{Tr}\,(\bigoplus_{i\in I} X_i) \simeq \bigoplus_{i\in I} D\,\mathrm{Tr}\,X_i$ *where* I *is finite and all* X_i*'s are in* $\mathrm{mod}\,\Lambda$*.*

(iv) $D\,\mathrm{Tr}\,X = 0$ *if and only if* X *is projective.*

(v) $D\,\mathrm{Tr}\,X$ *has no nonzero injective direct summands for all* X *in* $\mathrm{mod}\,\Lambda$*.*

(vi) *For all* X *in* $\mathrm{mod}\,\Lambda$, $(\mathrm{Tr}\,D)D\,\mathrm{Tr}\,X \simeq X_{\mathscr{P}}$ *where* $X_{\mathscr{P}}$ *are the nonprojective direct summands of* X*.*

(vii) *If* X *and* Y *have no nonzero projective direct summands, then* $D\,\mathrm{Tr}\,X \simeq D\,\mathrm{Tr}\,Y$ *if and only if* $X \simeq Y$*.* □

Another important connection to note is that if Λ is in addition hereditatry and for X and Y in $\mathrm{mod}\,\Lambda$ with X, Y with no nonzero projective direct summands, then $\mathscr{P}(X,Y) = 0$. Let $f \in \mathscr{P}(X,Y)$, then there is some projective P in $\mathrm{mod}\,\Lambda$ and $h : X \longrightarrow P$ and $g : P \longrightarrow Y$ such that $f = gh$. Now this implies $h : X \longrightarrow \mathrm{Im}\,f \subseteq P$ is split epi, in other words $X \simeq \mathrm{Im}\,h \oplus Z$, with $\mathrm{Im}\,h$ projective. This yields $\mathrm{Im}\,h = 0$ and $f = 0$. In view of this we denote by $\mathrm{mod}_{\mathscr{P}}\,\Lambda$ the full subcategory of $\mathrm{mod}\,\Lambda$ in which the objects are the objects X in $\mathrm{mod}\,\Lambda$ with $X \simeq X_{\mathscr{P}}$, that is all objects with no nonzero projective direct summands. The assertation above the yields a useful property calculationwise for $D\,\mathrm{Tr}$.

Proposition 4.5. Λ *a hereditary artin algebra. Then there is an equivalence of categories between* $\mathrm{mod}_{\mathscr{P}}\,\Lambda$ *and* $\underline{\mathrm{mod}}\,\Lambda$*.* □

We now turn our attention to studying some important modules that show up in accordance with the functors $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$. Let Λ be a hereditary artin algebra, we say that a Λ-module Q is **preprojective** if there is a nonnegative integer such that $(D\,\mathrm{Tr}\,)^n Q$ is projective. Furthermore, Q is **indecomposable preprojective** if Q is indecomposable. Dually, we define a Λ-module J to be **preinjective** if $(\mathrm{Tr}\,D)^m J$ is injective for some nonnegative integer m. We say that J is **indecomposable preinjective** if it is indecomposable and preinjective. If an indecomposable module is neither preprojective nor preinjective, then we call it **regular**. An easy observation to make is that Q is preprojective if and only if $(D\,\mathrm{Tr}\,)^n Q = 0$ for some n. Dually, J is preinjective if and only if $(\mathrm{Tr}\ D)^m J = 0$, where $m \geq 0$

We give a complete characterization of finitely generated indecomposable preprojectives (preinjectives) for a hereditary artin algebra.

Proposition 4.6. *Let Λ be an hereditary artin algebra. Q is an indecomposable preprojective Λ-module if and only if there is an indecompsable projective Λ-module, P, such that $Q \simeq (\mathrm{Tr}\ D)^m P$ for some $m \geq 0$.*

Proof. If $Q \simeq (\mathrm{Tr}\ D)^m P$ then it is preprojective by definition, and indecomposable since the functor $D\,\mathrm{Tr}$ is additive. Conversely, if Q is indecomposable projective this is trivial. Assume Q non-projective indecomposable preprojective. Then there exists an indecomposable projective $P \neq 0$ and some $m \in \mathbb{N}$ such that $P \simeq (D\,\mathrm{Tr}\,)^m Q$. Applying the $\mathrm{Tr}\ D$ functor m times gives us $(\mathrm{Tr}\ D)^m P \simeq (\mathrm{Tr}\ D)^m (D\,\mathrm{Tr}\,)^m Q \simeq Q$. As desired. ☐

The dual of this proposition follows by a small observation, namely

Lemma 4.7. *If Λ is an artin algebra, then $(\mathrm{Tr}\ D)^n DX \simeq D(D\,\mathrm{Tr}\,)^n X$ for all X in $\mathrm{mod}\,\Lambda$.*

Proof. We prove this by induction on n. For $n = 1$ this is trivially true since $D^2 \simeq 1$ as functors. Assume that it holds for $n = k$, $k \geq 1$.

$$
\begin{aligned}
(\mathrm{Tr}\ D)^{k+1} DX = \mathrm{Tr}\ D(\mathrm{Tr}\ D)^k DX &\simeq (\mathrm{Tr}\ D) D(D\,\mathrm{Tr}\,)^k X \simeq D(D\,\mathrm{Tr}\,)(D\,\mathrm{Tr}\,)^k X \\
&= D(D\,\mathrm{Tr}\,)^{k+1} X
\end{aligned}
$$

which concludes the proof. ☐

Hence we get the dual result of Proposition 4.6.

Proposition 4.8. *Λ as above. J is a indecomposable preinjective Λ-module if and only if there is a indecompsable injective Λ-module, I, such that $J \simeq (D\,\mathrm{Tr}\,)^n I$ for some $n \geq 0$.* ☐

5 The Coxeter Transformation

Let Γ be a quiver and assume Λ is finite dimensional and $\mathrm{gl.dim}\,\Lambda < \infty$, where Λ is the path algebra of Γ. Let $S_1, \ldots, S_m$ be a complete list of simple Λ-modules. It is easy to check that the set of the dimension vectors of the simple modules, $\{\dim S_1, \ldots, \dim S_m\}$, give a basis for the abelian group $\mathbb{Z}^m$. However, if $P_1, \ldots, P_m$ is a complete list of projective indecomposable Λ-modules, such that $P_j \longrightarrow S_j \longrightarrow 0$ is a projective cover, then $\{\dim P_i\}_{i=1}^m$ is also a basis for $\mathbb{Z}^m$. This is obtained by the fact that Λ has a finite global dimension, say, n. Then there is a long exact sequence $0 \longrightarrow Q_{i_n} \longrightarrow \cdots \longrightarrow Q_{i_1} \longrightarrow Q_{i_0} \longrightarrow S_i \longrightarrow 0$ with Q_{i_j} projective for each simple S_i. This yields, $\dim S_i = \sum_{j=0}^n (-1)^j Q_{i_j}$. Clearly, $\dim Q_{i_j}$ is in $\mathrm{span}\{\dim P_i\}_{i=1}^n$, this yields that $\{\dim P_i\}$ is a basis for $\mathbb{Z}^m$. Similarily we get that $\{\dim I_j\}_{j=1}^m$ is a basis for $\mathbb{Z}^m$, where $I_1, \ldots, I_m$ are all the indecomposable injectives in $\mathrm{mod}\,\Lambda$, and $0 \longrightarrow S_j \longrightarrow I_j$ is a injective envelope. Using the notation above we can define the **Coxeter transformation**, c, to be $c(\dim P_j) = -\dim I_j$ for $j = 1, \ldots, m$. Since $I_j \simeq D\,P_j^*$ we see that $c(\dim P_j) = -\dim D\,P_j^*$. We present some properties of the Coxeter transformation here.

Proposition 5.1. *Let Λ be a hereditary artin algebra and let c be the Coxeter transformation. Then for indecomposable Λ-module X we have the following:*

(i) If X is nonprojective, then $c(\dim X) = \dim D \operatorname{Tr} X$.

(ii) X is projective if and only if $c(\dim X)$ is negative.

(iii) $c(\dim X)$ is either positive or negative.

(iv) If X is noninjective, then $c^{-1}(\dim X) = \dim \operatorname{Tr} DX$.

(v) X is injective if and only if $c^{-1}(\dim X)$ is negative.

(vi) $c^{-1}(\dim X)$ is either positive or negative.

Proof. (i) Let X be an indecomposable nonprojective Λ-module and let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a minimal projective presentation. Applying $(\ )^*$ results in the exact sequence $0 \longrightarrow X^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$. However, $X^* = 0$ since there are no nonzero maps from an indecomposable nonprojective module to an hereditary ring. Thus we are left with $0 \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$. This yields the short exact sequence $0 \longrightarrow D \operatorname{Tr} X \longrightarrow DP_1^* \longrightarrow DP_0^* \longrightarrow 0$, and calculating the dimension vectors give

$$\dim D \operatorname{Tr} X = \dim DP_1^* - \dim DP_0^* = \\ c(-\dim P_1) + c(\dim P_0) = c(\dim P_0 - \dim P_1) = c(\dim X)$$

(ii) If X is projective, then $c(\dim X)$ is trivially negative by definition. If $c(\dim X)$ is negative then X is trivially projective by (i).

(iii) This follows by (ii). The rest of the propostion follows by the duality D. □

Some easy made observations follow from this proposition, we give them in the following corollary.

Corollary 5.2. *Let X and Y be indecomposable Λ-modules for a hereditary artin algebra. If c is the Coxeter transformation and $\dim X = \dim Y$.*

(i) X is projective if and only if Y is projective.

(ii) If X is projective, then $X \simeq Y$.

(iii) X is preprojective if and only if $c^n(\dim X)$ is negative for some $n \in \mathbb{N}$.

(iv) If X is preprojective, then $X \simeq Y$.

(v) X is injective if and only if Y is injective.

(vi) If X is injective, then $X \simeq Y$.

(vii) X is preinjective if and only if $c^{-m}(\dim X)$ is negative for some $m \in \mathbb{N}$.

(viii) If X is preinjective, then $X \simeq Y$.

Proof. (i) If X and Y are indecomposable, then by (ii) in the previous proposition we get that X is an indecomposable projective module if and only if $c(\dim X)$ is negative, since $c(\dim X) = c(\dim Y)$ we get that X is projective if and only if Y is projective.

(ii) Let X be projective and let $\dim Y = \dim X$, then by (i) we get that Y is projective. Since $\dim Y = \dim X$ there is a nonzero map from X to Y, say $f : X \longrightarrow Y$. Since Λ is hereditary we get that $X \simeq \operatorname{Im} f \oplus \operatorname{Ker} f$, now since X is indecomposable and f nonzero, we get that $\operatorname{Ker} f = 0$. Hence $X \simeq Y$, since $0 \longrightarrow X \longrightarrow Y \longrightarrow Y/\operatorname{Im} f \longrightarrow 0$ and $\dim Y/\operatorname{Im} f = 0$.

(iii) If X is indecomposable, we know that X is preprojective if and only if $(D\,\mathrm{Tr}\,)^m X$ is projective for some nonnegative m, then by the previous proposition we get that this is equivivalent to $c^n(\dim X)$ being negative for some natural number n.

(iv) For X and Y indecomposable with $\dim X = \dim Y$ and let X be preprojective. By (iii) we know that $c^n(\dim X) = c^n(\dim Y)$ is negative with $n \in \mathbb{N}$. Let n be the smallest such number, then $\dim(D\,\mathrm{Tr}\,)^{n-1}X = c^{n-1}(\dim X) = c^{n-1}(\dim Y) = \dim(D\,\mathrm{Tr}\,)^{n-1}Y$ is positive. Thus $(D\,\mathrm{Tr}\,)^{n-1}X$ and $(D\,\mathrm{Tr}\,)^{n-1}Y$ are indecomposable projective and by (ii) are isomorphic. Thus $X \simeq Y$.

The rest are just dual statements of the ones proven and follow trivially. □

In the case of the Kronecker algebra we have that $(1,2) \longmapsto (-1,0)$ and $(0,1) \longmapsto (-2,-1)$ by the Coxeter transformation. Hence we get the following matrix:

$$c = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = I - \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$

From this it is easy to see the powers of c

$$c^n = I - n \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}$$

for $n \in \mathbb{Z}$.

6 Preprojective and Preinjective indecomposables

Recall that the indecomposable preprojective and preinjective modules over an artin algebra occur as $D\,\mathrm{Tr}$ and $\mathrm{Tr}\ D$ shifts of the indecomposable projectives and injectives. Thus we are interested in investigating shifts of the modules found in section 3. We do this in the manner of the Coxeter transformation. Let P_1, P_2, I_1, I_1 be the modules presented in Section 3, hence

$$\dim(\mathrm{Tr}\ D)^t P_1 = c^{-t}(\dim P_1) = \left(I + t\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2t+1 \\ 2t+2 \end{pmatrix}$$

$$\dim(\mathrm{Tr}\ D)^t P_2 = \begin{pmatrix} 2t \\ 2t+1 \end{pmatrix}$$

$$\dim(D\,\mathrm{Tr}\,)^s I_1 = \begin{pmatrix} 2s+1 \\ 2s \end{pmatrix}$$

$$\dim(D\,\mathrm{Tr}\,)^s I_2 = \begin{pmatrix} 2s+2 \\ 2s+1 \end{pmatrix}$$

What we now have established is that the dimension vectors of the preprojectives are of the form $(n, n+1)$ and the dimension vectors of the preinjectives are $(m+1, m)$, where $n, m \geq 0$.

Now let Λ be the Kronecker algebra and let Q_n be the Λ-module corresponding to the following representation

$$k^n \overset{\binom{I}{0}}{\underset{\binom{0}{I}}{\rightrightarrows}} k^{n+1}$$

and let J_n be the Λ-module corresponding to the representation

$$k^{n+1} \underset{(0,I)}{\overset{(I,0)}{\rightrightarrows}} k^n$$

where I is the $n \times n$ identity matrix. We now claim that Q_n and J_m are indecomposable for all n, m.

Proof. It will suffice to show this for Q_n, since a similar argument will hold for J_m. So assume $Q_n = V \oplus W$, where $0 \neq V, W$ are in $\operatorname{mod} \Lambda$. Passing to representations we get

$$k^n \underset{\binom{0}{I}}{\overset{\binom{I}{0}}{\rightrightarrows}} k^{n+1} \;=\; V_1 \underset{f_\beta}{\overset{f_\alpha}{\rightrightarrows}} V_2 \;\oplus\; W_1 \underset{g_\beta}{\overset{g_\alpha}{\rightrightarrows}} W_2$$

Since V and W are direct summands in Q_n they are also subrepresentations of Q_n, and thus the dimensions of the corresponding vectorspaces should add up, i.e. $n = \dim_k V_1 + \dim_k W_1$ and $n+1 = \dim_k V_2 + \dim_k W_2$. Since V is a subrepresentation of Q_n we have the following commutative diagram

$$\begin{array}{ccc} k^n & \underset{\binom{0}{I}}{\overset{\binom{I}{0}}{\rightrightarrows}} & k^{n+1} \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ V_1 & \underset{f_\beta}{\overset{f_\alpha}{\rightrightarrows}} & V_2 \end{array}$$

where i_1, i_2 are monomorphisms. Let $\dim_k V_1 = t$, thus given a basis B for V_1 it is sent to a linearly independent set in k^n, which in turn is sent to a set of t linearly independent vectors in k^{n+1} by $\binom{0}{I}$. However, if we consider $\binom{I}{0}$, we get at least one new vector linearly independent of the ones obtained earlier. Since the diagram above commutes and $V_2 \subseteq k^{n+1}$ as vector spaces, we get that $\dim_k V_2 \geq \dim_k V_1 + 1$. A similar argument for the representation corresponding to W yields $\dim_k W_2 \geq \dim_k W_1 + 1$. Resulting in a contradiction, namely

$$n + 1 = \dim_k V_2 + \dim_k W_2 \geq \dim_k V_1 + 1 + \dim_k W_1 + 1 = n + 2.$$

□

Now that we have established that Q_n and J_m are indecomposable, we may use Corollary 5.2. Since $\dim Q_{2t} = \dim(\operatorname{Tr} D)^t P_2$, $\dim Q_{2t+1} = \dim(\operatorname{Tr} D)^t P_1$, $\dim J_{2s} = \dim(D \operatorname{Tr})^s I_1$ and $\dim J_{2s+1} = \dim(D \operatorname{Tr})^s I_2$, we obtain by Corollary 5.2 that $Q_{2t+i} \simeq (\operatorname{Tr} D)^t P_{i+1}$ and $J_{2s+i} \simeq (D \operatorname{Tr})^s I_{i+1}$ for $i = 0, 1$. Moreover, these are all the preprojective and preinjective indecomposable Λ-modules up to isomorphism. We summarize.

Proposition 6.1. *Notation as above. Let X be an indecomposable Λ-module.*

(i) X is preprojective if and only if $X \simeq Q_n$ for some non-negative integer n.

(ii) X is preprojective if and only if $X \simeq J_m$ for some non-negative integer m. □

7 Regular indecomposables

We now turn our attention towards the regular Λ-modules, Λ still beeing the Kronecker algebra, however we now have to assume k to be algebraically closed field. Let $R_{a,b}$ be the Λ-module corresponding to the representation $k \overset{a}{\underset{b}{\rightrightarrows}} k$. We claim now that $R_{a,b}$ is indecomposable for $(a,b) \neq (0,0)$ and $R_{a,b} \simeq R_{a',b'}$ if and only if $(a,b) = (ta', tb')$ for some $t \in k \setminus \{0\}$

Proof. First observe if $(a,b) = (0,0)$ then $R_{a,b}$ decomposes to the two simple Λ-modules, i.e.

$$k \overset{0}{\underset{0}{\rightrightarrows}} k \;=\; k \rightrightarrows 0 \;\oplus\; 0 \rightrightarrows k$$

So let $(a,b) \neq (0,0)$ and assume that $R_{a,b} = S \oplus T$ for some $0 \neq S, T$ in $\operatorname{mod} \Lambda$. If we pass to representations we get that S is a subrepresentation of $R_{a,b}$ since S is a direct summand of $R_{a,b}$. Hence we have the following commutative diagram

$$\begin{array}{ccc} k & \overset{a}{\underset{b}{\rightrightarrows}} & k \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_2} \\ V_1 & \overset{f_\alpha}{\underset{f_\beta}{\rightrightarrows}} & V_2 \end{array}$$

with i_j monomorphisms for $j = 1,2$ and $V_1 \overset{f_\alpha}{\underset{f_\beta}{\rightrightarrows}} V_2$ beeing the representation corresponding to S. This yields $V_1, V_2 = 0, k$. However, we cannot have $V_1 = V_2 = k$, since this would imply that $T = 0$. Since $S \neq 0$, we arrive at either $S_1 : k \rightrightarrows 0$ or $S_2 : 0 \rightrightarrows k$, nevertheless the first one is not a subrepresentation and the second cannot be a direct summand since it would imply that the first also was a direct summand of $R_{a,b}$ when $(a,b) \neq (0,0)$, this is because the above diagram needs to commute, hence $R_{a,b}$ is indecomposable. Let $R_{a,b}$ and $R_{a',b'}$ be indecomposable, and assume that we have the following commutative diagram

$$\begin{array}{ccc} k & \overset{a}{\underset{b}{\rightrightarrows}} & k \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle y} \\ k & \overset{a'}{\underset{b'}{\rightrightarrows}} & k \end{array}$$

where $x, y \in k$. By the commutativity of the diagram we get $ax = ya'$ and $bx = yb'$, thus $(a,b) = (ta', tb')$ for $t = x^{-1}y$, that is $R_{a,b} \simeq R_{a',b'}$ if and only if $(a,b) = (ta', tb')$ for nonzero $t \in k$. $\square$

In light of this discussion, we introduce some notation for further use. Let R_λ be the module corresponding to the representation $k \overset{\lambda}{\underset{1}{\rightrightarrows}} k$ for $\lambda \in k$. And let R_ω be the module corresponding to the representation $k \overset{1}{\underset{0}{\rightrightarrows}} k$. It is easy to see that $R_\lambda \simeq R_{a,b}$ when $a, b \neq 0$, just put $\lambda = b^{-1}a$. We are now able to describe the dimension vectors of the indecomposable Λ-modules. We present this in the following proposition:

Proposition 7.1. *Let Λ be the Kronecker algebra. Let X be indecomposable and let* $\operatorname{\mathbf{dim}} X = (s,t)$.

(i) If $s < t$ then $s = t - 1$ and X is preprojective.

(ii) If $s > t$ then $s = t + 1$ and X is preinjective.

(iii) If X is regular, then $s = t$ and $R_\lambda \subseteq X$ submodule for some $\lambda \in k \cup \{\omega\}$.

Proof. (i) If $s < t$ then the Coxeter transformation yields $c^s \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} - 2s \begin{pmatrix} t-s \\ t-s \end{pmatrix}$. Hence $c^s \dim X$ is negative, which by Proposition 5.1 (ii) gives that X is preprojective. However, this means that $X \simeq Q_n$ by Proposition 6.1, and thus $\dim X = (s, s+1)$.

(ii) If $t < s$, then applying the Coxeter transformation $-s$ times gives X preinjective and $X \simeq J_m$ for some m, i.e. $s = t + 1$.

(iii) If X is regular then $c^n \dim X$ is never negative for $n \in \mathbb{Z}$. In view of (i) and (ii) it follows that $\dim X = (s, s)$. Let $V \underset{B}{\overset{A}{\rightrightarrows}} V$ be the representation corresponding to X. We assertain that either $\operatorname{Ker} B = 0$ or $\operatorname{Ker} B \neq 0$. Assume that $\operatorname{Ker} B = 0$, that is, B is invertible. Now consider $B^{-1}A$, since k is algebraically closed there is nonzero eigenvector $v \in V$ and eigenvalue $\lambda \in k$ such that $(B^{-1}A)v = \lambda v$, that is $Av = \lambda Bv$. Now consider the following diagram

$$\begin{array}{ccc} V & \underset{B}{\overset{A}{\rightrightarrows}} & V \\ {\scriptstyle v}\uparrow & & \uparrow{\scriptstyle Bv} \\ k & \underset{1}{\overset{\lambda}{\rightrightarrows}} & k \end{array}$$

A quick check will show that this diagram does commute. Furthermore, it will be an embedding, thus R_λ is a subrepresentation of X, that is R_λ is a submodule of X.

Now assume $\operatorname{Ker} B \neq 0$, then there exists $0 \neq v \in V$ such that $Bv = 0$. Moreover, $Av \neq 0$, since X is assumed to be indecomposable. Else we would have $I_1 : k \rightrightarrows 0$ as subrepresentation, nevertheless this means that I_1 a direct summand of X since I_1 is injective. Consider the following diagram

$$\begin{array}{ccc} V & \underset{B}{\overset{A}{\rightrightarrows}} & V \\ {\scriptstyle v}\uparrow & & \uparrow{\scriptstyle Av} \\ k & \underset{0}{\overset{1}{\rightrightarrows}} & k \end{array}$$

It follows that R_ω is then a submodule of X. □

There are some observations to be made here, we give them in form of the next proposition:

Proposition 7.2. *Notation as above. Let $\lambda, \mu \in k \cup \omega$.*

(i) $\operatorname{End}_\Lambda(R_\lambda) \simeq k$.

(ii) $\operatorname{Hom}_\Lambda(R_\lambda, R_\mu) = 0$ *when* $\lambda \neq \mu$.

(iii) $\operatorname{Ext}^1_\Lambda(R_\lambda, R_\mu) = 0$ *for* $\lambda \neq \mu$.

Proof. (i) and (ii) are straightforward to check. For (iii) we form the minimal projective presentation of R_λ. $0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow R_\lambda \longrightarrow 0$, where P_1, P_2 are as in Section 3. Applying $\operatorname{Hom}_\Lambda(\ \ , R_\mu)$ for $\mu \neq \lambda$. This results in the following exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_\Lambda(R_\lambda, R_\mu) \longrightarrow \operatorname{Hom}_\Lambda(P_1, R_\mu) \longrightarrow \operatorname{Hom}_\Lambda(P_2, R_\mu) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_\Lambda(R_\lambda, R_\mu) \longrightarrow \operatorname{Ext}^1_\Lambda(P_1, R_\mu) \longrightarrow \operatorname{Ext}^1_\Lambda(P_2, R_\mu) \longrightarrow \cdots \end{aligned}$$

From (ii) we get that the first term is zero, also $\operatorname{Ext}^1_\Lambda(P_i, R_\mu) = 0$ for $i = 1, 2$ since P_i is projective. Consequently, we are left with the short exact sequence $0 \longrightarrow \operatorname{Hom}_\Lambda(P_1, R_\mu) \longrightarrow \operatorname{Hom}_\Lambda(P_2, R_\mu) \longrightarrow \operatorname{Ext}^1_\Lambda(R_\lambda, R_\mu) \longrightarrow 0$. Nevertheless, we see that $\dim_k \operatorname{Hom}_\Lambda(P_1, R_\mu) = \dim_k \operatorname{Hom}_\Lambda(P_2, R_\mu) = 1$, this, in turn, implies that $\dim_k \operatorname{Ext}^1_\Lambda(R_\lambda, R_\mu) = 0$, that is $\operatorname{Ext}^1_\Lambda(R_\lambda, R_\mu) = 0$. □

We now generalize the representations R_λ and R_ω to higher dimensions. For $j \geq 1$ define the following. Let $R_{\lambda,j}$ be the Λ-module corresponding to the representation $k^j \overset{J_\lambda}{\underset{I}{\rightrightarrows}} k^j$, where

$$J_\lambda = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

i.e. a matrix consisting of one Jordan cell, and I is the identy matrix. Denote the module corresponding to the representation $k^j \overset{I}{\underset{J_0}{\rightrightarrows}} k^j$ by $R_{\omega,j}$. Here $J_0 = J_\lambda$ for $\lambda = 0$. The remainder of the section is dedicated to proving that all the regular indecomposables are of the form $R_{\lambda,j}$, where $\lambda \in k \cup \{\omega\}$ and $j \geq 1$. We give a formal statement in the next proposition.

Proposition 7.3. *If X is a regular indecomposable, then $X \simeq R_{\lambda,j}$ for some $\lambda \in k \cup \{\omega\}$ and $j \geq 1$.*

Before we indulge in the proof of this proposition we need the following lemma.

Lemma. *Let Λ be a hereditary algebra. Given R indecomposable regular and Q indecomposable preprojective, for R, Q in* $\operatorname{mod} \Lambda$*. Then there are no nonzero maps from R to Q.*

Proof. Let $f : R \longrightarrow Q$ be nonzero. Then if Q is projective then we have $\operatorname{Im} f \subseteq Q$ projective since Λ is hereditary. This means we have the following commutative diagram:

$$\begin{array}{ccc} & & \operatorname{Im} f \\ & \overset{g}{\swarrow} & \Big\| \, id \\ X & \xrightarrow{f} & \operatorname{Im} f \end{array}$$

In other words $fg = id$. We now claim that $X = \operatorname{Ker} f \oplus \operatorname{Im} g$. Let $x \in \operatorname{Ker} f \cap \operatorname{Im} g$, i.e. $\exists\, y \in \operatorname{Im} g$ such that $g(y) = x$ and $f(x) = 0$. Combining these two yields, $y = id(y) = fg(y) = f(x) = 0$, which in turn results in $x = 0$ since g is a homorphism. Also, observe that $x = gf(x) + (x - gf(x))$ for all $x \in X$. Furthermore, for $y \in \operatorname{Ker} g$ we have that $0 = fg(y) = y$, thus g is a monomorphism. This means that $\operatorname{Im} f$ is a direct summand of

R, impossible since R is indecomposable. Now if Q is nonprojective, then there is some projective P and $m \in \mathbb{N}$ such that $(D\operatorname{Tr})^m Q \simeq P$. Then we have $(D\operatorname{Tr})^m R \xrightarrow{(D\operatorname{Tr})^m f} P$. Thus $(D\operatorname{Tr})^m R$ has a projective direct summand. By applying $\operatorname{Tr} D$ functor m times we get a contradiction against R being indecomposable. $\square$

Now we are properly equipped to prove Proposition 7.3.

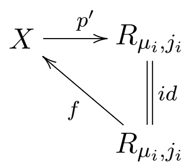
Proof of Proposition 7.3. We prove this by induction on the dimension of X. Let X be regular indecomposable, then we know by Proposition 7.1 (iii) that $\dim X = (m, m)$ for some $m \in \mathbb{N}$. We have allready shown that for $m = 1$, this is truely the case. Now assume that for $n < m$ the hypothesis holds and let $\dim X = (m, m)$. Remember that for a regular indecomposable there is some $\lambda \in k \cup \{\omega\}$ such that $R_\lambda \subset X$. This gives rise to the short exact sequence $0 \longrightarrow R_\lambda \longrightarrow X \longrightarrow Y \longrightarrow 0$, with $\dim Y = (m-1, m-1)$. Let $Y = \bigoplus_{i \in I} Y_i$ where I is finite and Y_i indecomposable. Then none of the Y_i's are preprojective, this is due to the preceding lemma. Consequently, by a dimension count, we cannot have Y_i preinjective either. Thus Y_i is regular for each $i \in I$, and by the induction hypothesis we get that $Y_i \simeq R_{\mu_i, j_i}$. We now show that $\mu_i = \lambda$ for all i. Note that we have the short exact sequence $0 \longrightarrow R_\mu \longrightarrow R_{\mu, j+1} \longrightarrow R_{\mu, j} \longrightarrow 0$. Let us now apply $\operatorname{Hom}_\Lambda(\ \ , R_\lambda)$, $\lambda \neq \mu$, on this sequence.

$$0 \longrightarrow \operatorname{Hom}_\Lambda(R_{\mu,j}, R_\lambda) \longrightarrow \cdots \longrightarrow \operatorname{Ext}^1_\Lambda(R_{\mu,j}, R_\lambda) \longrightarrow \operatorname{Ext}^1_\Lambda(R_{\mu,j+1}, R_\lambda) \longrightarrow \operatorname{Ext}^1_\Lambda(R_\mu, R_\lambda)$$

By Proposition 7.2 we have that $\operatorname{Ext}^1_\Lambda(R_\mu, R_\lambda) = 0$. Now if $\operatorname{Ext}^1_\Lambda(R_{\mu,j}, R_\lambda) = 0$ we would have $\operatorname{Ext}^1_\Lambda(R_{\mu,j+1}, R_\lambda) = 0$. However, this we are able to see by induction on j. Observe that from the exact sequence $0 \longrightarrow R_\mu \longrightarrow R_{\mu,2} \longrightarrow R_\mu \longrightarrow 0$, we get that $\operatorname{Ext}^1_\Lambda(R_{\mu,2}, R_\lambda) = 0$. Continuing in this fashion we arrive at $\operatorname{Ext}^1_\Lambda(R_{\mu,j}, R_\lambda) = 0$ for $\lambda \neq \mu$ and $j \geq 1$. If we now apply $\operatorname{Hom}_\Lambda(R_{\mu_i, j_i},\ \ )$ on $0 \longrightarrow R_\lambda \longrightarrow X \longrightarrow Y \longrightarrow 0$ we get

$$0 \longrightarrow \operatorname{Hom}_\Lambda(R_{\mu_i,j_i}, R_\lambda) \longrightarrow \operatorname{Hom}_\Lambda(R_{\mu_i,j_i}, X) \longrightarrow \operatorname{Hom}_\Lambda(R_{\mu_i,j_i}, Y) \longrightarrow \operatorname{Ext}^1_\Lambda(R_{\mu_i,j_i}, R_\lambda)$$

Now, if $\lambda \neq \mu_i$ then the last term in the above sequence is zero. This then means that for all $g \in \operatorname{Hom}_\Lambda(R_{\mu_i,j_i}, Y)$ there is a $f \in \operatorname{Hom}_\Lambda(R_{\mu_i,j_i}, X)$ such that $pf = g$, when $p : X \longrightarrow Y$ is the map in the underlying sequence. Especially, this means that we have the following commutative diagram:



Where $p' = \pi_i p$ and $\pi_i : Y \longrightarrow R_{\mu_i,j_i}$ is the canonical projection. We now claim that $X = \operatorname{Im} f \oplus \operatorname{Ker} p'$. First we see that if $x \in \operatorname{Im} f \cap \operatorname{Ker} p'$ then $x = 0$, this is beacuase $0 = p'(x) = p'f(y) = y$ for some $y \in R_{\mu_i,j_i}$ such that $f(y) = x$. Secondly, for every $x \in X$ we have $x = fp'(x) + (x - fp'(x))$. Moreover, f is a monomorphism since for $f \in \operatorname{Ker} f$ we have $y = p'f(y) = 0$. Thus $\operatorname{Im} f \simeq R_{\mu_i,j_i}$, a contradiction to X being indecomposable. This means that $\mu_i = \lambda$ for all i. We are now at the following situation, $0 \longrightarrow R_\lambda \longrightarrow X \longrightarrow \bigoplus R_{\lambda, j_i} \longrightarrow 0$. Without loss of generality we may assume that $\lambda \in k$, a similar argument will do in the case $\lambda = \omega$. Passing to representations yields the commutative diagram

$$
\begin{array}{ccc}
k & \overset{\lambda}{\underset{1}{\rightrightarrows}} & k \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
k^j & \overset{A}{\underset{B}{\rightrightarrows}} & k^j \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
k^{j-1} & \overset{J'}{\underset{I}{\rightrightarrows}} & k^{j-1}
\end{array}
$$

where I is the identy matrix and J' is the matrix consiting of the Jordan cells from all of the R_{λ,j_i} and i is the inclusion map to, say, the first coordinate and π is the natural projection such that $\pi i = 0$. Let us now consider the bottom arrows in the above digram. Since the two maps at the end are isomorphisms we get that B also is an isomorphism. Even more so, we see that

$$B = \begin{pmatrix} 1 & b \\ 0 & I \end{pmatrix}$$

with b a $1 \times (j-1)$ vector and I the $(j-1)$ identity matrix. The inverse is then given by

$$B^{-1} = \begin{pmatrix} 1 & -b \\ 0 & I \end{pmatrix}$$

If we now turn our attention towards the top arrows in the diagram above, and by a similar argument we arrive at

$$A = \begin{pmatrix} \lambda & c \\ 0 & J_\lambda^* \end{pmatrix}$$

where J_λ^* is a $(j-1) \times (j-1)$ matrix consisting of Jordan cells with λ on the diagonal. Now if we compute $B^{-1}A$ we get

$$B^{-1}A = \begin{pmatrix} \lambda & d \\ 0 & J_\lambda^* \end{pmatrix}$$

The important thing to notice here is that $\det(xI - B^{-1}A) = (x-\lambda)^j$. This then means that $B^{-1}A$ is similar to a matrix, J, on the form

$$J = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_q} \end{pmatrix}$$

where J_{λ_i} are Jordan cells with λ on the diagonal, for $1 \geq i \geq q$. Observe that

$$k^j \overset{A}{\underset{B}{\rightrightarrows}} k^j \simeq k^j \overset{B^{-1}A}{\underset{I}{\rightrightarrows}} k^j \simeq k^j \overset{J}{\underset{I}{\rightrightarrows}} k^j$$

Let $q \geq 2$ then X would decompose, and we would arrive at a contradiction. Thus $q = 1$ and $X \simeq R_{\lambda,j}$. □

References

[ARS97] M. Auslander, I. Reiten, and S. Smalø. *Representation Theory of Artin Algebras.* Cambridge University Press, 1997.