Torsion and complete dualizable objects in tt-categories with a Noetherian ring action

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New Directions in Group Theory and Triangulated Categories
March 19, 2026

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Motivation: tt-completion

Model situation—derived torsion and completion

- Let R be a commutative Noetherian ring, $\mathfrak{a} \subseteq R$ and ideal and

$$t_{\mathfrak{a}} = \varinjlim_{n} \operatorname{Hom}_R(R/\mathfrak{a}^n, -)\colon \operatorname{Mod} R \longrightarrow \operatorname{Mod} R$$

  the functor that assigns to M the largest submodule supported in $V(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$.
- Then we have a recollement (where $U = \operatorname{Spec}(R) \setminus V(\mathfrak{a})$)

$$\begin{tikzcd}
\mathrm{D}_{V(\mathfrak{a})}(R) \arrow[r, hook, bend left, "i_! = \mathrm{inc}"] \arrow[r, hook, bend right, "i_*"'] & \mathrm{D}(R) \arrow[l, two heads, "i^*"'] \arrow[r, two heads, bend left, "p^*"] \arrow[r, two heads, bend right, "p^!"'] & \mathrm{D}(U) \arrow[l, tail, "p_*"']
\end{tikzcd}$$

- Then $\Gamma_{\mathfrak{a}} := i_! i^* = \mathbb{R} t_{\mathfrak{a}}$ (local cohomology).
- [Greenlees-May 1990's, Alonso-Jeremías-Lipman, Porta-Shaul-Yekutieli, Positselski, …] $\Lambda^{\mathfrak{a}} := i_* i^* = \mathbb{L}\widehat{(-)}_{\mathfrak{a}}$, where $\widehat{M}_{\mathfrak{a}} := \varprojlim_n R/\mathfrak{a}^n \otimes_R M$.

The relevant abstract structure

Following [Barthel-Heard-Valenzuela 2018], [Benson-Iyengar-Krause-Pevtsova 2024], ...

$$\begin{tikzcd}
\mathrm{D}_{V(\mathfrak{a})}(R) \arrow[r, bend left, "i_!=\mathrm{inc}"] \arrow[r, bend right, "i_*"'] & \mathrm{D}(R) \arrow[l, two heads, "i^*"'] \arrow[r, bend left, two heads, "p^*"] \arrow[r, bend right, two heads, "p^!"'] & \mathrm{D}(U) \arrow[l, tail, "p_*"']
\end{tikzcd}$$

- Notation: $\Gamma_{\mathfrak{a}} := i_! i^* = \mathbb{R}t_{\mathfrak{a}}$, $\Lambda^{\mathfrak{a}} := i_* i^* = \mathbb{L}\widehat{(-)}_{\mathfrak{a}}$,
  $L_{\mathfrak{a}} := p_* p^*$, $V_{\mathfrak{a}} := p_* p^!$.
- There are three distinguished triangulated subcategories of $\mathrm{D}(R)$ forming a "torsion-torsion-free triple" in $\mathcal{T} := \mathrm{D}(R)$:

$$(\Gamma_{\mathfrak{a}}\mathcal{T}, L_{\mathfrak{a}}\mathcal{T} = V^{\mathfrak{a}}\mathcal{T}, \Lambda^{\mathfrak{a}}\mathcal{T}).$$

- Both $\Gamma_{\mathfrak{a}}\mathcal{T}$ and $\Lambda^{\mathfrak{a}}\mathcal{T}$ are closed symmetric monoidal:
  $\left(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes^{\mathbb{L}}_{R}, \Gamma_{\mathfrak{a}}R, \Gamma_{\mathfrak{a}}\mathbb{R}\mathrm{Hom}_R\right)$,
  $\left(\Lambda^{\mathfrak{a}}\mathcal{T}, \Lambda^{\mathfrak{a}}(- \otimes^{\mathbb{L}}_{R} -), \Lambda^{\mathfrak{a}}R, \mathbb{R}\mathrm{Hom}_R\right)$.

The Matlis-Greenlees-May equivalence

- Recall that for $\mathfrak{a} \subseteq R$ and $\mathcal{T} = \mathrm{D}(R)$ we have

$$\begin{tikzcd}
\mathrm{D}_{V(\mathfrak{a})}(R) \arrow[r, hook, bend left, "i_!=\mathrm{inc}"] \arrow[r, hook, bend right, "i_*"'] & \mathrm{D}(R) \arrow[l, two heads, "i^*" description] \arrow[r, two heads, bend left, "p^*"] \arrow[r, two heads, bend right, "p^!"'] & \mathrm{D}(U) \arrow[l, tail, "p_*" description]
\end{tikzcd}$$

- The full subcategory $\Lambda^{\mathfrak{a}}\mathcal{T} \subset \mathrm{D}(R)$ is also given cohomologically: $X \in \Lambda^{\mathfrak{a}}\mathcal{T}$ iff $H^n(X) \in \operatorname{Mod} R$ is derived complete for all n.
- Here, M is derived complete if $M \xrightarrow{\sim} \mathbb{L}_0\widehat{M}_{\mathfrak{a}}$. Equivalently:

$$\widehat{R^{(J)}}_{\mathfrak{a}} \to \widehat{R^{(I)}}_{\mathfrak{a}} \to M \to 0$$

- Matlis-Greenlees-May equivalence

$$\big(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes^{\mathbb{L}}_R, \Gamma_{\mathfrak{a}}R, \Gamma_{\mathfrak{a}}\mathbb{R}\mathrm{Hom}_R\big) \mathrel{\substack{\xleftarrow{\;\Gamma_{\mathfrak{a}}\;} \\ \sim \\ \xrightarrow[\;\Lambda^{\mathfrak{a}}\;]{}}} \big(\Lambda^{\mathfrak{a}}\mathcal{T}, \widehat{\otimes}^{\mathbb{L}}_R, \Lambda^{\mathfrak{a}}R, \mathbb{R}\mathrm{Hom}_R\big).$$

Finiteness conditions

Definition
Let $(\mathcal{T}, \otimes, \mathbb{1}, \underline{\mathrm{Hom}})$ be a closed symmetric monoidal category with coproducts:

$$\mathrm{Hom}_{\mathcal{T}}(x \otimes y, z) \cong \mathrm{Hom}_{\mathcal{T}}(x, \underline{\mathrm{Hom}}(y, z)).$$

1. An object $c \in \mathcal{T}$ is compact if

$$\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(c, x_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(c, \coprod_i x_i).$$

2. An object $d \in \mathcal{T}$ is dualizable if

$$\underline{\mathrm{Hom}}(d, \mathbb{1}) \otimes x \xrightarrow{\sim} \underline{\mathrm{Hom}}(d, x).$$

Example
If $(\mathcal{T}, \otimes, \mathbb{1}, \underline{\mathrm{Hom}}) = (\mathrm{D}(R), \otimes_R^{\mathbb{L}}, R, \mathbb{R}\mathrm{Hom}_R)$. Then $\mathcal{T}^c = \mathcal{T}^d =$ perfect complexes. One says that $\mathcal{T}$ is compactly rigidly generated.

Torsion and complete dualizable complexes

- Recall that for $\mathfrak{a} \subseteq R$ and $\mathcal{T} = \mathrm{D}(R)$ we have

$$\begin{tikzcd}
\Gamma_{\mathfrak{a}}\mathcal{T} \arrow[r, tail, bend left, "i_! = \mathrm{inc}"] \arrow[r, tail, bend right, "i_*"'] & \mathcal{T} \arrow[l, two heads, "i^*" description] \arrow[r, two heads, bend left, "p^*"] \arrow[r, two heads, bend right, "p^!"'] & L_{\mathfrak{a}}\mathcal{T} \arrow[l, tail, "p_*" description]
\end{tikzcd}$$

- Now $\Gamma_{\mathfrak{a}}\mathcal{T} = \mathrm{D}_{V(\mathfrak{a})}(R)$ is also compactly generated (by a Koszul complex $\bigotimes_{i=1}^{s}(R \xrightarrow{a_i} R)$ if $\mathfrak{a} = (a_1, \ldots, a_s)$).
- Also $(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes_R^{\mathbb{L}}, \Gamma_{\mathfrak{a}}R, \Gamma_{\mathfrak{a}}\mathbb{R}\mathrm{Hom}_R)$ is tensor-triangulated with internal Hom's.
- Problem: $\Gamma_{\mathfrak{a}}$ is typically not compact! E.g. if $R = \mathbb{Z}$ and $\mathfrak{a} = (p)$, then $\Gamma_{\mathfrak{a}}R \cong E(\mathbb{Z}/(p))[-1]$ (the Prüfer module).
- So $(\Gamma_{\mathfrak{a}}\mathcal{T})^c$ is not tensor-triangulated, but $(\Gamma_{\mathfrak{a}}\mathcal{T})^c \subseteq (\Gamma_{\mathfrak{a}}\mathcal{T})^d$ and the latter is a small tensor-triangulated category.
- By the Matlis-Greenlees-May equivalence: $(\Gamma_{\mathfrak{a}}\mathcal{T})^d \simeq (\Lambda^{\mathfrak{a}}\mathcal{T})^d$.

Completion in tt-geometry

Theorem ([Balmer-Sanders 2024, Benson-Iyengar-Krause-Pevtsova 2022])
Let R be a commutative Noetherian ring and $\mathfrak{a} \subseteq R$. Then

$$(\Gamma_{\mathfrak{a}}\,\mathrm{D}(R))^d \simeq (\Lambda^{\mathfrak{a}}\,\mathrm{D}(R))^d \simeq \mathrm{D}^{\mathrm{per}}(\widehat{R}_{\mathfrak{a}}).$$

Context

- [Balmer 2005] suggested to view small tensor triangulated categories as commutative ring-like objects and defined the spectrum and other concepts. This was the foundation of tensor triangular geometry.
- This allows to transfer ideas from algebraic geometry to other fields: modular representation theory of finite groups, spectra in the sense of topology, ...
- The above (and also [Naumann-Pol-Ramzi 2024]) suggest that $\Gamma_{\mathfrak{a}} \colon \mathcal{T}^c = \mathcal{T}^d \to (\Gamma_{\mathfrak{a}}\mathcal{T})^d$ might be a good way to generalize $\widehat{(-)}_{\mathfrak{a}} \colon \mathrm{D}^{\mathrm{per}}(R) \to \mathrm{D}^{\mathrm{per}}(\widehat{R}_{\mathfrak{a}})$.

The setup and further examples

The setup

- From now on, $\mathcal{T}$ will be a compactly generated triangulated category. We will assume that $\mathcal{T}$ carries a compatible symmetric monoidal structure $(\mathcal{T}, \otimes, \mathbb{1})$ (i.e. $\mathcal{T}$ is tensor-triangulated, [Hovey-Palmieri-Strickland 1997])
- It follows from the Brown representability that internal Hom's exist: $\mathrm{Hom}_{\mathcal{T}}(-, \underline{\mathrm{Hom}}(x, y)) \cong \mathrm{Hom}_{\mathcal{T}}(- \otimes x, y)$.
- We will also assume that $\mathcal{T}^c = \mathcal{T}^d$ (i.e. that $\mathcal{T}$ is rigidly compactly generated tensor triangulated).
- We will also assume that $\mathcal{T}$ is R-linear over a graded commutative notherian ring, i.e. the graded Hom groups

  $$\mathrm{Hom}_{\mathcal{T}}(x, y) = \{f : x \to y[n]\}$$

  are R-modules and composition is bilinear.
- Finally, assume that $\mathcal{T}$ is Noetherian: $\mathrm{Hom}(x, y)$ is Noetherian over R for all $x, y \in \mathcal{T}^c$.

Examples

1. "Trivial": R commutative Noetherian, $(\mathcal{T}, \otimes, \mathbb{1}) = (\mathsf{D}(R), \otimes_R^{\mathbb{L}}, R)$.
2. More generally: S can be a graded commutative dg algebra or an $\mathbb{E}_\infty$-ring spectrum whose cohomology/homotopy ring $R = H^*(S)$ is graded commutative Noetherian.
3. Modular representation theory of finite groups [Benson-Iyengar-Krause-Pevtsova]: $(\mathcal{T}, \otimes, \mathbb{1}) = (\mathsf{K}(\mathsf{Inj}kG), \otimes_k, k)$. In this case, $R = H^*(G; k) = \bigoplus_{n=0}^{\infty} \mathrm{Ext}_{kG}^n(k, k)$.
4. Combine the above, [Barthel-Benson-Iyengar-Krause-Pevtsova]: S a commutative Noetherian ring, G a finite flat group scheme, $(\mathcal{T}, \otimes, \mathbb{1})$ such that $\mathcal{T}^c$ is the bounded derived category of G-lattices over S. Again $R = H^*(G; S)$ (Noetherian by [Lau 2023], [van der Kallen 2023]).

Abstract torsion and complete objects

Theorem
Let $\mathcal{T}$ and R be as above, $\mathfrak{a} \subseteq R$ be a homogeneous ideal. Put

$$\Gamma_{\mathfrak{a}}\mathcal{T} := \{x \in \mathcal{T} \mid \forall c \in \mathcal{T}^c : \mathrm{Hom}_{\mathcal{T}}(c, x) \text{ is } \mathfrak{a}\text{-power torsion } \}$$

- *Then we have*

$$\begin{tikzcd}
\Gamma_{\mathfrak{a}}\mathcal{T} \arrow[r, bend left, "i_! = \mathrm{inc}"] \arrow[r, bend right, "i_*"'] & \mathcal{T} \arrow[l, two heads, "i^*" description] \arrow[r, bend left, two heads, "p^*"] \arrow[r, bend right, two heads, "p^!"'] & L_{\mathfrak{a}}\mathcal{T}, \arrow[l, tail, "p_*" description]
\end{tikzcd}$$

- *the torsion and completion functors $\Gamma_{\mathfrak{a}} = i_! i^*$ and $\Lambda^{\mathfrak{a}} = i_* i^*$.*
- *the "torsion-torsion-free triple" $(\Gamma_{\mathfrak{a}}\mathcal{T}, L_{\mathfrak{a}}\mathcal{T}, \Lambda^{\mathfrak{a}}\mathcal{T})$,*
- *the "MGM equivalence":*

$$\big(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes, \Gamma_{\mathfrak{a}}\mathbb{1}, \Gamma_{\mathfrak{a}}\underline{\mathrm{Hom}}\big) \underset{\Lambda^{\mathfrak{a}}}{\overset{\Gamma_{\mathfrak{a}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\sim}}}} \big(\Lambda^{\mathfrak{a}}\mathcal{T}, \widehat{\otimes}, \Lambda^{\mathfrak{a}}\mathbb{1}, \underline{\mathrm{Hom}}\,\big),$$

- *[Maillard-Š.] and a cohomological description of $\Lambda^{\mathfrak{a}}\mathcal{T}$:*
  $\Lambda^{\mathfrak{a}}\mathcal{T} := \{x \in \mathcal{T} \mid \forall c \in \mathcal{T}^c : \mathrm{Hom}_{\mathcal{T}}(c, x)$ *is derived* $\mathfrak{a}$*-complete* $\}$.

Properties of $\Gamma_{\mathfrak{a}}\mathcal{T}$

- As mentioned, $(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes, \Gamma_{\mathfrak{a}}\mathbb{1})$ is tensor triangulated.
- It is again compactly generated by Koszul objects $c/\!\!/\mathrm{a}$, where $c \in \mathcal{T}^c$ and $\mathrm{a} = (a_1, \dots, a_s)$ is a set of homogeneous generators for $\mathfrak{a}$. Here $c/\!\!/\mathrm{a} := c \otimes \left(\bigotimes_{i=1}^{s} \mathrm{cone}(\mathbb{1} \xrightarrow{a_i} \Sigma^{|a_i|}\mathbb{1}) \right)$.
- We have $(\Gamma_{\mathfrak{a}}\mathcal{T}) \cap \mathcal{T}^c = (\Gamma_{\mathfrak{a}}\mathcal{T})^c \subseteq (\Gamma_{\mathfrak{a}}\mathcal{T})^d$, but the inclusion is typically not an equality. The reason is that the tensor unit

  $$\Gamma_{\mathfrak{a}}\mathbb{1} = \mathrm{hocolim}(\mathbb{1}/\!\!/\mathrm{a} \to \mathbb{1}/\!\!/\mathrm{a}^2 \to \mathbb{1}/\!\!/\mathrm{a}^3 \to \cdots)$$

  is typically not compact.
- $\left((\Gamma_{\mathfrak{a}}\mathcal{T})^d, \otimes, \Gamma_{\mathfrak{a}}\mathbb{1}\right)$ is an (essentially) small tensor triangulated category with internal Hom's and there is a tt-functor

  $$\Gamma_{\mathfrak{a}} \colon \mathcal{T}^c = \mathcal{T}^d \to (\Gamma_{\mathfrak{a}}\mathcal{T})^d.$$

Completion on modules of homomorphisms

A canonical action of the completed ring

- Let $\widehat{R}$ be the graded adic completion of R at $\mathfrak{a}$.
- We know that

$$\mathrm{Hom}_{\mathcal{T}}(\Gamma_{\mathfrak{a}}\mathbb{1}, \Gamma_{\mathfrak{a}}\mathbb{1}) \cong \mathrm{Hom}_{\mathcal{T}}(\Lambda^{\mathfrak{a}}\mathbb{1}, \Lambda^{\mathfrak{a}}\mathbb{1}) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \Lambda^{\mathfrak{a}}\mathbb{1})$$

is a derived complete R-module, so we have

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathbb{1}) \arrow[d, "\Gamma_{\mathfrak{a}}"] \\ \widehat{R} \arrow[r, dotted, "\exists!"] & \mathrm{Hom}_{\mathcal{T}}(\Gamma_{\mathfrak{a}}\mathbb{1}, \Gamma_{\mathfrak{a}}\mathbb{1}). \end{tikzcd}$$

- It follows that $\Gamma_{\mathfrak{a}}\mathcal{T}$ admits a canonical action of the (graded commutative Noetherian) ring $\widehat{R}$.

Finiteness of the action on dualizable objects

Theorem (Maillard-Š.)
Let $\mathcal{T}$ and $\mathfrak{a} \subseteq R$ be as above. Then $\operatorname{Hom}_{\Gamma_\mathfrak{a}\mathcal{T}}(x, y)$ *is a Noetherian* $\widehat{R}$*-module for each pair* $x, y \in (\Gamma_\mathfrak{a}\mathcal{T})^d$.

The proof is based on arguments of [Benson-Iyengar-Krause-Pevtsova] and the following version of the Nakayama lemma:

Lemma (Porta-Shaul-Yekutieli 2015, Positselski 2017)
Let M be a derived $\mathfrak{a}$-complete graded R-module. If $M/\mathfrak{a}M = 0$, then $M = 0$.

In particular, if $M/\mathfrak{a}M$ is a Noetherian R-module, then M is a Noetherian $\widehat{R}$*-module.*

Completion on modules of homomorphisms

The category $(\Gamma_{\mathfrak{a}}\mathcal{T}, \otimes, \Gamma_{\mathfrak{a}}\mathbb{1})$ shares many features with $(\mathcal{T}, \otimes, \mathbb{1})$:

- $\Gamma_{\mathfrak{a}}\mathcal{T}$ is compactly generated, tensor triangulated, has a Noetherian $\widehat{R}$-action.
- The main difference: $(\Gamma_{\mathfrak{a}}\mathcal{T})^c \subseteq (\Gamma_{\mathfrak{a}}\mathcal{T})^d$ and $\Gamma_{\mathfrak{a}}\mathbb{1} \notin (\Gamma_{\mathfrak{a}}\mathcal{T})^c$.

The relation between $\mathcal{T}^c$ and $(\Gamma_{\mathfrak{a}}\mathcal{T})^d$ is even tighter:

Theorem (Maillard-Š.)
Let $x, y \in \mathcal{T}^c$. Then the functor $\Gamma_{\mathfrak{a}} \colon \mathcal{T}^c \to (\Gamma_{\mathfrak{a}}\mathcal{T})^d$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(x,y)^{\wedge}_{\mathfrak{a}} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_{\mathfrak{a}}\mathcal{T}}(\Gamma_{\mathfrak{a}}x, \Gamma_{\mathfrak{a}}y).$$

Idea of the proof.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(x,y)^{\wedge}_{\mathfrak{a}} &= \varprojlim_n R/(\mathrm{a}^n) \otimes_R \mathrm{Hom}_{\mathcal{T}}(x,y) \\ &\xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{\mathcal{T}}(x/\!\!/\mathrm{a}^n, y) \xleftarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(\mathrm{hocolim}_n\, x/\!\!/\mathrm{a}^n, y).\end{aligned}$$

The dualizables as a triangulated “Cauchy” completion

Ruminations about torsion compacts and dualizables

Question

Can we reconstruct $(\Gamma_{\mathfrak{a}}\mathcal{T})^d$ from its full triangulated subcategory $(\Gamma_{\mathfrak{a}}\mathcal{T})^c$ by some form of completion (as one constructs $\widehat{R}$ from its torsion quotients $R/\mathfrak{a}^n$)?

Some "evidence" supporting that the question makes sense:

1. [Krause 2020] and [Neeman ∼2020] found methods to reconstruct $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ from $\mathrm{D}^{\mathrm{per}}(R)$ for nice enough rings R.
2. The relation $(\Gamma_{\mathfrak{a}}\mathcal{T})^c \subseteq (\Gamma_{\mathfrak{a}}\mathcal{T})^d$ is reminiscent to that of $\mathrm{D}^{\mathrm{per}}(R) \subseteq \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$:

Lemma

(a) *If $c \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$ and $d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^d$, then $c \otimes d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$.*

(b) *If $c \otimes d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$ for all $c \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$, then $d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^d$.*

(c) *If $c \otimes d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$ for all $d \in (\Gamma_{\mathfrak{a}}\mathcal{T})^d$, then $c \in (\Gamma_{\mathfrak{a}}\mathcal{T})^c$.*

Triangulated reconstruction result

Theorem (Maillard-Š.)
Let $\mathcal{T}$ and $\mathfrak{a} \subseteq R$ be as above. Then we can recover $(\Gamma_{\mathfrak{a}}\mathcal{T})^d$ as an $\widehat{R}$-linear tensor-triangulated category from the approximate unital R-linear tensor-triangulated full category $(\Gamma_{\mathfrak{a}}\mathcal{T})^c$. More precisely,

1. *the functor category* $\mathrm{Mod}(\Gamma_{\mathfrak{a}}\mathcal{T})^c$ *becomes a closed symmetric monoidal category with the Day convolution product, so that the Yoneda embedding* $y\colon (\Gamma_{\mathfrak{a}}\mathcal{T})^c \to \mathrm{Mod}(\Gamma_{\mathfrak{a}}\mathcal{T})^c$ *becomes a strong monoidal functor.*
2. *The restricted Yoneda functor* $y\colon \Gamma_{\mathfrak{a}}\mathcal{T} \to \mathrm{Mod}(\Gamma_{\mathfrak{a}}\mathcal{T})^c$, $x \mapsto \mathrm{Hom}_{\mathcal{T}}(-,x)|_{(\Gamma_{\mathfrak{a}}\mathcal{T})^c}$ *induces an equivalence*
$$y\colon (\Gamma_{\mathfrak{a}}\mathcal{T})^d \xrightarrow{\sim} \Big(\mathrm{Mod}(\Gamma_{\mathfrak{a}}\mathcal{T})^c\Big)^d.$$
3. *The triangles in* $(\Gamma_{\mathfrak{a}}\mathcal{T})^d$ *are recovered as suitable colimits of triangles in* $(\Gamma_{\mathfrak{a}}\mathcal{T})^c$.

More on the context of the recostruction result

- Exactly as Neeman's reconstruction of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ from $\mathrm{D}^{\mathrm{per}}(R)$, the proof is completely enhancement-free.
- Our setup has a few features very different from Neeman's:
  (a) Neeman's proof relies heavily on metrics from t-structures, while our setup is not t-structure friendly at all (it can easily happen $\Sigma^n \cong \mathrm{Id}_{\mathcal{T}}$, e.g. if R has invertible elements of non-zero degree).
  (b) Neeman approximates from $x \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ by a direct system $x_1 \to x_2 \to x_3 \to \cdots$ in $\mathrm{D}^{\mathrm{per}}(R)$ such that

$$\mathrm{Hom}(x_1, y) \leftarrow \mathrm{Hom}(x_2, y) \leftarrow \mathrm{Hom}(x_3, y) \leftarrow \cdots$$

  eventually stabilizes. In our situation, these inverse systems only have the Mittag-Leffler property (i.e. are very close to systems of surjections).
  (c) Unlike Neeman, we substantially use the monoidal structure.

Consequences of strong generation

Strong generators

Definition
Let $\mathcal{T}^c$ be a small idempotent complete triangulated category.

- If $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{T}^c$ are additive subcategories, we write
$$\mathcal{S}_1 \star \mathcal{S}_2 = \{x \in \mathcal{T}^c \mid \exists\ s_1 \to x \to s_2 \to \ \text{with } s_i \in \mathcal{S}_i\}.$$
- If $\mathcal{S} \subseteq \mathcal{T}^c$ is additive and $n \geq 1$, we put
$$\mathcal{S}^{\star n} = \underbrace{\mathcal{S} \star \cdots \star \mathcal{S}}_{n \text{ times}}.$$
- If $\mathcal{S} \subseteq \mathcal{T}^c$ is additive, we put $\mathsf{thick}_1(\mathcal{S}) = \mathsf{add}\{\bigcup \mathcal{S}[i] \mid i \in \mathbb{Z}\}$. For $n \geq 1$, we write $\mathsf{thick}_n(\mathcal{S}) = \mathsf{add}(\mathsf{thick}_1(\mathcal{S})^{\star n})$, and $\mathsf{thick}(\mathcal{S}) = \bigcup_{n \geq 1} \mathsf{thick}_n(\mathcal{S})$ (= the smallest thick subcategory containing $\mathcal{S}$).
- We say that $\mathcal{T}^c$ is strongly generated (or has finite Rouquier dimension) if $\exists g \in \mathcal{T}\ \exists n \geq 0 : \mathsf{thick}_{n+1}(g) = \mathcal{T}$.

Examples with strong generation

1. R regular commutative Noetherian of finite Krull dimension, $(\mathcal{T}, \otimes, \mathbb{1}) = (\mathsf{D}(R), \otimes^{\mathbb{L}}_R, R)$.
2. More generally: $\mathcal{T} = \mathsf{D}(S)$, where S is a graded commutative dg algebra of $\mathbb{E}_\infty$-ring spectrum such that $R = H^*S$ is regular Noetherian of finite Krull dimension.
3. $(\mathcal{T}, \otimes, \mathbb{1}) = (\mathsf{K}(\mathsf{Inj}kG), \otimes_k, k)$ where k is a field and G is a finite group.
4. Let S be a nice enough regular commutative Noetherian ring of finite Krull dimension, G a finite flat commutative group scheme, $(\mathcal{T}, \otimes, \mathbb{1})$ such that $\mathcal{T}^c$ is the bounded derived category of G-lattices over S. Then $\mathcal{T}^c$ is strongly generated using a recent result by Neeman for the commutative ring SG.

Strong generation descends to torsion dualizables

Theorem (Maillard-Š.)
Let $\mathcal{T}$ and $\mathfrak{a} \subseteq R$ be as above. Assume that g is a strong generator of $\mathcal{T}^c$. Then:

1. $\Gamma_{\mathfrak{a}} g$ *is a strong generator of* $(\Gamma_{\mathfrak{a}}\mathcal{T})^d$ *and* $\Lambda^{\mathfrak{a}} g$ *is a strong generator of* $(\Lambda^{\mathfrak{a}}\mathcal{T})^d$.
2. *An object* $x \in \Lambda^{\mathfrak{a}}\mathcal{T}$ *is dualizable if and only if* $\mathrm{Hom}_{\mathcal{T}}(c, x)$ *is a finitely generated* $\widehat{R}$*-module for each* $c \in \mathcal{T}^c$ *(equivalently for* $c = g$*).*

“Perfect” pairing via representability

Theorem (Maillard-Š.)
Let $\mathcal{T}$ and $\mathfrak{a} \subseteq R$ be as above and assume that such that $\mathcal{T}^c$ admits a strong generator. Then

$$\mathrm{Hom}_{\mathcal{T}} \colon (\Gamma_{\mathfrak{a}}\mathcal{T})^{c,\mathrm{op}} \times (\Gamma_{\mathfrak{a}}\mathcal{T})^{d} \to \mathrm{tors} R \subseteq \mathrm{mod}\, R$$

is a perfect pairing in the sense that

1. *the restricted Yoneda functor* $(\Gamma_{\mathfrak{a}}\mathcal{T})^{d} \to \mathrm{Fun}_R((\Gamma_{\mathfrak{a}}\mathcal{T})^{c,\mathrm{op}}, \mathrm{tors} R)$ *is fully faithful and the essential image consists precisely of the cohomological functors, and*
2. *the restricted Yoneda functor* $(\Gamma_{\mathfrak{a}}\mathcal{T})^{c,\mathrm{op}} \to \mathrm{Fun}_R((\Gamma_{\mathfrak{a}}\mathcal{T})^{d}, \mathrm{tors} R)$ *is fully faithful and the essential image consists precisely of the homological functors.*

Thank you for your attention!