FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Cohen-Macaulay modules over simple singularities

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Study programme: Mathematics (N1101)
Study branch: Mathematical Structures

Prague 2022

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I would like to thank doc. RNDr. Jan Šťovíček, Ph.D. for his supervision of my thesis as well as great help with my study in Prague.

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Abstract: The thesis is focused on the maximal Cohen-Macaulay modules over simple singularities. Previous results on the topic are summarised, and in particular it is shown that a hypersurface is MCM-finite if and only if it is a simple singularity. The stable Auslander-Reiten quivers of simple singularities are drawn for better understanding of the category of maximal Cohen-Macaulay modules over a simple singularity.

Keywords: Cohen-Macaulay module   simple singularity

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Introduction

The study of Cohen-Macaulay modules is one of the active subjects in commutative algebra and representation theory. This thesis deals with a particularly useful case: the maximal Cohen-Macaulay modules over a ring of simple hypersurface singularity. As is remarked in Yoshino [1990], the classification problem of Cohen-Macaulay modules over a given Cohen-Macaulay ring has two aspects: the first is the algebraic or representation-theoretic side, in which the theory developed by Maurice Auslander and Idun Reiten serves as a powerful tool; the second is a geometric side, where many have made great contribution, including Horst Knörrer, Ragnar-Olaf Buchweitz, David Eisenbud, etc. This thesis focuses on the algebraic aspect, to give a clear presentation of the stable category of maximal Cohen-Macaulay modules over a simple singularity using Auslander-Reiten sequences.

The first chapter starts from the basics of commutative algebra and introduces the concept of maximal Cohen-Macaulay modules over Cohen-Macaulay rings in Section 1.1. In Section 1.2, the focus is on maximal Cohen-Macauly modules over a ring of hypersurface, where a very important tool, matrix factorisation, is imported from Eisenbud [1980], and we show the correspondence between maximal Cohen-Macaulay modules and matrix factorisations, known as Eisenbud's matrix factorisation theorem, see (1.6). This equivalence of categories is extensively and sometimes even implicitly used throughout the thesis when we consider some maximal Cohen-Macaulay module or some matrix factorisation.

The second chapter is devoted to the introduction to Auslander-Reiten theory, which aims to represent a certain category by a quiver carrying the data of some generators and relations. In Section 2.1, we give a detailed explanation of the properties of Auslander-Reiten sequences, or almost split sequences, in particular the relation with irreducible morphisms. We also mention the existence of Auslander-Reiten sequences in the category of maximal Cohen-Macaulay modules over an isolated singularity, a famous result from Auslander and Unger [2006]. In Section 2.2, we discuss the data in the Auslander-Reiten quiver of the category of maximal Cohen-Macaulay modules over some Henselian Cohen-Macaulay local ring. In particular, we show that the number of arrows between two vertices in the quiver can be determined by inspecting certain Auslander-Reiten sequences. In addition, we mention the local finiteness of the Auslander-Reiten quiver of an isolated singularity, as well as a version of Brauer-Thrall theorem for the category maximal Cohen-Macaulay modules, known as Dieterich-Yoshino's theorem from Dieterich [1987] and Yoshino [1987].

The third chapter brings the context to our main focus: the simple singularities. In Section 3.1, we follow the method of Buchweitz et al. [1987] to give a proof of the claim that if an analytic hypersurface admits only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, then it is a simple singularity. We also present a proof from Herzog [1978] that looses the hypersurface assumption, to show that it suffices to assume that the ring is Gorenstein. In Section 3.2, we introduce Knörrer's periodicity and give a thorough proof of it, see (3.14). This famous result from Knörrer [1987] allows us to reduce the dimension of the ring by two when proving that it admits only a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules. Once we finish the discussion of 1-dimensional and 2-dimensional cases later, we shall be able to conclude that any hypersurface is MCM-finite, if and only if it is a simple singularity, or equivalently if it is classified by the Dynkin types A_n for $n \geq 1$, D_n for $n \geq 4$, E_6, E_7 and E_8.

The last chapter aims to give a concrete presentation of the (stable) category of maximal Cohen-Macaulay modules over a simple singularity. In other words, we determine the (stable) Auslander-Reiten quiver of the category. By Knörrer's periodicity (3.14), it suffices to look into the 1-dimensional and 2-dimensional cases. In Section 4.1, we consider the 2-dimensional case, where we use the isomorphism between the Auslander-Reiten quiver and the McKay graph of Klein groups to reduce the problem to a classical result from Klein [1893]. This isomorphism is often referred to as the algebraic McKay correspondence, discovered in Auslander [1986] and Auslander and Reiten [1987]. In Section 4.2, we consider the 1-dimensional case, and we apply the techniques of matrix factorisations and Auslander-Reiten sequences to determine the vertices and arrows in the Auslander-Reiten quiver, as is so in Yoshino [1990]. Finally, we adopt the notation of orbit quivers used in Iyama [2018] to present the result in a simpler form, and we illustrate how the quotients of the extended quivers by some automorphisms correspond to the Auslander-Reiten determined before.

Although this thesis mainly uses the technique from commutative algebra, it is worth mentioning that in recent years the theory is put into the context of modern representation theory in the language of homological algebra, see for example the triangulated equivalences in Kajiura et al. [2007], and this point of view also plays a major role in the survey Iyama [2018].

1. Cohen-Macaulay modules

1.1 Cohen-Macaulay rings and modules

Throughout the thesis any *ring* R is assumed to be *commutative* and *unital.* Any *module* M over a ring R is assumed to be *unital* and *finitely generated.*

We denote by $\mathsf{mod}(R)$ the category of finitely generated R-modules, considered as a subcategory of $\mathsf{Mod}(R)$, the category of all R-modules that are possibly not finitely-generated.

A ring R is *local*, if it admits a unique maximal ideal $\mathfrak{m}_R$. Recall that:

> Any projective module over a local ring is free.

See Matsumura [1989], Thm. 2.5. □

For a ring R, its *(Krull) dimension* is denoted by $\mathsf{dim}(R)$. For an R-module M, its *dimension* is defined as $\mathsf{dim}(M) := \mathsf{dim}(R/\mathsf{Ann}(M))$, where

$$\mathsf{Ann}(M) := \{r \in R \mid rm = 0, \forall m \in M\}$$

is the *annihilator* of M, considered as an ideal of R.

Let R be a local noetherian ring with maximal ideal $\mathfrak{m}_R$. We denote by $\kappa_R := R/\mathfrak{m}_R$ the *residue field* of R. Then $\mathfrak{m}_R/\mathfrak{m}_R^2$ is a κ_R-vector space. R is *regular*, if $\mathsf{dim}(R) = [\mathfrak{m}_R/\mathfrak{m}_R^2 : \kappa_R]$.

Jean-Pierre Serre has shown that:

> A notherian local ring R is regular, if and only if R is of finite *global dimension*, i.e.
>
> $$\mathsf{gl.dim}(R) := \sup\{\mathsf{proj.dim}(M) \mid M \in \mathsf{Mod}(R)\} < \infty.$$
>
> Moreover, in this case, $\mathsf{dim}(R) = \mathsf{gl.dim}(R) < \infty$.

See Matsumura [1989], Thm. 19.2. □

We also know that:

> Every regular local ring is a unique factorisation domain.

See Auslander and Buchsbaum [1959]. □

Let R be a ring, I an ideal of R, and $M \in \mathsf{mod}(R)$, such that $IM \subsetneq M$. The *I-depth* of M is defined as

$$\mathsf{depth}_I(M) := \min\{i : \mathsf{Ext}^i_R(R/I, M) \neq 0\}.$$

If R is moreover local, then the $\mathfrak{m}_R$-depth of M is simply called the *depth* of M. In other words,

$$\mathsf{depth}(M) := \min\{i : \mathsf{Ext}^i_R(\kappa_R, M) \neq 0\}.$$

According to David Rees, we can reformulate the definition in a more elementary way. Let M be a finitely generated R-module for a local ring R, then $x \in \mathfrak{m}_R$ is M-*regular*, if $M \xrightarrow{x} M$ is injective; otherwise x is a *zero divisor* of M. A sequence $\{x_1, \cdots, x_n\}$ of elements in $\mathfrak{m}_R$ is a *regular sequence*[1] of M, if x_{i+1} is $M/(x_1, \cdots, x_i)M$-regular, for any $i = 0, \cdots, n-1$. Then:

> The depth of M equals the maximum length of regular sequences of M in $\mathfrak{m}_R$.

See Bruns and Herzog [1998], Thm. 1.2.5. □

The most basic property of depth is the following:

> Let M be a nonzero finitely generated module over a noetherian local ring R, then
>
> $$\mathsf{depth}(M) \leq \mathsf{dim}(M). \tag{1.1}$$

See Bruns and Herzog [1998], Thm 1.2.12. □

> Let R be a noetherian local ring, and $0 \to M \to N \to P \to 0$ be a short exact sequence of nonzero R-modules. Then
> - $\mathsf{depth}(N) \geq \min\{\mathsf{depth}(M), \mathsf{depth}(P)\}$;
> - $\mathsf{depth}(P) \geq \min\{\mathsf{depth}(N), \mathsf{depth}(M) - 1\}$;
> - $\mathsf{depth}(M) \geq \min\{\mathsf{depth}(N), \mathsf{depth}(P) + 1\}$.

The statements are proven immediately once we consider the long exact sequence by applying the functor $\mathsf{Hom}_R(\kappa_R, -)$. □

Auslander-Buchsbaum formula. The Auslander-Buchsbaum formula allows to calculate the projective dimension of a module by its depth:

> Let R be a noetherian local ring with maximal ideal $\mathfrak{m}_R$, and M be a finitely generated R-module. If $\mathsf{proj.dim}_R(M) < +\infty$, then
>
> $$\mathsf{proj.dim}_R(M) + \mathsf{depth}(M) = \mathsf{depth}(R). \tag{1.2}$$

See Auslander and Buchsbaum [1957]. □

[1] For an arbitrary ring R, the definition of an M-regular sequence also requires $M/(x_1, \cdots, x_n)M \neq 0$. Thanks to Nakayama lemma this amounts to $M \neq 0$ when R is local, though.

Let R be an integral domain. The *rank* of an R-module M, denoted by $\mathsf{rank}_R(M)$, is the maximal number of elements of M that are linearly independent over R. Here a subset T of M is *R-linearly independent*, if the natural morphism $R^{(T)} \to M$ is injective.

Let K be the field of fractions of R. Then $\mathsf{rank}_R(M) = \mathsf{dim}_K(M \otimes_R K)$.

Suppose M is a free R-module, then $M \simeq R^{(n)}$, and $\mathsf{rank}_R(M) = n$.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules, then

$$\mathsf{rank}_R(M_1) + \mathsf{rank}_R(M_3) = \mathsf{rank}_R(M_2).$$

In fact, since the field of fractions K is flat, we can apply $- \otimes_R K$ to the original exact sequence, and the result follows from the rank-nullity theorem in linear algebra. □

Let M be a module over some ring R. Given a chain of its R-submodules of the form

$$M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M,$$

we say this chain has length n.

The *length* of M, denoted by $l(M)$, is defined as the largest length of any chain of R-submodules of M, or $+\infty$, if no such largest length exists.

We have the following results:

- If N is an R-submodule of M,

$$l(M) = l(N) + l(M/N).$$

- If $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ is an exact sequence of R-modules of finite length, then

$$\sum_{i=1}^{n} (-1)^i l(M_i) = 0.$$

See Matsumura [1989], p.12. □

A local ring R is *Henselian*, if any commutative R-algebra which is finitely generated as an R-module is a direct product of local R-algebras.

For any Henselian local ring R and $M \in \mathsf{mod}(R)$, M is indecomposable if and only if $\mathsf{End}_R(M)$ is a local algebra. This imples that $\mathsf{mod}(R)$ is Krull-Schmidt, i.e. any R-module can be written uniquely as a finite direct sum of indecomposable R-modules.

See Yoshino [1990], (1.18). □

An important example is given by an *analytic algebra* over a valued field K, i.e. a finite algebra over a convergent power series ring $K\{x_1, \cdots, x_n\}$. Recall that

K is equipped with a *valuation* $v : K \to [0, +\infty) \subset \mathbb{R}$ satisfying

- $v(x) = 0 \Leftrightarrow x = 0$;
- $v(xy) = v(x)v(y)$, $v(x+y) \leq v(x) + v(y)$, $\forall x, y \in K$.

Also recall that a formal power series $f = \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ over K in variables $\{x_1, \cdots, x_n\}$ is *convergent*, if $\exists r_1, \cdots, r_n, N \in (0, +\infty)$ such that

$$v(a_{i_1 \cdots i_n}) r_1^{i_1} \cdots r_n^{i_n} \leq N, \quad \forall i_1, \cdots, i_n.$$

Any local analytic algebra over a valued field K is a Henselian ring.

See Nagata [1962], Thm. 45.5. □

Maximal Cohen-Macaulay module. Let R be a noetherian local ring. A finitely generated R-module M is *maximal Cohen-Macaulay* (MCM)[2], if

$$\mathsf{depth}(M) = \mathsf{dim}(R).$$

The ring R is a *Cohen-Macaulay ring*, if it is a MCM module over itself, i.e. if $\mathsf{depth}(R) = \mathsf{dim}(R)$.

The category of MCM R-modules is denoted by $\mathsf{MCM}(R)$, which is naturally a subcategory of $\mathsf{mod}(R)$.

We clearly have:

Any regular local ring is a CM ring.

In fact, for a regular local ring R, we can pick a minimal set $\{x_1, \cdots, x_n\}$ of generators for $\mathfrak{m}_R$, where $n = \mathsf{dim}(R)$. Recall that R is a UFD, and then it is easy to show that $x_1, \cdots, x_n$ is a regular sequence of R, and hence $\mathsf{depth} R = \mathsf{dim}(R)$. □

The following homological characterisation of MCM modules are useful:

For a finitely generated R-module M, the followings are equivalent:
- M is an MCM R-module;
- $\mathsf{Ext}^i_R(\kappa_R, M) = 0$ for any $i < \mathsf{dim}(R)$.

Given $M \in \mathsf{MCM}(R)$, we know by definition $\mathsf{dim}(R) = \mathsf{depth}(M)$. By definition of depth, we know that for any $i < \mathsf{depth}(M) = \mathsf{dim}(R)$, $\mathsf{Ext}^i_R(\kappa_R, M) = 0$. Conversely, we know from (1.1) that $\mathsf{depth}(M) \leq \mathsf{dim}(M) = \mathsf{dim}(R/\mathsf{Ann}(M)) \leq \mathsf{dim}(R)$. Meanwhile, $\mathsf{depth}(M) = \min\{i : \mathsf{Ext}^i_R(\kappa_R, M) \neq 0\} \geq \mathsf{dim}(R)$. Hence $\mathsf{depth}(R) = \mathsf{dim}(R)$. □

An easy corollary is:

[2]There are also Cohen-Macaulay modules that are not maximal, which are out of topic in the thesis.

For any short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are MCM, then so is M; if M and N are MCM, then so is L.

In fact, just consider the long exact sequence

$$\cdots \to \mathsf{Ext}^i_R(\kappa_R, L) \to \mathsf{Ext}^i_R(\kappa_R, M) \to \mathsf{Ext}^i_R(\kappa_R, N) \to \cdots$$

where $i < \mathsf{dim}(R)$. $\square$

For a CM local ring R, given an exact sequence of R-modules

$$0 \to M \xrightarrow{\alpha_n} F_{n-1} \xrightarrow{\alpha_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{\alpha_1} F_0$$

where each F_i is free and $n \geq \mathsf{dim}(R)$, then M is a MCM R-module.

First notice that for free modules F_i, $\mathsf{proj.dim}(F_i) = 0$, and thus by (1.2) we get $\mathsf{depth}(F_i) = \mathsf{depth}(R) = \mathsf{dim}(R)$, $\forall i$. Now consider the short exact sequence

$$0 \to M \xrightarrow{\alpha_n} F_{n-1} \to \mathsf{Coker}(\alpha_n) \to 0,$$

from which we get $\mathsf{depth}(M) \geq \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_n)) + 1\}$. Similarly consider

$$0 \to \mathsf{Coker}(\alpha_n) \to F_{n-2} \to \mathsf{Coker}(\alpha_{n-1}) \to 0,$$

to get $\mathsf{depth}(\mathsf{Coker}(\alpha_n)) \geq \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_{n-1})) + 1\}$. Combining them, we get

$$\begin{aligned}
\mathsf{depth}(M) &\geq \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_n)) + 1\} \\
&\geq \min\{\mathsf{dim}(R), \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_{n-1})) + 1\} + 1\} \\
&\geq \min\{\mathsf{dim}(R), \min\{\mathsf{dim}(R) + 1, \mathsf{depth}(\mathsf{Coker}(\alpha_{n-1})) + 2\}\} \\
&= \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_{n-1})) + 2\}.
\end{aligned}$$

Then it is not hard to continue until we get

$$\begin{aligned}
\mathsf{depth}(M) &\geq \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_1)) + n\} \\
&\geq \min\{\mathsf{dim}(R), \mathsf{depth}(\mathsf{Coker}(\alpha_1)) + \mathsf{dim}(R)\} \\
&= \mathsf{dim}(R).
\end{aligned}$$

$\square$

In other words, for any R-module M of finite type and any $n \geq \mathsf{dim}(R)$, the n-th syzygy of M is a MCM R-module. We dismiss any free direct summand in it and denote by $\mathsf{syz}^n_R(M)$ the *reduced n-th syzygy* of M, and then $\mathsf{syz}^n_R(M)$ is also MCM.

The following results are well-known and useful:

- If R is a regular local ring, then all MCM R-modules are free.
- If R is a 1-dimensional *reduced*[a] local ring, then an R-module is MCM if and only if it is *torsion-free*, i.e. the natural homomorphism $M \to \mathsf{Hom}_R(\mathsf{Hom}_R(M, R), R)$ is a monomorphism.
- If R is a 2-dimensional normal local domain[b], then an R-module M is MCM if and only if it is *reflexive*, i.e. the natural homomorphism $M \to \mathsf{Hom}_R(\mathsf{Hom}_R(M, R), R)$ is an isomorphism.

[a] i.e. having no nonzero nilpotent elements.
[b] A ring is *normal* if its localisations at prime ideals are integrally closed domains.

See Yoshino [1990], (1.5). □

A noetherian local ring R is *Gorenstein*, if

$$\mathsf{inj.dim}(R_R) < \infty.$$

A noetherian ring R is *Gorenstein*, if $R_\mathfrak{p}$ is a Gorenstein local ring for any $\mathfrak{p} \in \mathsf{Spec}(R)$.

For a noetherian local ring R of dimension n, the followings are equivalent:

- R is Gorenstein, i.e. $\mathsf{inj.dim}(R_R) < \infty$;
- $\mathsf{inj.dim}(R_R) = n$;
- $\mathsf{Ext}^i_R(\kappa_R, R) = \begin{cases} \kappa_R, & i = n; \\ 0, & i \neq n; \end{cases}$;
- $\mathsf{Ext}^i_R(\kappa_R, R) = 0$ for some $i > n$;
- $\mathsf{Ext}^i_R(\kappa_R, R) = \begin{cases} \kappa_R, & i = n; \\ 0, & i < n. \end{cases}$

See Matsumura [1989], Thm 18.1. □

It follows directly that:

- Every regular local ring is Gorenstein.
- Every Gorenstein ring is Cohen-Macaulay.
- Every Gorenstein ring R admits a canonical module ω_R that is isomorphic to R itself.

Given a regular local ring R, we know that

$$\mathsf{gl.dim}(R) = \sup\{\mathsf{inj.dim}(M) \mid M \in \mathsf{Mod}(R)\} < \infty.$$

In particular, $\mathsf{inj.dim}(R_R) < \infty$, i.e. R is Gorenstein. The other statements are rather obvious. We remark that for any CM ring R, $\omega_R \in \mathsf{mod}(R)$ is a *canonical module* of R if $\omega_R \in \mathsf{MCM}(R)$ and

$$\mathsf{Ext}^i_R(\kappa_R, \omega_R) = \begin{cases} \kappa_R & \text{if } i = \mathsf{dim}(R); \\ 0 & \text{otherwise.} \end{cases}$$

□

For an arbitrary CM ring, if a canonical module exists, then it is unique up to isomorphism. Moreover we have:

Let R be a CM ring with canonical module ω_R, and $M \in \mathsf{MCM}(R)$. Then $\mathsf{Hom}_R(M, \omega_R) \in \mathsf{MCM}(R)$ as well and $M \simeq \mathsf{Hom}_R(\mathsf{Hom}_R(M, \omega_R), \omega_R)$. Moreover, $\mathsf{Ext}^i_R(M, \omega_R) = 0$ for $i \geq 1$. In particular, $\mathsf{Hom}_R(-, \omega_R)$ is an exact auto-functor on $\mathsf{MCM}(R)$.

See Yoshino [1990], (1.13). □

1.2 MCM modules over a hypersurface

In this section we are interested in the MCM modules over a *hypersurface singularity* R. In other words, $R = S/(f)$ is a fixed CM local ring, where S is a regular local ring, and $f \in \mathfrak{m}_S$. If $f = 0$, we know that all MCM R-modules are actually free. So from now on we assume $f \neq 0$. In this thesis, we are particularly interested in an *analytic hypersurface*, which is a hypersurface $R = S/(f)$ where S is a regular analytic algebra over certain field K. We shall simply call it a hypersurface if no confusion is caused.

Given $M \in \mathsf{MCM}(R)$, and it has a structure of S-module naturally. By Auslander-Buchsbaum formula (1.2), $\mathsf{proj.dim}_S(M) = \mathsf{depth}(S) - \mathsf{depth}(M) = \mathsf{dim}(S) - \mathsf{dim}(R) = 1$. Thus M has a free S-resolution of the form

$$0 \to S^{(n_1)} \xrightarrow{\phi} S^{(n_2)} \xrightarrow{\pi} M \to 0.$$

Moreover, $\mathsf{rank}_S(M) = 0$. In fact, suppose $S \xrightarrow{\iota} M$ is an S-linear morphism, then $0 \neq (f) \subset \mathsf{Ker}(\iota)$, hence ι cannot be injective. Applying the additivity of rank, we get $n_1 = n_2 =: n$. So M has a free S-resolution of the form

$$0 \to S^{(n)} \xrightarrow{\phi} S^{(n)} \xrightarrow{\pi} M \to 0. \tag{1.3}$$

Consider the submodule $fS^{(n)} \leq S^{(n)}$. Since $fM = 0$, we get $fS^{(n)} \subset \mathsf{Ker}(\pi) = \mathsf{Im}(\phi) = \phi(S^{(n)})$. Thus $\forall x \in S^{(n)}$, $\exists! y \in S^{(n)}$, such that $fx = \phi(y)$. We then define $\psi : S^{(n)} \to S^{(n)}$ by $\psi(x) = y$. Clearly ψ is a linear endomorphism, and

$$\phi \circ \psi = f1_{S^{(n)}}.$$

So $\phi \circ \psi \circ \phi = f1_{S^{(n)}} \circ \phi = \phi \circ f1_{S^{(n)}}$, which implies, since ϕ is monomorphic, that

$$\psi \circ \phi = f1_{S^{(n)}}.$$

By fixing a base for $S^{(n)}$, the maps ϕ, ψ can be regarded as matrices, i.e. $\phi, \psi \in S^{n \times n}$.

Matrix factorisation. A pair of matrices $(\phi, \psi) \in S^{n\times n} \times S^{n\times n}$ satisfying $\phi \circ \psi = \psi \circ \phi = f 1_{S^{(n)}}$ is called a *matrix factorisation* of f, where $n \in \mathbb{N}$.

A morphism of matrix factorisations $(\phi_1, \psi_1) \to (\phi_2, \psi_2)$ is a pair of matrices $(\alpha, \beta) \in S^{n_2\times n_1} \times S^{n_2\times n_1}$, such that $\alpha \circ \phi_1 = \phi_2 \circ \beta$, i.e. the following diagram commutes:

$$\begin{array}{ccc} S^{(n_1)} & \xrightarrow{\phi_1} & S^{(n_1)} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ S^{(n_2)} & \xrightarrow{\phi_2} & S^{(n_2)} \end{array}$$

Note that in this case, we can show that $\beta \circ \psi_1 = \psi_2 \circ \alpha$ holds as well. In fact,

$$\begin{aligned} f\beta \circ \psi_1 &= \psi_2 \circ \phi_2 \circ \beta \circ \psi_1 \\ &= \psi_2 \circ \alpha \circ \phi_1 \circ \psi_1 \\ &= f\psi_2 \circ \alpha. \end{aligned}$$

In other words, actually the following bigger diagram commutes as well:

$$\begin{array}{ccccc} S^{(n_1)} & \xrightarrow{\psi_1} & S^{(n_1)} & \xrightarrow{\phi_1} & S^{(n_1)} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ S^{(n_2)} & \xrightarrow{\psi_2} & S^{(n_2)} & \xrightarrow{\phi_2} & S^{(n_2)} \end{array}$$

It is easy to check that, (α, β) is an isomorphism if and only if α and β are both isomorphisms. We often identify isomorphic matrix factorisations.

We denote by $\mathsf{MF}_S(f)$ the category of matrix factorisations of f. It is clearly additive, because there is a natural direct sum defined by $(\phi_1, \psi_1) \oplus (\phi_2, \psi_2) := (\phi_1 \oplus \phi_2, \psi_1 \oplus \psi_2)$.

A matrix factorisation (ϕ, ψ) of f is *reduced*, if $\phi, \psi \in (S \setminus S^\times)^{n\times n}$. The trivial factorisations $(1, f)$ and $(f, 1)$ are clearly not reduced.

Let $(\phi, \psi) \in \mathsf{MF}_S(f)$, then $\phi, \psi : S^{(n)} \to S^{(n)}$ are obviously monomorphic. For example, if $\psi(x) = 0$ for some $x \in S^{(n)}$, then $fx = \phi(\psi(x)) = 0$ as well. Notice that S is a regular local ring, hence a UFD, so $f \neq 0$ is not a zero divisor, and therefore $x = 0$.

Up to now we have mainly been considering any $M \in \mathsf{MCM}(R)$ as an S-module. We now show a famous result regarding the free R-resolutions of M, discovered by David Eisenbud:

> Every MCM module M over a hypersurface R has a 2-periodic free R-resolution.

Let $(\phi, \psi) \in \mathsf{MF}_S(f)$. Denote by $\bar{\phi}, \bar{\psi} \in R^{n\times n}$ the matrices ϕ, ψ modulo (f), respectively. We now claim that there is an exact sequence of R-modules of the following form:

$$\cdots \to R^{(n)} \xrightarrow{\bar{\phi}} R^{(n)} \xrightarrow{\bar{\psi}} R^{(n)} \xrightarrow{\bar{\phi}} R^{(n)} \to \cdots \tag{1.4}$$

and this gives an R-free resolution of M:

$$\Phi(\phi, \psi): \quad \cdots \to R^{(n)} \xrightarrow{\bar{\phi}} R^{(n)} \xrightarrow{\bar{\psi}} R^{(n)} \xrightarrow{\bar{\phi}} R^{(n)} \xrightarrow{\bar{\pi}} M \to 0, \tag{1.5}$$

where $\bar{\pi}$ denotes the map $\pi : S^{(n)} \to M$ modulo (f).

In fact, (1.4) is a complex by definition of $(\phi, \psi) \in \mathsf{MF}_S(f)$. Suppose $\bar{x} \in R^{(n)}$ such that $\bar{\phi}(\bar{x}) = 0$, then $\phi(x) \in fS^{(n)} = \phi(\psi(S^{(n)}))$. Since ϕ is monomorphic, $x \in \psi(S^{(n)})$, i.e. $\bar{x} \in \mathsf{Im}(\bar{\psi})$. Then it is clear that (1.4) is exact. Now (1.5) is exact simply by combining (1.4) and (1.3). □

For $(\phi, \psi) \in \mathsf{MF}_S(f)$, we put $\mathsf{Coker}(\phi, \psi) := \mathsf{Coker}(\phi)$, considered as an R-module. For $(\alpha, \beta) : (\phi, \psi) \to (\phi', \psi')$, β induces a morphism $\mathsf{Coker}(\phi, \psi) \to \mathsf{Coker}(\phi', \psi')$, denoted by $\mathsf{Coker}(\alpha, \beta)$. Then it is clear that $\mathsf{Coker} : \mathsf{MF}_S(f) \to \mathsf{mod}(R)$ is an additive exact functor.

Moreover, by the periodicity of 1.5 for $M := \mathsf{Coker}(\phi, \psi)$, we can cut the long exact sequence short:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow R^{(n)} & \xrightarrow{\bar{\phi}} & R^{(n)} & \xrightarrow{\quad\bar{\psi}\quad} & R^{(n)} & \xrightarrow{\quad\bar{\phi}\quad} & \cdots & \xrightarrow{\bar{\phi}} & R^{(n)} & \xrightarrow{\bar{\pi}} & \mathsf{Coker}(\phi, \psi) \longrightarrow 0 \\
 & & & \searrow^{\bar{\pi}} & \uparrow & & & & & & \\
 & & 0 & \dashrightarrow & \mathsf{Coker}(\phi, \psi) & \longrightarrow 0 & & & & &
\end{array}$$

In other words, $\mathsf{Coker}(\phi, \psi)$ is the $2i$-th syzygy of itself, $\forall i \geq 1$. Hence $\mathsf{Coker}(\phi, \psi)$ is an MCM R-module. Thus we have shown that $\mathsf{Coker} : \mathsf{MF}_S(f) \to \mathsf{MCM}(R)$ is an additive exact functor.

We let $\mathcal{I}$ be the ideal of $\mathsf{MF}_S(f)$ generated by the morphisms factoring through direct sums of $(1, f)$, and $\mathcal{J}$ be the ideal generated by the morphisms factoring through direct sums of $(1, f)$ and $(f, 1)$. Notice that $\mathsf{Coker}(1, f) = 0$ and $\mathsf{Coker}(f, 1) = R$.

Now we are ready to state and prove the main theorem by David Eisenbud on matrix factorisations:

Eisenbud's matrix factorisation theorem. If $R = S/(f)$ is a hypersurface, then Coker induces an equivalence of categories:

$$\mathsf{MF}_S(f)/\mathcal{I} \simeq \mathsf{MCM}(R), \tag{1.6}$$

which induces another equivalence

$$\mathsf{MF}_S(f)/\mathcal{J} \simeq \underline{\mathsf{MCM}}(R), \tag{1.7}$$

where $\underline{\mathsf{MCM}}(R) := \mathsf{MCM}(R)/\mathcal{F}$ is the stable category of $\mathsf{MCM}(R)$, and $\mathcal{F}$ denotes the ideal of $\mathsf{MCM}(R)$ generated by the morphisms factoring through free R-modules.

As we have already shown, the cokernel induces a functor again denoted by $\mathsf{Coker} : \mathsf{MF}_S(f)/\mathcal{I} \to \mathsf{MCM}(R)$. Conversely, we define a functor $F : \mathsf{MCM}(R) \to \mathsf{MF}_S(f)/\mathcal{I}$ as follows: given $M \in \mathsf{MCM}(R)$, we have shown that we can get a matrix factorisation $F(M) = (\phi, \psi)$. This will be a well-defined object in $\mathsf{MF}_S(f)/\mathcal{I}$, if we take such a ϕ that the resolution 1.3 is minimal, see Yoshino [1990], (7.4). Given a morphism $g : M_1 \to M_2$ in $\mathsf{MCM}(R)$, by projectivity we have $\alpha, \beta : S^{(n_1)} \to S^{(n_2)}$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{(n_1)} & \xrightarrow{\phi_1} & S^{(n_1)} & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & S^{(n_2)} & \xrightarrow{\phi_2} & S^{(n_2)} & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

Then (α, β) gives a morphism, denoted by $F(g) : F(M_1) \to F(M_2)$. To show that this is well-defined, assume (α', β') is another pair of matrices making the same diagram commute, then $\phi_2 \circ (\beta - \beta') = (\alpha - \alpha') \circ \phi_1$. So $\exists \mu : S^{(n_1)} \to S^{(n_2)}$ such that $\alpha - \alpha' = \phi_2 \circ \mu$ and $\beta - \beta' = \mu \circ \phi_1$. Thus

$$(\alpha, \beta) - (\alpha', \beta') = (\phi_2, 1) \circ (\mu, \mu \circ \phi_1) : (\phi_1, \psi_1) \to (\phi_2, \psi_2),$$

where $(\mu, \mu \circ \phi_1) : (\phi_1, \psi_1) \to (1_{n_2}, f \circ 1_{n_2})$ and $(\phi_2, 1) : (1_{n_2}, f \circ 1_{n_2}) \to (\phi_2, \psi_2)$. Therefore, $(\alpha, \beta) - (\alpha', \beta') = 0 \in \mathsf{MF}_S(f)/\mathcal{I}$, and hence $F(g)$ is uniquely defined. It is then easy to check that $F : \mathsf{MCM}(R) \to \mathsf{MF}_S(f)/\mathcal{I}$ is a functor. In addition, it is straightforward to check that F is a quasi-inverse to $\mathsf{Coker} : \mathsf{MF}_S(f)/\mathcal{I} \to \mathsf{MCM}(R)$. So 1.6 and consequently 1.7 hold. □

Given any $(\phi, \psi) \in \mathsf{MF}_S(f)$. According to Yoshino [1990], p. 58, if the matrix ϕ has a unit entry, then up to isomorphism, (ϕ, ψ) will have $(1, f)$ as a direct summand; likewise if ψ has a unit entry, then up to isomorphism, (ϕ, ψ) will have $(f, 1)$ as a direct summand. This implies that any (ϕ, ψ) can be written as

$$(\phi, \psi) = (\phi_0, \psi_0) \oplus (f, 1)^{(p)} \oplus (1, f)^{(q)}, \tag{1.8}$$

where $p, q \in \mathbb{N}$ and (ϕ_0, ψ_0) is reduced.

An easy observation is:

> If $(\phi, \psi) \in \mathsf{MF}_S(f)$ is reduced, then $\mathsf{Coker}(\phi, \psi)$ has no free direct summand.

In fact, if $M := \mathsf{Coker}(\phi, \psi)$ has a direct summand R, then in (1.3), $\exists \phi'$ such that $\phi = \phi' \oplus f$, hence (ϕ, ψ) is isomorphic to $(\phi', \psi') \oplus (f, 1)$ for some ψ', a contradiction! □

We can further show that the decomposition (1.8) is unique up to isomorphism. To do so, suppose $(\phi, \psi) = (\phi_0', \psi_0') \oplus (f, 1)^{(p')} \oplus (1, f)^{(q')}$ is another such decomposition. Letting $M := \mathsf{Coker}(\phi, \psi)$, $M_0 := \mathsf{Coker}(\phi_0, \psi_0)$, $M_0' := \mathsf{Coker}(\phi_0', \psi_0')$, by (1.6) we get $M \simeq M_0 \oplus R^{(p)} \simeq M_0' \oplus R^{(p')}$. As we have just shown, neither M_0 nor M_0' has any free direct summand, so $p = p'$ and $M_0 \simeq M_0'$. By comparing the size of matrices, we get $q = q'$ as well. That $(\phi_0, \psi_0) \simeq (\phi_0', \psi_0')$ follows from $M_0 \simeq M_0'$ and the proof of (1.6).

Another important result[3] is:

[3] From (1.9) we also get that in general the matrix factorisations (ϕ, ψ) and (ψ, ϕ) are not isomorphic.

Let $M = \mathsf{Coker}(\phi, \psi)$ be an indecomposable MCM R-module which is not free, where $(\phi, \psi) \in \mathsf{MF}_S(f)$ is reduced. Then its first reduced syzygy $\mathsf{syz}^1_R M$ is also indecomposable and not free, moreover,

$$\mathsf{syz}^1_R M \simeq \mathsf{Coker}(\psi, \phi). \tag{1.9}$$

Since M is indecomposable, clearly (ϕ, ψ) is also indecomposable. Now it is obvious to see that (ψ, ϕ) is another indecomposable reduced matrix factorisation of f, and hence $\mathsf{Coker}(\psi, \phi)$ is indecomposable and not free. Now we can extract from (1.4) a short exact sequence as follows:

$$0 \to \mathsf{Coker}(\psi, \phi) \to R^{(n)} \to \mathsf{Coker}(\phi, \psi) \to 0,$$

from which we have $\mathsf{syz}^1_R M \simeq \mathsf{Coker}(\psi, \phi)$ since $\mathsf{Coker}(\psi, \phi)$ is shown to be indecomposable and not free. □

Given another MCM R-module $N = \mathsf{Coker}(\phi', \psi')$ for some matrix factorisation (ϕ', ψ'), if there exists $g \in \mathsf{Hom}_R(N, \mathsf{syz}^1_R(M))$, by Eisenbud's matrix factorisation theorem, there is a morphism of matrix factorisations $(\alpha, \beta) : (\phi', \psi') \to (\psi, \phi)$, such that $\mathsf{Coker}(\alpha, \beta) = g$. By definition this implies $\beta \circ \psi' = \phi \circ \alpha$. Notice that

$$\begin{pmatrix} \phi & \beta \\ 0 & \phi' \end{pmatrix} \begin{pmatrix} \psi & -\alpha \\ 0 & \psi' \end{pmatrix} = \begin{pmatrix} \phi\psi & -\phi\alpha + \beta\psi' \\ 0 & \phi'\psi' \end{pmatrix} = f1,$$

so $(\begin{pmatrix} \phi & \beta \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} \psi & -\alpha \\ 0 & \psi' \end{pmatrix}) \in \mathsf{MF}_S(f)$ as well. Denote by $L := \mathsf{Coker}(\begin{pmatrix} \phi & \beta \\ 0 & \phi' \end{pmatrix})$, and we have a short exact sequence

$$0 \to M \xrightarrow{j} L \xrightarrow{q} N \to 0,$$

where $j = \mathsf{Coker}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ and $q = \mathsf{Coker}(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix})$.

Any extension in $\mathsf{Ext}^1_R(N, M)$ can be obtained this way, since the middle element in such a short exact sequence is also an MCM R-module as we have shown before.

2. Auslander-Reiten theory

2.1 Almost split sequences and irreducible morphisms

Fix a Henselian CM local ring $(R, \mathfrak{m}_R)$ with residue field κ_R for the whole section. We continue to denote by $\mathsf{MCM}(R)$ the category of all finitely generated MCM R-modules, a full subcategory of $\mathsf{mod}(R)$. Recall that $M \in \mathsf{MCM}(R)$ is indecomposable if and only if $\mathsf{End}_R(M)$ is a local ring.

For any indecomposable $M \in \mathsf{MCM}(R)$, we define $\mathfrak{S}(M)$ to be the set of all non-split exact sequences s in $\mathsf{MCM}(R)$ of the form

$$s : 0 \to N_s \to E_s \to M \to 0,$$

where N_s is indecomposable. By definition any $s \in \mathfrak{S}(M)$ gives a non-trivial element in $\mathsf{Ext}^1_R(M, N_s)$.

First we observe that:

> If the indecomposable $M \in \mathsf{MCM}(R)$ is not free, then $\mathfrak{S}(M) \neq \varnothing$.

In fact, since M is not free, there exists a non-split epimorphism $E \twoheadrightarrow M$ for some free R-module E. Let $N = \mathsf{Ker}(E \to M)$, we get a non-split short exact sequence of the form

$$s : 0 \to N \to E \to M \to 0.$$

Now we decompose N into indecomposable direct summands $N = \bigoplus_i N_i$, and let $E_i := \frac{E}{\bigoplus_{j \neq i} N_j}$. Then for any i,

$$s_i : 0 \to N_i \to E_i \to M \to 0$$

is a short exact sequence, since $E_i/N_i \simeq \frac{E}{\bigoplus_j N_j} = \frac{E}{N} \simeq M$. Now we claim that, since s is not split, $\exists i$ such that s_i is not split either[1].

In fact, if each s_i splits, then for each i, the inclusion $\iota_i : N_i \hookrightarrow E_i$ in s_i admits a left inverse $r_i : E_i \to N_i$. Then we have a map $r : E \twoheadrightarrow E_i \xrightarrow{r_i} N_i \hookrightarrow N$ by composing r_i with the quotient map $E \to E_i$ and the inclusion $N_i \to N$. Denote by $\iota : N \hookrightarrow E$ the inclusion in s, and we can easily see that

$$r \circ \iota = N \xrightarrow{\iota} E \twoheadrightarrow E_i \xrightarrow{r_i} N_i \hookrightarrow N = 1_N.$$

This can be checked by looking at each N_i. Therefore, s splits as well, a contradiction! □

Now we would like to inspect the ordering structure on $\mathfrak{S}(M)$. For any $s, t \in \mathfrak{S}(M)$, where $M \in \mathsf{MCM}(R)$ is indecomposable, we write $s \geq t$ if there exists $f \in \mathsf{Hom}_R(N_s, N_t)$ such that $\mathsf{Ext}^1_R(M, f)(s) = t$. By definition this means there exists a commutative diagram in the following form:

[1] This indeed follows from Yoshino [1990], (1.22).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_s & \longrightarrow & E_s & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow 1 & & \\ 0 & \longrightarrow & N_t & \longrightarrow & E_t & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Moreover if such an f is an isomorphism, we write $s \simeq t$.

Notice that $\geq$ is clearly reflexive and transitive on $\mathfrak{S}(M)$, and hence so is $\simeq$. Also, $\simeq$ is clearly symmetric on $\mathfrak{S}(M)$, thus $\simeq$ is an equivalence relation on $\mathfrak{S}(M)$. We shall prove that:

> For any indecomposable $M \in \mathsf{MCM}(R)$, $\geq$ is a partial order on $\mathfrak{S}(M)/\simeq$.

It suffices to show that $\geq$ is anti-symmetric on $\mathfrak{S}(M)$ modulo $\simeq$. Suppose $s, t \in \mathfrak{S}(M)$ satisfy $s \geq t$ and $t \geq s$. Then there exist R-linear map $f : N_s \to N_t$ and $g : N_t \to N_s$ such that $\mathsf{Ext}^1_R(M, f)(s) = t$ and $\mathsf{Ext}^1_R(M, g)(t) = s$. Letting $h := g \circ f : N_s \to N_s$, we see that $\mathsf{Ext}^1_R(M, h)(s) = s$. We claim that $h \in \mathsf{End}_R(N_s)$ is an isomorphism.

To prove that, consider the R-subalgebra

$$R[h] := \{a_0 + a_1 h + \cdots + a_n h^n \mid n \in \mathbb{N}, a_0, \cdots, a_n \in R\}$$

of $\mathsf{End}_R(N)$. Since R is noetherian, Hilbert's basis theorem implies that $R[h]$ is finitely generated as an R-module. Since R is Henselian, $R[h]$ is a direct product of local R-algebras. If $R[h]$ is not local, then it contains non-trivial idempotents[2] including $e := (1, 0, \cdots, 0)$. Then $e(1-e) = e - e^2 = 0$ implies that both e and $1-e$ are zero divisor in $R[h]$ and hence in $\mathsf{End}_R(N)$. Thus neither e nor $1-e$ is a unit in $\mathsf{End}_R(N)$, so $e, 1-e \in \mathsf{radEnd}_R(N)$, which implies $1 = e + (1-e) \in \mathsf{radEnd}_R(N)$, a contradiction! Therefore, $R[h]$ is local.

Observe that there is a natural map $\phi : \mathsf{End}_R(N) \to \mathsf{Ext}^1_R(M, N)$, given by $\alpha \mapsto \mathsf{Ext}^1_R(M, \alpha)(s)$. We still denote by ϕ its restriction on $R[h]$. Since s does not split, $\phi(1) = \mathsf{Ext}^1_R(M, 1)(s) = s \neq 0$, i.e. $1 \notin \mathsf{Ker}(\phi)$. On the other hand, $\phi(1-h) = \mathsf{Ext}^1_R(M, 1)(s) - \mathsf{Ext}^1_R(M, h)(s) = s - s = 0$, so $1 - h \in \mathsf{Ker}(\phi)$. Thus $\mathsf{Ker}(\phi)$ is a proper ideal of $R[h]$, hence $\mathsf{Ker}(\phi) \subset \mathsf{rad}(R[h])$. This implies that $1 - h$ is not a unit, and thus h is a unit for $R[h]$ is local, which completes the proof. □

Now we prove another important property of $\mathfrak{S}(M)$:

> Let $M \in \mathsf{MCM}(R)$ be indecomposable and let $s : 0 \to N_s \xrightarrow{\iota_s} E_s \xrightarrow{\pi_s} M \to 0$ and $t : 0 \to N_t \xrightarrow{\iota_t} E_t \xrightarrow{\pi_t} M \to 0$ be any two elements in $\mathfrak{S}(M)$. Then there is an element $u \in \mathfrak{S}(M)$ with $s \geq u$ and $t \geq u$.

Consider the following short exact sequence:

$$0 \to \mathsf{Ker}(\pi_s, \pi_t) \to E_s \oplus E_t \xrightarrow{(\pi_s, \pi_t)} M \to 0.$$

[2] i.e. idempotents that are not equal to 0 or 1.

Recall that since neither π_s nor π_t is a split epimorphism, the sequence is not split. Actually, if (π_s, π_t) were split, there would be some $\alpha : M \to E_s$ and some $\beta : M \to E_t$ satisfying $\pi_s \circ \alpha + \pi_t \circ \beta = 1_M$. Since $\mathsf{End}_R(M)$ is local, either $\pi_s \circ \alpha$ or $\pi_t \circ \beta$ would be an automorphism of M, which means either s or t splits.

For simplicity we let $N := \mathsf{Ker}(\pi_s, \pi_t)$, $E := E_s \oplus E_t$, and $\phi := (\pi_s, \pi_t)$. Decompose N into indecomposable direct summands $N = \bigoplus_i N_i$ and denote by $E_i := E/\bigoplus_{j \neq i} N_j$, we get short exact sequences of the form

$$u_i : 0 \to N_i \to E_i \to M \to 0.$$

And we know from Yoshino [1990], (1.22) that there exists some i such that $u_i \in \mathfrak{S}(M)$. Denote by u this non-split short exact sequence, and we claim that $s \geq u$ and $t \geq u$. In fact, consider the following diagram:

$$
\begin{array}{lccccccccc}
s: & 0 & \longrightarrow & N_s & \xhookrightarrow{\iota_s} & E_s & \xrightarrow{\pi_s} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
 & 0 & \longrightarrow & N & \hookrightarrow & E & \xrightarrow{(\pi_s, \pi_t)} & M & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
u: & 0 & \longrightarrow & N_i & \hookrightarrow & E_i & \xrightarrow{\pi_i} & M & \longrightarrow & 0
\end{array}
$$

where $\mu : N_s \to N$ is given by $z \mapsto (\iota_s(z), 0)$. Then it is easy to check that the diagram commutes and hence $s \geq u$. Similarly we can show $t \geq u$. $\square$

An easy corollary is that any minimal element in $\mathfrak{S}(M)$ is also minimum.

In fact, if $s \in \mathfrak{S}(M)$ is minimal, then for any $t \in \mathfrak{S}(M)$, we have $s \not\geq t$. Now by what we have just proved, $\exists u \in \mathfrak{S}(M)$ with $s \geq u$ and $t \geq u$. But $s \geq u$ forces $s \simeq u$. Thus $t \geq s$. $\square$

Auslander-Reiten sequence. From now on, we say $s \in \mathfrak{S}(M)$ is an *almost split sequence*, or an *Auslander-Reiten sequence* ending in M if s is the minimum element in $\mathfrak{S}(M)$. Clearly, the Auslander-Reiten sequence ending in any indecomposable $M \in \mathsf{MCM}(R)$ is unique up to $\simeq$ if it exists. In this case, we denote by $\tau(M) := N_s$, called the *Auslander-Reiten translation* of M.

The following equivalent definition shows why an Auslander-Reiten sequence is "almost split".

> Let $M \in \mathsf{MCM}(R)$ be indecomposable and let $s : 0 \to N_s \xrightarrow{\iota_s} E_s \xrightarrow{\pi_s} M \to 0$ be in $\mathfrak{S}(M)$. Then s is the Auslander-Reiten sequence ending in M if and only if for any $L \in \mathsf{MCM}(R)$ and any $p \in \mathsf{Hom}_R(L, M)$ that is not a split epimorphism, there exists $f \in \mathsf{Hom}_R(L, E_s)$ such that $p = \pi_s \circ f$.

For the "if" part, we need only show that s is minimal. Let $t : 0 \to N_t \xrightarrow{\iota_t} E_t \xrightarrow{\pi_t} M \to 0$ be an element of $\mathfrak{S}(M)$ with $s \geq t$. Since π_t is not a split epimorphism, $\exists f : E_t \to E_s$ such that $\pi_t = \pi_s \circ f$. Then clearly $\mathsf{Ext}^1_R(M, f|_{N_t})(t) = s$, i.e. $t \geq s$. So $s \simeq t$, and hence s is minimal.

For the "only if" part, let $p : L \to M$ be a R-homomorphism that is not a split epimorphism, and then the exact sequence

$$u : 0 \to \mathsf{Ker}(\pi_s, p) \to E_s \oplus L \xrightarrow{(\pi_s,p)} M \to 0$$

does not split. Since $\iota_s(N_s) \subset \mathsf{Ker}(\pi_s, p)$, the following diagram commutes:

$$\begin{array}{cccccccccc}
s: & 0 & \longrightarrow & N_s & \xrightarrow{\iota_s} & E_s & \xrightarrow{\pi_s} & M & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \iota_s} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
u: & 0 & \longrightarrow & \mathsf{Ker}(\pi_s,p) & \hookrightarrow & E_s \oplus L & \xrightarrow{(\pi_s,p)} & M & \longrightarrow & 0
\end{array}$$

In other words, $\mathsf{Ext}^1_R(M, \iota_s)(s) = u$. Now we decompose $Q := \mathsf{Ker}(\pi_s, p)$ into indecomposable direct summands $Q = \bigoplus_i Q_i$, and let $E_i := \frac{E_s \oplus L}{\bigoplus_{j \neq i} Q_j}$, so that $\forall i$, $u_i : 0 \to Q_i \to E_i \to M \to 0$ is a short exact sequence. We have seen that there exists some i such that $t := u_i$ does not split. So $t \in \mathfrak{S}(M)$ and $s \geq t$, since $\mathsf{Ext}^1_R(M, Q) = \sum_i \mathsf{Ext}^1_R(M, Q_i)$. Meanwhile, s being the Auslander-Reiten sequence ending in M, it is minimal in $\mathfrak{S}(M)$, so $s \simeq t$. In particular, there exist $g \in \mathsf{Hom}_R(Q, N_s)$ and $f' \in \mathsf{Hom}_R(E_s \oplus L, E_s)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \hookrightarrow & E_s \oplus L & \xrightarrow{(\pi_s,p)} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & N_s & \xrightarrow{\iota_s} & E_s & \xrightarrow{\pi_s} & M & \longrightarrow & 0
\end{array}$$

Now we denote by f the composition $L \hookrightarrow E_s \oplus L \xrightarrow{f'} E_s$ of f' with the natural inclusion. Then it is clear that $p = \pi_s \circ f$. $\square$

Irreducible morphism. Let $M, N \in \mathsf{MCM}(R)$ and $f \in \mathsf{Hom}_R(M, N)$. f is an *irreducible morphism* if

- f is neither a split monomorphism nor a split epimorphism; and
- whenever $f = h \circ g$ is a factorisation in $\mathsf{MCM}(R)$, then either g is a split monomorphism or h is a split epimorphism.

Let $M \in \mathsf{MCM}(R)$ be indecomposable and $s : 0 \to N \to E \xrightarrow{p} M \to 0$ be the Auslander-Reiten sequence ending in M. Then p is an irreducible morphism.

Clearly, p is a non-split epimorphism. Assume that $p = (E \xrightarrow{g} X \xrightarrow{h} M)$ for some $X \in \mathsf{MCM}(R)$, where h is not a split epimorphism, then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{p} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j|_N} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & Q & \hookrightarrow & G & \xrightarrow{(p,h)} & M & \longrightarrow & 0
\end{array}$$

where $G := E \oplus X$, $Q := \mathsf{Ker}(p,h)$ and $j = \begin{pmatrix} 0 \\ g \end{pmatrix}$. Since p and h are non-split, we know that (p,h) is non-split as well. Decompose $Q = \bigoplus_i Q_i$ into indecomposable direct summands, and denote by $G_i := G/\bigoplus_{j \neq i} Q_j$. We know from Yoshino [1990], (1.22) that there exists i such that $s_i : 0 \to Q_i \to G_i \to M \to 0$ is a non-split short exact sequence, hence $s_i \in \mathfrak{S}(M)$. Combining the diagram above with the one below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \hookrightarrow & G & \xrightarrow{(p,h)} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q|_Q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & Q_i & \hookrightarrow & G_i & \twoheadrightarrow & M & \longrightarrow & 0 \end{array}$$

where q is the natural projection, we obtain $s \geq s_i$. This implies $s \simeq s_i$ for s is minimal. Then $q \circ j$ is an isomorphism. Writing $q = (q_1, q_2) : E \oplus X \to G_i$, we see that $(q_1, q_2) \begin{pmatrix} 0 \\ g \end{pmatrix} = q_2 \circ g : E \to G_i$ is an isomorphism. In particular,

$$1_E = (q_2 \circ g)^{-1}(q_2 \circ g) = [(q_2 \circ g)^{-1} \circ q_2] \circ g,$$

which implies g is a split monomorphism. □

As a corollary, we get:

> Let $M,\ L \in \mathsf{MCM}(R)$ be indecomposable and assume there exists an Auslander-Reiten sequence $s : 0 \to N \to E \xrightarrow{p} M \to 0$ ending in M. Then the following are equivalent:
> - there is an irreducible morphism $L \to M$;
> - L is isomorphic to a direct summand of E.

First let $f : L \to M$ be an irreducible morphism. In particular, f is not a split epimorphism, and then there exists $g : L \to E$ such that $f = p \circ g$, for s is almost split. Since p is not a split epimorphism, we get g is a split monomorphism, i.e. L is isomorphic to a direct summand of E.

Conversely, suppose that $E \simeq L \oplus L'$ and let $p = (\alpha, \beta) : L \oplus L' \to M$. We claim that $\alpha : L \to M$ is an irreducible morphism. If α is an isomorphism, then p is isomorphic to the canonical retraction $L \oplus L' \twoheadrightarrow L$, a contradiction! So α is a non-isomorphism between indecomposable modules, hence α is neither a split monomorphism nor a split epimorphism. Now assume that $\alpha = (L \xrightarrow{h} X \xrightarrow{k} M)$ for some $X \in \mathsf{MCM}(R)$, where k is not a split epimorphism. Then the following diagram clearly commutes:

$$\begin{array}{ccc} L \oplus L' & \xrightarrow{\quad p \quad} & M \\ & {\scriptstyle h \oplus 1_{L'}} \searrow \quad \nearrow {\scriptstyle (k,\beta)} & \\ & X \oplus L' & \end{array}$$

since $\begin{pmatrix} k & \beta \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k \circ h & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \end{pmatrix} = p$. If (k, β) is a split epimorphism, there exists $\mu : M \to X$ and $\nu : M \to L'$ such that

$$\begin{pmatrix} k & \beta \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = k \circ \mu + \beta \circ \nu = 1_M.$$

Given that M is indecomposable, $\mathsf{End}_R(M)$ is local, and that k is not a split epimorphism, we get $k \circ \mu \in \mathsf{radEnd}_R(M)$, and thus $\beta \circ \nu$ is an isomorphism. So $p \circ \begin{pmatrix} 0 \\ \nu \end{pmatrix} = \beta \circ \nu$ is also an isomorphism, and thus p is a split epimorphism as β, a contradiction! Therefore, (k, β) can not be a split epimorphism, and thus $h \oplus 1_{L'}$ is a split monomorphism for p is an irreducible morphism. In other words, $\exists \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} : X \oplus L' \to L \oplus L'$ such that

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives $\lambda_{11} \circ h = 1$, i.e. h is a split monomorphism. □

In other words, any irreducible morphism $g : L \to M$ is obtained by $g = p \circ h$ for some split monomorphism $h : L \to E$.

Recall that $\mathsf{MCM}(R)$ admits a duality, and thus we define dually the following.

For any indecomposable $N \in \mathsf{MCM}(R)$, let $\mathfrak{S}'(N)$ denote the set of all non-split exact sequences s in $\mathsf{MCM}(R)$ of the form

$$s : 0 \to N \to P_s \to M_s \to 0,$$

where M_s is indecomposable. By definition any $s \in \mathfrak{S}'(N)$ gives a non-trivial element in $\mathsf{Ext}^1_R(M_s, N)$.

If R has the canonical module ω_R, we denote by $(-)^* := \mathsf{Hom}_R(-, \omega_R)$ the canonical exact auto-functor on $\mathsf{MCM}(R)$, and then for any non-split short exact sequence $s : 0 \to N \to E \to M \to 0$ in $\mathsf{MCM}(R)$, $s \in \mathfrak{S}'(N)$ if and only if $s \in \mathfrak{S}(M)$, if and only if $s^* \in \mathfrak{S}(N^*)$.

We list below the dual version of the statements regarding $\mathfrak{S}(M)$ we have proved in the section.

Fix an indecomposable $N \in \mathsf{MCM}(R)$.

- If $N \not\simeq \omega_R$, then $\mathfrak{S}'(N) \neq \varnothing$.
- For any $s, t \in \mathfrak{S}'(N)$, write $s \geq' t$ if there exists $f \in \mathsf{Hom}_R(M_t, M_s)$ such that $\mathsf{Ext}^1_R(f, N)(s) = t$; write $s \simeq' t$ if such an f is an isomorphism. Then $\geq'$ is a partial order on $\mathfrak{S}'(N)$ modulo $\simeq'$.
- For any $s, t \in \mathfrak{S}'(N)$, there exists $u \in \mathfrak{S}'(N)$ such that $s \geq' u$ and $t \geq' u$. Therefore, any minimal element in $\mathfrak{S}'(N)$ is minimum.

We call $s \in \mathfrak{S}'(N)$ an *almost split sequence*, or an *Auslander-Reiten sequence* starting from N if s is the minimum element in $\mathfrak{S}'(N)$. Clearly, the Auslander-Reiten sequence starting from any indecomposable $N \in \mathsf{MCM}(R)$ is unique up to $\simeq'$ if it exists. In this case, we denote by $\tau^{-1}(N) := M_s$.

The following are the dual to the statements concerning the Auslander-Reiten sequence ending in some indecomposable $M \in \mathsf{MCM}(R)$ proven before.

> Fix an indecomposable $N \in \mathsf{MCM}(R)$.
>
> - Any $s : 0 \to N \xrightarrow{\alpha_s} P_s \xrightarrow{\beta_s} M_s \to 0$ in $\mathfrak{S}'(N)$ is the Auslander-Reiten sequence starting from N, if and only if for any $L \in \mathsf{MCM}(R)$ and any $r \in \mathsf{Hom}_R(N, L)$ that is not a split monomorphism, there exists $f \in \mathsf{Hom}_R(P_s, L)$ such that $r = f \circ \alpha_s$.
> - If $s : 0 \to N \xrightarrow{\alpha_s} P_s \xrightarrow{\beta_s} M_s \to 0$ is the Auslander-Reiten sequence starting from N, then α_s is an irreducible morphism.
> - Let $L \in \mathsf{MCM}(R)$ be indecomposable, and $s : 0 \to N \xrightarrow{\alpha_s} P_s \xrightarrow{\beta_s} M_s \to 0$ is the Auslander-Reiten sequence starting from N. Then there exists an irreducible morphism $N \to L$, if and only if L is isomorphic to a direct summand of P_s. In other words, any irreducible morphism $g : N \to L$ is obtained by $g = h \circ \alpha_s$ for some split epimorphism $h : P_s \to L$.

The mutually dual definitions of Auslander-Reiten sequences are coherent in the following sense:

> Let $s : 0 \to N \to E \to M \to 0$ be a non-split short exact sequence in $\mathsf{MCM}(R)$, where N, M are indecomposable. Then s is the Auslander-Reiten sequence ending in M, if and only if it is the Auslander-Reiten sequence starting from N.

Suppose s is the Auslander-Reiten sequence ending in M, and we would like to show that s is minimal with respect to $\geq'$. Let $t \in \mathfrak{S}'(N)$ with $s \geq' t$. By definition, there exists $f : M_t \to M$ such that $\mathsf{Ext}^1_R(f, N)(s) = t$. In other words, there exists a commutative diagram of the form

$$\begin{array}{lccccccccc} s: & 0 & \longrightarrow & N & \xrightarrow{\iota} & E & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle f} & & \\ t: & 0 & \longrightarrow & N & \xrightarrow{\alpha_t} & P_t & \xrightarrow{\beta_t} & M_t & \longrightarrow & 0 \end{array}$$

Assume that f is not an isomorphism, then it is not even a split epimorphism for M and M_t are both indecomposable. Since s is the almost split sequence ending in M, there exists $\theta : M_t \to E$ such that $\pi \circ \theta = f$. Notice that $\pi \circ (g - \theta \circ \beta_t)$, in other words, $\mathsf{Im}(g - \theta \circ \beta_t) \subset \mathsf{Ker}(\pi) = \mathsf{Im}(\iota)$, which implies that $\exists \phi : P_t \to N$ such that $\iota \circ \phi = g - \theta \circ \beta_t$. Then $\iota \circ \phi \circ \alpha_t = (g - \theta \circ \beta_t) \circ \alpha_t = g \circ \alpha_t = \iota$, so $\phi \circ \alpha_t = 1_N$ for ι is a monomorphism. This contradicts the fact that t does not split! Thus, f must be an isomorphism, hence $s \simeq' t$. Therefore, we have shown that s is also the Auslander-Reiten sequence starting from N. The other direction is analogous. □

The category $\mathsf{MCM}(R)$ is said to *admit Auslander-Reiten sequences*, if for any indecomposable $M \in \mathsf{MCM}(R)$ which is not free, there exists an Auslander-Reiten sequence ending in M.

Recall that if R has the canonical module ω_R, then the canonical dual is an exact anti-equivalence of $\mathsf{MCM}(R)$, i.e. an equivalence $\mathsf{MCM}(R) \to \mathsf{MCM}(R)^{op}$. Thus in this case, $\mathsf{MCM}(R)$ admits Auslander-Reiten sequences if and only if for any indecomposable $N \in \mathsf{MCM}(R)$ that is not isomorphic to ω_R, there exists an Auslander-Reiten sequence starting from N.

Isolated singularity. Given any Henselian CM local ring R with canonical module ω_R, we say that R is an *isolated singularity*, if for any prime ideal $\mathfrak{p}$ of R which is not maximal, the localisation $R_\mathfrak{p}$ is a regular local ring. The following results are well-known:

- If $\mathsf{dim}(R) = 1$, R is an isolated singularity if and only if it is reduced.
- If $\mathsf{dim}(R) = 2$, R is an isolated singularity if and only if it is a normal integral domain.

See Yoshino [1990], p. 16. □

Auslander transpose. As before let R be a Henselian CM local ring with canonical module ω_R. Given any $M \in \mathsf{mod}(R)$, there is a free presentation of the form

$$F_1 \xrightarrow{f} F_0 \to M \to 0.$$

We denote by $\mathsf{Tr}(M) = \mathsf{Coker}(\mathsf{Hom}_R(F_0, R) \xrightarrow{-\circ f} \mathsf{Hom}_R(F_1, R))$, called the *Auslander transpose* of M, which depends on the choice of the free presentation. Nevertheless, by horseshoe lemma it is easy to see that $\mathsf{Tr}(M)$ is unique up to free direct summand.

The following well-known theorem by M. Auslander gives an equivalent condition for $\mathsf{MCM}(R)$ to admit Auslander-Reiten sequences.

As before let R be a Henselian CM local ring with canonical module ω_R. Then $\mathsf{MCM}(R)$ admits Auslander-Reiten sequences, if and only if R is an isolated singularity.

See Auslander and Unger [2006], p. 200. □

Moreover, in the case of an isolated singularity, the Auslander-Reiten translation can be given by the Auslander transpose:

If R is an isolated singularity and $M \in \mathsf{MCM}(R)$ is not free, we have

$$\tau(M) = \mathsf{Hom}_R(\mathsf{syz}_R^{\mathsf{dim}(R)} \mathsf{Tr}(M), \omega_R), \tag{2.1}$$

See Yoshino [1990], (3.11). □

2.2 Auslander-Reiten quivers

Fix a Henselian CM local ring R with maximal ideal $\mathfrak{m}_R$, and we assume that

- the residual field $\kappa_R := R/\mathfrak{m}_R$ is algebraically closed; and
- R has a canonical module ω_R.

Radical morphism. Given $M, N \in \mathsf{MCM}(R)$, we consider the *radical morphisms*[3] between them, defined by

$$\mathsf{rad}_R(M,N) := \{f \in \mathsf{Hom}_R(M,N) \mid \forall g \in \mathsf{Hom}_R(N,M), 1 - gf \text{ is invertible}\}.$$

For $n \geq 2$, we also define

$$\mathsf{rad}^n_R(M,N) := \{f \in \mathsf{Hom}_R(M,N)\} \mid f = f_n \cdots f_1,\ f_i \text{ are radical morphisms}\}.$$

And for coherence, we set $\mathsf{rad}^0_R(M,N) := \mathsf{Hom}_R(M,N)$ and $\mathsf{rad}^1_R(M,N) := \mathsf{rad}_R(M,N)$.

Above all, we would like to show that:

> rad^n_R is a 2-sided ideal of the category $\mathsf{MCM}(R)$ for any $n \in \mathbb{N}$. In other words,
> - $\forall M \in \mathsf{MCM}(R)$, $0_M \in \mathsf{rad}^n_R(M,M)$;
> - $\forall M, N \in \mathsf{MCM}(R)$, if $f, g \in \mathsf{rad}^n_R(M,N)$, $\lambda, \mu \in \kappa_R$, then $\lambda f + \mu g \in \mathsf{rad}^n_R(M,N)$;
> - if $f \in \mathsf{rad}^n_R(M,N)$, $\phi \in \mathsf{Hom}_R(N,N')$, $\psi \in \mathsf{Hom}_R(M',M)$ for $M, N, M', N' \in \mathsf{MCM}(R)$, then $\phi \circ f \in \mathsf{rad}^n_R(M,N')$ and $f \circ \psi \in \mathsf{rad}^n_R(M',N)$.

Clearly when $n = 0$, $\mathsf{rad}^0_R = \mathsf{Hom}_R$ is a 2-sided ideal of $\mathsf{MCM}(R)$.

Now consider the case $n = 1$. Then $0_M \in \mathsf{rad}_R(M,M)$, $\forall M \in \mathsf{MCM}(R)$. Let $f \in \mathsf{rad}_R(M,N)$, $\phi \in \mathsf{Hom}_R(N,N')$, and $g \in \mathsf{Hom}_R(N',M)$, then $1 - g \circ \phi \circ f$ is invertible by definition, so $\phi \circ f \in \mathsf{rad}_R(M,N')$. If moreover $\psi \in \mathsf{Hom}_R(M',M)$ and $g' \in \mathsf{Hom}_R(N,M')$, then by definition $\exists h \in \mathsf{End}_R(M)$ such that $(1 - \psi \circ g' \circ f) \circ h = 1 = h \circ (1 - \psi \circ g' \circ f)$. Now

$$\begin{aligned}
&(1 + g' \circ f \circ h \circ \psi) \circ (1 - g' \circ f \circ \psi)\\
=&1 - g' \circ f \circ \psi + g' \circ f \circ h \circ \psi - g' \circ f \circ h \circ \psi \circ g' \circ f \circ \psi\\
=&1 - g' \circ f \circ \psi + g' \circ f \circ h \circ \psi - g' \circ f \circ (h - 1) \circ \psi\\
=&1 - g' \circ f \circ \psi + g' \circ f \circ h \circ \psi - g' \circ f \circ h \circ \psi + g' \circ f \circ \psi\\
=&1,
\end{aligned}$$

and similarly $(1 - g' \circ f \circ \psi) \circ (1 + g' \circ f \circ h \circ \psi) = 1$, so $f \circ \psi \in \mathsf{rad}_R(M',N)$.

Notice that $\mathsf{rad}_R(M,N)$ is clearly closed under scalar multiplication, so it remains to show that given $f, g \in \mathsf{rad}_R(M,N)$, $f - g \in \mathsf{rad}_R(M,N)$ as well. Given any $h \in \mathsf{Hom}_R(N,M)$, by definition there exists $\alpha \in \mathsf{End}_R(M)$ such that $\alpha \circ (1 - h \circ f) = 1 = (1 - h \circ f) \circ \alpha$. Then by definition there exists $\beta \in \mathsf{End}_R(M)$ as well,

[3] In fact, the radical can be defined as a 2-sided ideal of any preadditive category.

such that $\beta \circ (1 - (-\alpha \circ h) \circ g) = 1 = (1 - (-\alpha \circ h) \circ g) \circ \beta$. Now

$$\begin{aligned}\beta \circ \alpha \circ (1 - h \circ (f - g)) &= \beta \circ \alpha \circ (1 - h \circ f + h \circ g) \\ &= \beta \circ \alpha \circ (1 - h \circ f) + \beta \circ \alpha \circ h \circ g \\ &= \beta + \beta \circ \alpha \circ h \circ g \\ &= \beta \circ (1 + \alpha \circ h \circ g) \\ &= 1.\end{aligned}$$

On the other hand, we have shown that $g \circ \alpha \in \mathsf{rad}_R(M, N)$, so $\exists \gamma \in \mathsf{End}_R(M)$ such that $(1 - (-h) \circ (g \circ \alpha)) \circ \gamma = 1 = \gamma \circ (1 - (-h) \circ (g \circ \alpha))$. Then

$$\begin{aligned}&(1 - h \circ (f - g)) \circ \alpha \circ \gamma \\ =&(1 - h \circ f) \circ \alpha \circ \gamma + h \circ g \circ \alpha \circ \gamma \\ =&\gamma + h \circ g \circ \alpha \circ \gamma \\ =&(1 + h \circ g \circ \alpha) \circ \gamma \\ =&1.\end{aligned}$$

Therefore, $1 - h \circ (f - g)$ is invertible, and hence $f - g \in \mathsf{rad}_R(M, N)$. So rad_R is a 2-sided ideal of $\mathsf{MCM}(R)$.

For $n \geq 1$, if $f \in \mathsf{rad}_R^{n+1}(M, N)$, we have a decomposition $f = f_{n+1} \circ \cdots \circ f_1$, where $f_i \in \mathsf{rad}_R(X_{i-1}, X_i)$, $X_0 = M$, $X_{n+1} = N$. Since rad_R is an ideal, $f_2 \circ f_1 \in \mathsf{rad}_R(X_0, X_2)$, and thus $f = f_{n+1} \circ \cdots \circ (f_2 \circ f_1) \in \mathsf{rad}_R^n(M, N)$. Hence $\mathsf{rad}_R^n(M, N) \supset \mathsf{rad}_R^{n+1}(M, N)$, $\forall n \in \mathbb{N}$. By induction, we can show that rad_R^n is a 2-sided ideal of $\mathsf{MCM}(R)$ for $n \geq 2$. □

If we decompose M, N into indecomposable direct summands $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$, then any morphism $f : M \to N$ is also decomposed as $f = (f_{ij})$, where $f_{ij} = \pi_j^N \circ f \circ \iota_i^M : M_i \to N_j$, π_j denotes the canonical projection onto the j-th direct summand, and ι_i denotes the natural embedding from the i-th direct summand. Since rad_R is a 2-sided ideal, we get:

$$f \in \mathsf{rad}_R(M, N) \;\Leftrightarrow\; f_{ij} \in \mathsf{rad}_R(M_i, N_j), \forall i, j.$$

The $\Rightarrow$ direction directly follows from the definition, and so does the $\Leftarrow$ direction when we write $f = \sum_i \sum_j \iota_j^N \circ f_{ij} \circ \pi_i^M$. □

When M, N are both indecomposable, clearly $\mathsf{rad}_R(M, N)$ consists of all non-isomorphisms from M to N. Therefore, given any $M, N \in \mathsf{MCM}(R)$, $f \in \mathsf{rad}_R(M, N)$ if and only if f_{ij} is not an isomorphism for any i, j. In particular, when $M = N$ is indecomposable, $\mathsf{rad}_R(M, M) = \mathsf{radEnd}_R(M)$ is just the Jacobson radical of the endomorphism ring.

Given 2 indecomposable modules $M, N \in \mathsf{MCM}(R)$, we now show that:

Any morphism $f \in \mathsf{Hom}_R(M, N)$ is irreducible, if and only if $f \in \mathsf{rad}_R(M, N) \setminus \mathsf{rad}_R^2(M, N)$.

First assume f is irreducible, then $f \in \mathsf{rad}_R(M,N)$ by definition. Suppose that $f \in \mathsf{rad}^2_R(M,N)$, then $f = \beta \circ \alpha$, where $\alpha \in \mathsf{rad}_R(M,M')$ and $\beta \in \mathsf{rad}_R(M',N)$. We decompose $M' = \bigoplus_i M_i$ into indecomposable direct summands, and then $\alpha = (M \xrightarrow{\alpha_i} M_i)$ and $\beta = (M_i \xrightarrow{\beta_i} N)$ are also decomposed. Now α is a split monomorphism or β is a split epimorphism. Assume that there exists $\phi = (M_i \xrightarrow{\phi_i} M)$ such that $\phi \circ \alpha = 1$, then $1 = \sum_i \phi_i \circ \alpha_i$. We have shown that every α_i is a radical morphism, so $\phi_i \circ \alpha_i \in \mathsf{rad}_R(M,M) = \mathsf{radEnd}_R(M)$ for each i, and thus $1 \in \mathsf{radEnd}_R(M)$, a contradiction! So α is not a split monomorphism. Similarly we can show that β is not a split epimorphism. Thus $f \in \mathsf{rad}_R(M,N) \setminus \mathsf{rad}^2_R(M,N)$.

Conversely, assume $f \in \mathsf{rad}_R(M,N) \setminus \mathsf{rad}^2_R(M,N)$. By definition f is not an isomorphism. Now that M, N are indecomposable, we easily get f is neither a split monomorphism nor a split epimorphism. If $f = g \circ h$, where $h \in \mathsf{Hom}_R(M,X)$ and $g \in \mathsf{Hom}(X,N)$, we decompose $X = \bigoplus_i X_i$ into indecomposable direct summands, then $h = (M \xrightarrow{h_i} X_i)$ and $g = (X_i \xrightarrow{g_i} N)$ are decomposed as well, such that $f = \sum_i g_i \circ h_i$. Since $f \notin \mathsf{rad}^2_R(M,N)$, there exists i such that either g_i or h_i is invertible. Therefore, h is a split monomorphism or g is a split epimorphism, and thus f is irreducible. □

For indecomposable modules $M, N \in \mathsf{MCM}(R)$, we now define the R-module of irreducible morphisms from M to N as

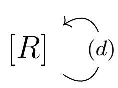
$$\mathsf{Irr}(M,N) := \mathsf{rad}_R(M,N)/\mathsf{rad}^2_R(M,N).$$

Given $f \in \mathsf{rad}_R(M,N)$ and $r \in \mathfrak{m}_R$, clearly $f \circ (M \xrightarrow{r\cdot} M) \in \mathsf{rad}^2_R(M,N)$. Therefore, $\mathsf{Irr}(M,N)$ is a κ_R-vector space. It is always finite-dimensional, since $\mathsf{rad}_R(M,N)$ is finitely generated over R. For brevity, we denote by

$$i(M,N) := \mathsf{dim}_{\kappa_R} \mathsf{Irr}(M,N).$$

Auslander-Reiten quiver. The *Auslander-Reiten quiver* of $\mathsf{MCM}(R)$ is defined as follows. Its vertices are the isomorphism classes of indecomposable MCM modules, and the number of arrows from $[M]$ to $[N]$ equals $i(M,N)$. In addition, if there is an Auslander-Reiten sequence $0 \to \tau(M) \to E \to M \to 0$, we usually connect $[M]$ and $[\tau(M)]$ by a dotted line. When it does not cause ambiguity, we refer to the Auslander-Reiten quiver of $\mathsf{MCM}(R)$ simply by the Auslander-Reiten quiver of R, and it is usually denoted by $\Gamma(R)$.

By definition we can determine the Auslander-Reiten quiver of a regular local ring R. In this case, the only indecomposable MCM R-module is R itself up to isomorphism. By $\mathsf{Hom}_R(R,R) \simeq R$, we get $\mathsf{rad}_R(R,R) \simeq \mathfrak{m}_R$ and $\mathsf{rad}^2_R(R,R) \simeq \mathfrak{m}^2_R$. Therefore, $\mathsf{Irr}(R,R) \simeq \mathfrak{m}_R/\mathfrak{m}^2_R$, and $i(R,R) = \mathsf{dim}_{\kappa_R} \mathfrak{m}_R/\mathfrak{m}^2_R = \mathsf{dim}(R)$. Since $\mathsf{Ext}^1_R(R,R) = 0$, there are no Auslander-Reiten sequences in this case. Therefore, the Auslander-Reiten quiver of R consists of d loops at a single vertex $[R]$, as shown below, where $d := \mathsf{dim}(R)$.



The following results help to determine the number of arrows between two vertices in an Auslander-Reiten quiver.

Let $M, N \in \mathsf{MCM}(R)$ be indecomposable modules.

- Given an Auslander-Reiten sequence ending in M as follows:
$$0 \to \tau(M) \xrightarrow{\iota} E \xrightarrow{\pi} M \to 0,$$
then $i(N, M)$ equals the number of copies of N appeared, up to isomorphism, in the decomposition of E into indecomposable direct summands.
- Dually, given an Auslander-Reiten sequence starting from N as follows:
$$0 \to N \to P \to \tau^{-1}(N) \to 0,$$
then $i(N, M)$ equals the number of copies of M appeared, up to isomorphism, in the decomposition of P into indecomposable direct summands.

By the duality established in the last section, we need only prove the first statement. Consider the R-module $\mathsf{Hom}_R(N, E)/\mathsf{rad}_R(N, E)$. Given $f \in \mathsf{Hom}_R(N, E)$ and $r \in \mathfrak{m}_R$, clearly the composition map $f \circ (N \xrightarrow{r\cdot} N) \in \mathsf{rad}_R(N, E)$. This shows that $\mathsf{Hom}_R(N, E)/\mathsf{rad}_R(N, E)$ is a κ_R-vector space. Since N is indecomposable, we may also view it as the space of split monomorphisms from N into E. Notice that, since N is finitely generated over R, if we decompose $E = \bigoplus_i E_i$ into indecomposable direct summands, then

$$\frac{\mathsf{Hom}_R(N, E)}{\mathsf{rad}_R(N, E)} \simeq \bigoplus_i \frac{\mathsf{Hom}_R(N, E_i)}{\mathsf{rad}_R(N, E_i)}.$$

For any i, if E_i is not isomorphic to N, clearly $\mathsf{Hom}_R(N, E_i)/\mathsf{rad}_R(N, E_i) = 0$. On the other hand, $\mathsf{Hom}_R(N, N)/\mathsf{rad}_R(N, N) = \mathsf{End}_R(N)/\mathsf{radEnd}_R(N)$. Since N is indecomposable, $\mathsf{End}_R(N)/\mathsf{radEnd}_R(N)$ is a division ring containing the algebraically closed field κ_R in the centre, which is meanwhile finite-dimensional over κ_R. If $\exists \alpha \in \frac{\mathsf{End}_R(N)}{\mathsf{radEnd}_R(N)} \setminus \kappa_R$, then the commutative ring $\kappa_R[\alpha] = \bigoplus_{t=0}^{\infty} \kappa_R \alpha^t$ is an intermediate integral domain that is finite-dimensional over the field κ_R, which implies that $\kappa_R[\alpha] \supsetneq \kappa_R$ is a field, a contradiction! Hence

$$\frac{\mathsf{Hom}_R(N, N)}{\mathsf{rad}_R(N, N)} = \frac{\mathsf{End}_R(N)}{\mathsf{radEnd}_R(N)} \simeq \kappa_R.$$

Therefore, if there are n copies of N appeared, up to isomorphism, in the decomposition of E into indecomposable direct summands, then

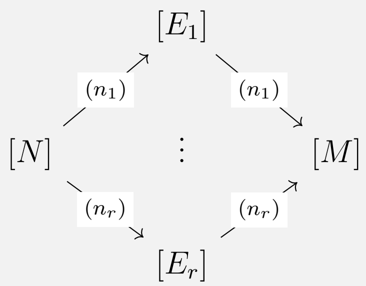
$$\mathsf{dim}_{\kappa_R} \frac{\mathsf{Hom}_R(N, E)}{\mathsf{rad}_R(N, E)} = n.$$

We want to establish an isomorphism between $\frac{\mathsf{Hom}_R(N,E)}{\mathsf{rad}_R(N,E)}$ and $\mathsf{Irr}(N, M)$, two vector spaces over κ_R. So consider the map $\phi : \mathsf{Hom}_R(N, E)/\mathsf{rad}_R(N, E) \xrightarrow{\pi\circ -} \mathsf{Irr}(N, M)$. Since the Auslander-Reiten sequence is almost split, ϕ is a well-defined epimorphism. To show that ϕ is also a monomorphism, let $h \in \mathsf{Hom}_R(N, E)$ be a morphism with $\pi \circ h \in \mathsf{rad}^2(N, M)$. By definition, there exists $\alpha \in \mathsf{rad}_R(N, X)$ and $\beta \in \mathsf{rad}_R(X, M)$ such that $\pi \circ h = \beta \circ \alpha$. Since β is not a split epimorphism, it factors through E. In other words, $\exists \gamma \in \mathsf{Hom}_R(X, E)$ such that $\beta = \pi \circ \gamma$. Thus we get $\pi \circ (h - \gamma \circ \alpha) = 0$, which means $\mathsf{Im}(h - \gamma \circ \alpha) \subset \mathsf{Ker}(\pi) = \tau(M) = \mathsf{Im}(\iota)$. In other words, $\exists g \in \mathsf{Hom}_R(N, \tau(M))$ such that $h - \gamma \circ \alpha = \iota \circ g$. Since α is not a split monomorphism, nor is $\gamma \circ \alpha$, so $\gamma \circ \alpha \in \mathsf{rad}_R(N, E)$. If $\iota \circ g$ had a left inverse, then so did g, but $g : N \to \tau(M)$ is a morphism of indecomposable modules, so g would be an isomorphism, which would mean that ι were a split monomorphism, a contradiction! Hence $\iota \circ g \in \mathsf{rad}_R(N, E)$ as well. Therefore, $h = \gamma \circ \alpha + \iota \circ g \in \mathsf{rad}_R(N, E)$, and we can conclude that ϕ is also a monomorphism. □

We then get the following straightforward corollaries.

Let $0 \to N \to E \to M \to 0$ be an Auslander-Reiten sequence in $\mathsf{MCM}(R)$.

- For any indecomposable $L \in \mathsf{MCM}(R)$, $i(L, M) = i(N, L)$.
- If we decompose $E = \bigoplus_{i=1}^r E_i^{n_i}$ into mutually non-isomorphic indecomposable direct summands E_i's, then the Auslander-Reiten quiver of R is locally in the following form:



In particular, we have shown that if there is an Auslander-Reiten sequence ending in (resp. starting from) M, then there are only finitely many arrows in $\Gamma(R)$ ending in (resp. starting from) $[M]$. This gives an important property of Auslander-Reiten quivers as follows.

If R is an isolated singularity, then its Auslander-Reiten quiver $\Gamma(R)$ is a *locally finite graph*, i.e. each vertex in $\Gamma(R)$ has only finitely many adjacent arrows.

Let $[M]$ be a vertex in $\Gamma(R)$. If M is neither isomorphic to R nor isomorphic to the canonical module ω_R, we have shown that there exists an Auslander-Reiten sequence ending in M, as well as one starting from it. Hence $[M]$ only has finitely many adjacent arrows. It is left to show that the number of arrows ending in $[R]$ and the number of arrows starting from $[\omega_R]$ are also finite, which follows from Yoshino [1990], p. 33, (4.21). □

Dieterich-Yoshino's theorem. Let K be a perfect valued field, R a local analytic K-algebra with maximal ideal $\mathfrak{m}_R$, which assumed to be a CM ring. Denote by $\Gamma := \Gamma(R)$ the Auslander-Reiten quiver for the category of MCM R-modules. An important result by Dieterich [1987] and Yoshino [1987] provides a version of Brauer-Thrall theorem for MCM modules:

> Let Γ° be a finite connected component of Γ. Assume that R has only an isolated singularity, then $\Gamma = \Gamma^\circ$ is a finite graph. In particular, R only has finitely many isomorphism classes of indecomposable MCM modules.

See Yoshino [1990], p.45, (6.2). □

3. Simple singularities

3.1 MCM-finite hypersurfaces

Simple singularity. Let $R = S/(f)$ be a hypersurface, and denote by

$$c(f) = \{I_S < S_S \mid f \in I^2\}.$$

If $c(f)$ is finite, R is a *simple singularity*.

Notice that by definition $I \in c(f)$ entails $I \subset \mathfrak{m}_S$.

In this section, we would like to show that for any analytic hypersurface $R = S/(f)$, if R admits only finitely many isomorphism classes of indecomposable MCM modules, or in short if R is *MCM-finite*, then it is a simple hypersurface singularity. See Buchweitz et al. [1987] for a brief version of the proof.

We fix an analytic hypersurface $R = S/(f)$, where $S \simeq K\{z_0, \cdots, z_n\}$ is a regular analytic algebra over an algebraically closed field K of characteristic 0, and let M be an MCM R-module without free direct summands, (ϕ, ψ) be a reduced matrix factorisation of f corresponding to M, i.e. $M = \mathsf{Coker}(\phi, \psi)$, where $\phi, \psi : S^{(r)} \to S^{(r)}$. Then we get an induced map $\Delta_\phi : S^{(r)} \otimes_S S^{(r)\prime} \to S$, given by

$$\Delta_\phi(f \otimes g) = g(\phi(f)),\ \forall f \in S^{(r)}, \forall g \in S^{(r)\prime}.$$

Clearly, $\mathfrak{I}(\phi) := \mathsf{Im}(\Delta_\phi)$ is an ideal of S. If we fix an S-basis $e_1, \cdots, e_r$ of $S^{(r)}$, then there is a natural dual basis on $S^{(r)\prime} = \mathsf{Hom}_S(S^{(r)}, S)$, denoted by $e'_1, \cdots, e'_r$, satisfying $e'_i(e_j) = \delta_{ij}$, $\forall 1 \leq i, j \leq r$. In this case, ϕ is represented by an $(r \times r)$-matrix over S, say $\phi = (\phi_{ij})_{1 \leq i,j \leq r}$, and we observe that

$$\mathfrak{I}(\phi) = \langle \Delta_\phi(e_i, e_j^*) \mid 1 \leq i, j \leq r\rangle = \langle \phi_{ji} \mid 1 \leq i, j \leq r\rangle. \tag{3.1}$$

Since (ϕ, ψ) is reduced, $\mathsf{Im}(\phi) \subset \mathfrak{m}_S S^{(r)}$. In other words,

$$\langle (\phi_{1i}, \cdots, \phi_{ri})^T \mid 1 \leq i \leq r\rangle \subset \mathfrak{m}_S S^{(r)},$$

from which we can deduce $\mathfrak{I}(\phi) \subset \mathfrak{m}_S$.

Similarly one can define another ideal $\mathfrak{I}(\psi) := \mathsf{Im}(\Delta_\psi)$ of S, and we denote by $\mathfrak{I}(M) := \mathfrak{I}(\phi) + \mathfrak{I}(\psi)$ the ideal of S associated to M.

We first claim that $\mathfrak{I}(M)$ does not depend on the choice of the matrix factorisation. In other words, $\mathfrak{I}(M)$ only depends on the isomorphism class of $M = \mathsf{Coker}(\phi, \psi)$ as an R-module. This is immediate once we write

$$\mathfrak{I}(M) = \mathfrak{I}(\phi) + \mathfrak{I}(\psi) = \langle \phi_{ji}, \psi_{ji} \mid 1 \leq i, j \leq r\rangle, \tag{3.2}$$

since any two matrix factorisations corresponding to M are isomorphic. In fact, if $M = \mathsf{Coker}(\phi, \psi) = \mathsf{Coker}(\phi', \psi')$, by (3.2) we get $\mathfrak{I}(\phi) = \mathfrak{I}(\phi')$ and $\mathfrak{I}(\psi) = \mathfrak{I}(\psi')$.

Another instant observation is

$$\mathfrak{I}(M \oplus N) = \mathfrak{I}(M) + \mathfrak{I}(N).$$

Indeed, if $M = \mathsf{Coker}(\phi, \psi)$, $N = \mathsf{Coker}(\phi', \psi')$, then by Eisenbud's matrix factorisation theorem we get

$$\begin{aligned} M \oplus N &= \mathsf{Coker}((\phi, \psi) \oplus (\phi', \psi')) \\ &= \mathsf{Coker}(\phi \oplus \phi', \psi \oplus \psi') = \mathsf{Coker}(\begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix}), \end{aligned}$$

and the observation follows from (3.2).

As a result, if R is MCM-finite, then the set $\{\mathfrak{I}(M) \mid M \in \mathsf{MCM}(R)\}$ is finite.

For any MCM R-module M, we can take a corresponding matrix factorisation (ϕ, ψ) of f, satisfying $\phi \circ \psi = f 1_{S^{(n)}}$. In other words, $f \in \mathfrak{I}(\phi)\mathfrak{I}(\psi) \subset \mathfrak{I}(M)^2$, and hence

$$\mathfrak{I}(M) \in c(f).$$

Therefore, $\mathfrak{I}$ can be considered as a map from the set of isomorphism classes of MCM R-modules without free direct summands to $c(f)$. We claim that the map $\mathfrak{I}$ is surjective.

Given any $I \in c(f)$, by definition we have $f \in I^2$. Take a generating set $\{x_1, \cdots, x_r\}$ of I, then f has the following expression:

$$f = \sum_{i=1}^{r} x_i y_i, \quad y_i \in I.$$

We define two linear maps on the exterior algebra $\bigwedge^{\bullet} S^{(r)}$. First, we fix a basis $e_1, \cdots, e_r$ for the free S-module $S^{(r)}$, and then we define the maps on the following natural basis of $\bigwedge^t S^{(r)}$ for $1 \leq t \leq r$:

$$e_{i_1} \wedge \cdots \wedge e_{i_t}, \quad 1 \leq i_1 < \cdots < i_t \leq r.$$

The maps, denoted by δ_- and δ_+, are defined by

$$\begin{aligned} \delta_-(e_{i_1} \wedge \cdots \wedge e_{i_t}) &= \sum_{j=1}^{t} (-1)^{j-1} x_{i_j} (e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}), \\ \delta_+(e_{i_1} \wedge \cdots \wedge e_{i_t}) &= \sum_{j=1}^{r} y_j (e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}). \end{aligned}$$

We claim that δ_- and δ_+ are both differential maps on $\bigwedge^\bullet S^{(r)}$, with degree -1 and $+1$ respectively. In fact, $\forall \omega \in \bigwedge^\bullet S^{(r)}$, we have

$$\begin{aligned}
\delta_+^2(\omega) = \delta_+(\sum_{j=1}^r y_j(e_j \wedge \omega)) &= \sum_{j=1}^r y_j \delta_+(e_j \wedge \omega) \\
&= \sum_{j=1}^r y_j \sum_{j'=1}^r y_{j'}(e_{j'} \wedge e_j \wedge \omega) \\
&= \sum_{j'=1}^r y_{j'} \sum_{j=1}^r y_j(e_j \wedge e_{j'} \wedge \omega) = -\delta_+^2(\omega),
\end{aligned}$$

so $\delta_+^2 = 0$. Meanwhile,

$$\begin{aligned}
&\delta_-^2(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^t (-1)^{j-1} x_{i_j} \delta_-(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}) \\
&= \sum_{j=1}^t (-1)^{j-1} x_{i_j} \sum_{1 \le j' < j \le t} (-1)^{j'-1} x_{i_{j'}}(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_{j'}} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}) \\
&+ \sum_{j=1}^t (-1)^{j-1} x_{i_j} \sum_{j < j' \le t} (-1)^{j'} x_{i_{j'}}(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge \hat{e}_{i_{j'}} \wedge \cdots \wedge e_{i_t}) \\
&= 0,
\end{aligned}$$

and thus $\delta_-^2 = 0$.

Now let $\delta := \delta_+ + \delta_-$, then $\delta^2 = \delta_+ \circ \delta_- + \delta_- \circ \delta_+$. We then apply the right side to an arbitrary element in the basis:

$$\begin{aligned}
&\delta_+ \circ \delta_-(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^t (-1)^{j-1} x_{i_j} \delta_+(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}) \\
&= \sum_{j=1}^t (-1)^{j-1} x_{i_j} \sum_{j'=1}^r y_{j'}(e_{j'} \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}) \\
&= \sum_{j=1}^t (-1)^{j-1} x_{i_j} [(-1)^{j-1} y_{i_j}(e_{i_1} \wedge \cdots \wedge e_{i_t}) \\
&+ \sum_{j' \in \mathcal{J}'} y_{j'}(e_{j'} \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t})] \\
&= \sum_{j \in \mathcal{J}} x_j y_j (e_{i_1} \wedge \cdots \wedge e_{i_t}) \\
&+ \sum_{j=1}^t \sum_{j' \in \mathcal{J}'} (-1)^{j-1} x_{i_j} y_{j'}(e_{j'} \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_t}),
\end{aligned}$$

and meanwhile

$$\delta_- \circ \delta_+(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{r} y_j \delta_-(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t})$$
$$= \sum_{j \in \mathcal{J}'} y_j \delta_-(e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t})$$
$$= \sum_{j \in \mathcal{J}'} y_j [x_j(e_{i_1} \wedge \cdots \wedge e_{i_t}) - e_j \wedge \delta_-(e_{i_1} \wedge \cdots \wedge e_{i_t})]$$
$$= \sum_{j \in \mathcal{J}'} x_j y_j (e_{i_1} \wedge \cdots \wedge e_{i_t})$$
$$- \sum_{j \in \mathcal{J}'} \sum_{j'=1}^{t} (-1)^{j'-1} x_{i_{j'}} y_j (e_j \wedge e_{i_1} \wedge \cdots \wedge \hat{e}_{i_{j'}} \wedge \cdots \wedge e_{i_t}),$$

where $\mathcal{J} = \{i_1, \cdots, i_t\}$ and $\mathcal{J}' = \{1, \cdots, r\} \setminus \mathcal{J}$. Therefore,

$$\delta^2(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{r} x_j y_j (e_{i_1} \wedge \cdots \wedge e_{i_t}),$$

and hence

$$\delta^2 = f \ 1_{\bigwedge^\bullet S^{(r)}}.$$

In other words, (δ, δ) gives a matrix factorisation of f, and moreover by (3.1) we get

$$\mathfrak{I}(\delta) = \langle x_1, \cdots, x_r, y_1, \cdots, y_r \rangle = I.$$

Since $I \in c(f)$, $I \subset \mathfrak{m}_S$, and hence $\mathsf{Im}(\delta) \subset \mathfrak{m}_S \bigwedge^\bullet S^{(r)}$, so (δ, δ) is reduced and corresponds to an MCM R-module $M = \mathsf{Coker}(\delta, \delta)$ without free direct summands according to Eisenbud's matrix factorisation theorem. Obviously, $\mathfrak{I}(M) = \mathfrak{I}(\delta) = I$. So we have proven that $\mathsf{Im}(\mathfrak{I}) = c(f)$. □

An easy corollary of the surjectivity of $\mathfrak{I}$ is:

> If a hypersurface $R = S/(f)$ is MCM-finite, then it is a simple singularity.

In fact, if R is MCM-finite, then there are only finitely many isomorphism classes of MCM R-modules without free direct summands, so the image of $\mathfrak{I}$ should also be a finite set, i.e. $c(f)$ is finite. □

Herzog's theorem. In the remaining part of the section, we would like to demonstrate that the hypersurface assumption is superfluous. We shall see that, according to Herzog [1978], it suffices to assume that R is a Gorenstein ring.

We begin by showing a "quasi-converse" of Eisenbud's 2-periodicity:

> Let $R = S/I$ be a Gorenstein ring, where S is a regular local ring and $I \subset \mathfrak{m}_S^2$ is an ideal of S. If any $M \in \mathsf{MCM}(R)$ admits an almost periodic free resolution, then I is a principal ideal.

As a remark, a free resolution of M of the form

$$\cdots \to F_{n+1} \xrightarrow{\phi_{n+1}} F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

is *almost periodic*, if $\exists n, h \geq 1$, such that $\phi_{\mu+h} = \phi_\mu$, $\forall \mu \geq n$.

According Tate [1957], Thm. 6, to show that I is a principal ideal, it suffices to prove that there is an upper bound for the Betti numbers of κ_R. Recall that given a minimal free resolution of κ_R of the form

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 = R \to \kappa_R \to 0,$$

the n-th *Betti number* of κ_R is $\beta_n(\kappa_R) = \mathsf{rank}(F_n)$.

If we let $M := \mathsf{Ker}(F_{d-1} \to F_{d-2})$, where $d = \mathsf{dim}(R)$, then such a free resolution can be cut short:

$$0 \to M \to F_{d-1} \to F_{d-2} \to \cdots \to F_0 = R \to \kappa_R \to 0.$$

Then $M \in \mathsf{MCM}(R)$. By our assumption, every indecomposable direct summands of M admits an almost periodic free resolution, so the Betti numbers of every indecomposable direct summand of M have an upper bound, hence it follows that $\{\beta_n(\kappa_R) \mid 1 \leq n < \infty\}$ has an upper bound as well. □

We can now prove the following result from Herzog [1978], Satz 1.2:

Let $R = S/I$ be a Gorenstein ring, where S is a regular local ring and $I \subset \mathfrak{m}_S^2$ is an ideal of S. If R is MCM-finite, then I is principal, hence R is a hypersurface.

We need only show that any $M \in \mathsf{MCM}(R)$ admits a free resolution that is almost periodic. Without loss of generality we assume that M is indecomposable. The minimal free cover of M gives an exact sequence

$$0 \to F_1 \xrightarrow{\iota} F_0 \xrightarrow{\pi} M \to 0,$$

and by definition (Eisenbud [1995], p. 472), $\iota \otimes 1 : F_1 \otimes \kappa_R \to F_0 \otimes \kappa_R$ is the 0 map, so $\pi \otimes 1 : F_0 \otimes \kappa_R \to M \otimes \kappa_R$ is an isomorphism. We then claim that $F_1 \in \mathsf{MCM}(R)$ is also indecomposable.

Suppose to the contrary that $F_1 = N \oplus N'$ for $N, N' \neq 0$, and let $\iota = (\iota_1, \iota_2) : N \oplus N' \to F_0$. Since R is Gorenstein, we can take the dual of the sequence by the canonical module $\omega_R \simeq R$, and we obtain another exact sequence:

$$0 \to M^* \to F_0^* \xrightarrow{(\iota_1^*, \iota_2^*)^t} N^* \oplus N'^* \to 0,$$

where $(-)^* := \mathsf{Hom}_R(-, \omega_R)$. Since R is Gorenstein, F_0^* becomes a free cover of $N^* \oplus N'^*$. If neither N nor N' is free, then M^* contains $\mathsf{syz}^1 N^* \oplus \mathsf{syz}^1 N'^*$ as a direct summand. But $M^{**} \simeq M$, so M can not be indecomposable. So assume without loss of generality that N is free, then ι_1^* must be a split epimorphism and hence ι_1 is a split monomorphism. In particular, $\iota_1 \otimes 1 : N \otimes \kappa_R \to F_0 \otimes \kappa_R$ is a monomorphism. Since $\pi \circ \iota_1 = 0$, $\pi \otimes 1 : F_0 \otimes \kappa_R \to M \otimes \kappa_R$ cannot be an isomorphism. We have thus proven that N is indecomposable.

Now define an endomorphism $\mathfrak{F}$ on the finite set of isomorphism classes of indecomposable MCM R-modules, by setting $\mathfrak{F}(M) = F_1$. The finiteness of the domain of $\mathfrak{F}$ implies that there are positive integers n, h such that $\mathfrak{F}^{\mu+h}(M) = \mathfrak{F}^{\mu}(M)$ whenever $\mu \geq n$. In other words, the free resolutions of M are almost periodic. This proves Herzog's theorem. □

As an easy corollary, we get:

Let $R = S/I$ be a Gorenstein ring, where S is a regular local ring and $I \subset \mathfrak{m}_S^2$ an ideal of S. If R is MCM-finite, then it is a simple singularity.

3.2 Knörrer's periodicity

We intend to show the converse of Buchweitz-Gruel-Schreyer's theorem in Section 3.1, i.e. any simple singularity is MCM-finite. The key step is the periodicity statement proved in Knörrer [1987].

Let K be an algebraically closed valued field of characteristic 0, and S a regular analytic K-algebra. As all of the following statements and demonstrations work for any such S, without loss of generality we may assume $S := K\{z_0, \cdots, z_n\}$ to be the power series ring over K in $n+1$ variables[1]. Any simple singularity $R = S/(f)$ can be classified into the following types:

Let $S = K\{z_0, \cdots, z_n\}$, where K is an algebraically closed valued field of characteristic 0. If $R = S/(f)$ is a simple singularity, then after a suitable change of variables, f is equal to one of the following polynomials:

$$\begin{aligned}
A_k &: f = z_0^{k+1} + z_1^2 + \cdots + z_n^2, \;\; k \geq 1; \\
D_k &: f = z_0^{k-1} + z_0 z_1^2 + z_2^2 + \cdots + z_n^2, \;\; k \geq 4; \\
E_6 &: f = z_0^3 + z_1^4 + z_2^2 + \cdots + z_n^2; \\
E_7 &: f = z_0^3 + z_0 z_1^3 + z_2^2 + \cdots + z_n^2; \\
E_8 &: f = z_0^3 + z_1^5 + z_2^2 + \cdots + z_n^2.
\end{aligned}$$

See Yoshino [1990], (8.8). □

We then define $S_1 := S\{y\}$ and $R_1 := S_1/(f+y^2)$, where y is a new variable. Note that R_1 has the same type (A, D, E) as R, and $\mathsf{dim}(R_1) = \mathsf{dim}(R)+1 = n+1$. Also notice that $R = R_1/(y)$, so any R-module can be regarded as an R_1-module.

A famous result regarding MCM-finite CM local rings is:

If a Henselian CM local ring R is MCM-finite, then it is an isolated singularity.

[1]Recall that when the valuation on K is taken trivial, $S = K[[z_0, \cdots, z_n]]$.

See Auslander and Unger [2006], p. 234. $\square$

Since we are only interested in the MCM-finiteness of the ring R, we adopt the assumption that R is an isolated singularity.

First we need a lemma from Knörrer [1987], (1.3):

> If $X := (\phi : S^r \rightleftarrows S^r : \psi) \in \mathsf{MF}_S(f)$ and $TX \simeq X$, then $X \simeq (\phi_0, \phi_0) \in \mathsf{MF}_S(f)$ for some $\phi_0 : S^r \to S^r$. Here $T : \mathsf{MF}_S(f) \to \mathsf{MF}_S(f)$ denotes the involutive transformation $(\phi, \psi) \mapsto (\psi, \phi)$.

We introduce the idea of proof from Knörrer [1987], which will be useful later. Without loss of generality we assume that X is indecomposable, and let $(\alpha, \beta) : (\phi, \psi) \to (\psi, \phi)$ be an isomorphism of matrix factorisations. Then $(\beta \circ \alpha, \alpha \circ \beta)$ is an automorphism of X. As X is indecomposable, according to Swan [1968] (2.19) and Eisenbud's matrix factorisation theorem, we have

$$\mathsf{End}(X)/\mathsf{rad}(\mathsf{End}(X)) \simeq K.$$

So we may assume that $(\beta \circ \alpha, \alpha \circ \beta) = (1, 1) + (\rho_1, \rho_2)$, where $(\rho_1, \rho_2) \in \mathsf{rad}(\mathsf{End}(X))$. Since $\beta \circ \alpha = 1 + \rho_1$ and $\alpha \circ \beta = 1 + \rho_2$, we get

$$\alpha + \rho_2 \circ \alpha = \alpha \circ \beta \circ \alpha = \alpha + \alpha \circ \rho_1,$$

i.e. $\alpha \circ \rho_1 = \rho_2 \circ \alpha$. Similarly we get $\beta \circ \rho_2 = \rho_1 \circ \beta$.

Now we choose a (convergent) power series $P(x)$ for $(1 + x)^{-\frac{1}{2}}$ [2] and let

$$\begin{aligned} \alpha' &:= \alpha \circ (1 + \rho_1)^{-\frac{1}{2}} = (1 + \rho_2)^{-\frac{1}{2}} \circ \alpha, \\ \beta' &:= \beta \circ (1 + \rho_2)^{-\frac{1}{2}} = (1 + \rho_1)^{-\frac{1}{2}} \circ \beta. \end{aligned}$$

It is easy to check that (α', β') is also a morphism from X to TX in $\mathsf{MF}_S(f)$, and we have

$$\begin{aligned} \alpha' \circ \beta' &= (1 + \rho_2)^{-\frac{1}{2}} \circ \alpha \circ \beta \circ (1 + \rho_2)^{-\frac{1}{2}} \\ &= (1 + \rho_2)^{-\frac{1}{2}} \circ (1 + \rho_2) \circ (1 + \rho_2)^{-\frac{1}{2}} = 1, \end{aligned}$$

and similarly $\beta' \circ \alpha' = 1$. So (α', β') is an isomorphism since $r = \mathsf{rank}(\alpha' \circ \beta') \leq \min\{\mathsf{rank}(\alpha'), \mathsf{rank}(\beta')\}$. Thus $\exists \gamma \in \mathsf{Aut}_S(S^r)$ with $\gamma^2 = \alpha'$. See for example Gantmacher [1989], XI. Thm. 3. For any such γ, we let $\phi_0 := \gamma \circ \psi \circ \gamma$. Now that

$$\gamma \circ \psi \circ \gamma = \gamma^{-1} \circ \alpha' \circ \psi \circ \gamma = \gamma^{-1} \circ \phi \circ \beta' \circ \alpha' \circ \gamma^{-1} = \gamma^{-1} \circ \phi \circ \gamma^{-1},$$

we can show that ϕ_0 is the demanded morphism. In fact,

$$\phi_0^2 = \gamma \circ \psi \circ \gamma \circ \gamma^{-1} \circ \phi \circ \gamma^{-1} = f \cdot 1,$$

and (γ, γ^{-1}) gives an isomorphism from $X = (\phi, \psi)$ to (ϕ_0, ϕ_0). $\square$

[2] For example, near $x = 0$ the binomial series $P(x) = \sum_{t=0}^{\infty} \binom{-\frac{1}{2}}{t} x^t = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots = (1 + x)^{-\frac{1}{2}}$ has a radius of convergence 1.

The group $\mathbb{Z}/2\mathbb{Z}$ acts on R_1 by $\sigma : (z, y) \mapsto (z, -y)$, and we let $R_1[\sigma] := R_1 \oplus R_1 \cdot \sigma$ be the twisted group ring, where the multiplication is given by:

$$(r_1 + r_2 \cdot \sigma)\ (r_1' + r_2' \cdot \sigma) = r_1 r_1' + r_2 \sigma(r_2')\ +\ (r_1 r_2' + r_2 \sigma(r_1')) \cdot \sigma.$$

An $R_1[\sigma]$-module M is an R_1-module with an action of the involution σ, such that

$$\sigma(rm) = \sigma(r)\sigma(m),\ \forall r \in R_1, \forall m \in M.$$

This simply follows from the axiom

$$[(r_1 + r_2 \cdot \sigma)(r_1' + r_2' \cdot \sigma)] \cdot m = (r_1 + r_2 \cdot \sigma)[(r_1' + r_2' \cdot \sigma) \cdot m].$$

We denote by $\mathsf{MCM}_\sigma(R_1)$ the category of $R_1[\sigma]$-modules that are MCM R_1-modules.

Actually, R_1 itself admits 2 actions of σ: $\phi \mapsto \phi\sigma$ and $\phi \mapsto -\phi\sigma$, and we denote by R_1^+ and R_1^- the corresponding $R_1[\sigma]$-modules, respectively. In other words, $R_1^+ = R_1 \cdot \frac{1+\sigma}{2} \subset R_1[\sigma]$, and $R_1^- = R_1 \cdot \frac{1-\sigma}{2} \subset R_1[\sigma]$. Then clearly $R_1[\sigma] = R_1^+ \oplus R_1^-$ as $R_1[\sigma]$-module. Moreover, R_1^+ and R_1^- are the only 2 indecomposable projective MCM $R_1[\sigma]$-modules.

For $M \in \mathsf{MCM}_\sigma(R_1)$, we denote by M^σ (resp. $\overline{M^\sigma}$) the set of σ-invariant (resp. σ-antiinvariant) elements of M. Clearly M^σ and $\overline{M^\sigma}$ are both MCM modules over R_1^σ.

Meanwhile, take any $\bar{\phi} \in R_1^\sigma$. By definition this means $\phi(z, y) - \phi(z, -y) \in (f(z) + y^2)$, so $\phi = \phi(z, y^2)$, and thus $R_1^\sigma \simeq S\{y^2\}/(f + y^2) \simeq S\{-f\} = K\{z, -f(z)\} \simeq K\{z\} = S$. Therefore, M^σ and $\overline{M^\sigma}$ are free S-modules of equal rank.

Based on these discussions, we can define a functor $\mathfrak{A} : \mathsf{MCM}_\sigma(R_1) \to \mathsf{MF}_S(f)$. Given $M \in \mathsf{MCM}_\sigma(R_1)$, we can define two S-linear maps

$$\phi : M^\sigma \xrightarrow{y\cdot} \overline{M^\sigma}, \quad \psi : \overline{M^\sigma} \xrightarrow{-y\cdot} M^\sigma.$$

Notice that $\phi \circ \psi = \psi \circ \phi = -y^2 1 = f1$, so $(\phi, \psi) \in \mathsf{MF}_S(f)$ is a matrix factorisation of f, which we shall denote by $\mathfrak{A}(M)$. The construction is clearly functorial. Moreover, we claim that:

$\mathfrak{A} : \mathsf{MCM}_\sigma(R_1) \to \mathsf{MF}_S(f)$ is an equivalence of categories.

We show that by explicitly constructing a quasi-inverse to $\mathfrak{A}$. Let $(\phi : M_1 \to M_2, \psi : M_2 \to M_1)$ be a matrix factorisation of f, and $M := M_1 \oplus M_2$ be considered as S-module. Then M has a structure of an $R_1[\sigma]$-module, given by

$$y(m_1, m_2) = (-\psi(m_2), \phi(m_1)),$$
$$\sigma(m_1, m_2) = (m_1, -m_2), \quad \forall (m_1, m_2) \in M_1 \oplus M_2.$$

As before, to check that this indeed defines an $R_1[\sigma]$-module structure, it suffices to check

$$\sigma(y)\cdot\sigma(m_1,m_2)=\sigma(y(m_1,m_2)).$$

Moreover, since M is free over S, $\mathsf{proj.dim}_S(M)=0$. By Auslander-Buchsbaum formula, $\mathsf{depth}(M)=\mathsf{depth}(R_1)=\mathsf{dim}(R_1)$, and hence M is an MCM R_1-module, and $M_1\in\mathsf{MCM}_\sigma(R_1)$. This clearly defines a functor $\mathsf{MF}_S(f)\to\mathsf{MCM}_\sigma(R_1)$.

We now show that it is a quasi-inverse to $\mathfrak{A}$. Take $M\in\mathsf{MCM}_\sigma(R_1)$. $\mathfrak{A}(M)$ gives a matrix factorisation $M^\sigma\rightleftarrows\overline{M^\sigma}$, and then $M^\sigma\oplus\overline{M^\sigma}\simeq M$. Conversely take matrix factorisation $M_1\rightleftarrows M_2$ of f. This functor gives $M:=M_1\oplus M_2\in\mathsf{MCM}_\sigma(R_1)$, and then $M^\sigma\simeq M_1$, $\overline{M^\sigma}\simeq M_2$ by the multiplication rule given above. So after applying $\mathfrak{A}$, the matrix factorisation is isomorphic to the original one. From now on we shall denote by $\mathfrak{A}^{-1}$ this quasi-inverse functor. □

We make some remarks on the property of $\mathfrak{A}$ that will be useful soon. First we have $\mathfrak{A}(R_1^+)=(1,f)$, and $\forall M\in\mathsf{MCM}_\sigma(R_1)$, $\mathfrak{A}(M\otimes_S R_1^-)\simeq T\circ\mathfrak{A}(M)$, where $T:\mathsf{MF}_S(f)\to\mathsf{MF}_S(f)$ is the involutive transformation $(\phi,\psi)\mapsto(\psi,\phi)$. In fact, this follows from $(M\otimes_S R_1^-)^\sigma\simeq\overline{M^\sigma}$ and $\overline{(M\otimes_S R_1^-)^\sigma}\simeq M^\sigma$.

Moreover, we take a closer look at the composition $\mathsf{Coker}\circ\mathfrak{A}:\mathsf{MCM}_\sigma(R_1)\to\mathsf{MCM}(R)$. Given $M\in\mathsf{MCM}_\sigma(R_1)$,

$$\begin{aligned}\mathsf{Coker}\circ\mathfrak{A}(M)&=\mathsf{Coker}(\phi:M^\sigma\to\overline{M^\sigma})=\overline{M^\sigma}/yM^\sigma\\&\simeq\frac{\overline{M^\sigma}\oplus M^\sigma}{yM^\sigma\oplus M^\sigma}\simeq M/R_1M^\sigma.\end{aligned}$$

As a result, for any $M\in\mathsf{MCM}_\sigma(R_1)$, since $M\simeq M\otimes R_1^+$, we have

$$\mathsf{Coker}\circ\mathfrak{A}(M)\oplus\mathsf{Coker}\circ\mathfrak{A}(M\otimes R_1^-)\simeq\overline{M^\sigma}/yM^\sigma\oplus M^\sigma/y\overline{M^\sigma}\simeq M/yM,\quad(3.3)$$

where the 2nd isomorphism is given by $([m^-],[m^+])\mapsto[m^-+m^+]$.

Note that according to Auslander and Unger [2006] p. 200, cf. Section 2.1, since R is an isolated singularity, $\mathsf{MCM}(R)$ and $\mathsf{MCM}_\sigma(R_1)$ both admit almost split sequences. As a result, the Auslander-Reiten quiver of $\mathsf{MCM}(R)$ is isomorphic to the full subquiver of the Auslander-Reiten quiver of $\mathsf{MCM}_\sigma(R_1)$ obtained by deleting the vertex corresponding to R_1^+.

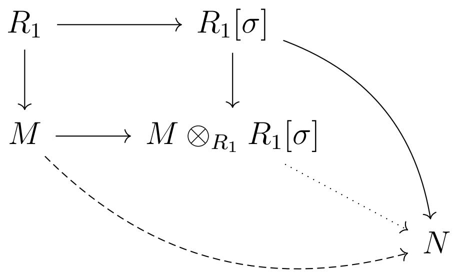
Now we wish to compare the modules in $\mathsf{MCM}_\sigma(R_1)$ and $\mathsf{MCM}(R_1)$. Denote by $\mathfrak{U}:\mathsf{MCM}_\sigma(R_1)\to\mathsf{MCM}(R_1)$ the forgetful functor, and we claim that $\mathfrak{U}$ admits a left adjoint

$$\begin{aligned}\mathfrak{E}:\mathsf{MCM}(R_1)&\to\mathsf{MCM}_\sigma(R_1),\\M&\mapsto M\otimes_{R_1}R_1[\sigma].\end{aligned}$$

In other words, we would like to show that there is a natural isomorphism

$$\mathsf{Hom}_{R_1[\sigma]}(M\otimes_{R_1}R_1[\sigma],N)\simeq\mathsf{Hom}_{R_1}(M,N),$$

on $M\in\mathsf{MCM}(R_1)$ and $N\in\mathsf{MCM}_\sigma(R_1)$, which is obvious, since $M\otimes_{R_1}R_1[\sigma]$ is characterised by the universal property of a pushout, as shown below:



Given any $M \in \mathsf{MCM}(R_1)$, notice that $\mathsf{depth}(M/yM) = \mathsf{depth}(M) - 1$, so $M/yM \in \mathsf{MCM}(R)$. Thus we can also define a restriction functor $\mathfrak{r} : \mathsf{MCM}(R_1) \to \mathsf{MCM}(R)$ by $M \mapsto M/yM$.

On the other hand, we define similar functors on the level of matrix factorisations. Given any $(\Phi : F_1 \rightleftarrows F_2 : \Psi) \in \mathsf{MF}_{S_1}(f + y^2)$, we have morphisms $\phi : F_1/yF_1 \rightleftarrows F_2/yF_2 : \psi$ induced by Φ and Ψ. It is easy to see that this gives a restriction functor $\mathfrak{R} : \mathsf{MF}(f + y^2) \to \mathsf{MF}(f)$.

Conversely, for any $(\phi : F_1 \rightleftarrows F_2 : \psi) \in \mathsf{MF}_S(f)$, we let $\mathfrak{G}(\phi, \psi) := (\Phi, \Phi)$, where

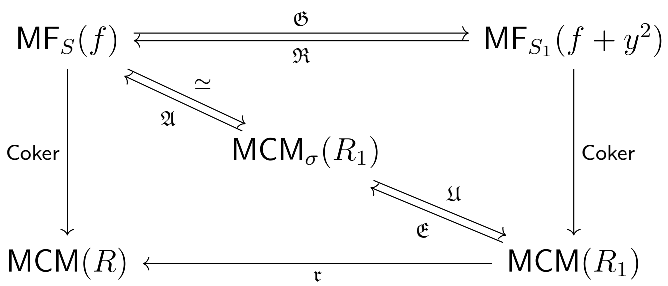
$$\Phi = \begin{pmatrix} y \cdot 1 & \psi \\ \phi & -y \cdot 1 \end{pmatrix} : (F_1 \oplus F_2) \otimes_S S_1 \to (F_1 \oplus F_2) \otimes_S S_1.$$

Since $\Phi^2 = (y^2 + f) \cdot 1$, $\mathfrak{G}(\phi, \psi) \in \mathsf{MF}_{S_1}(f + y^2)$. In the mean time, we associate $\mathfrak{G}(\alpha, \beta) := (\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix})$ to any morphism $(\alpha, \beta) \in \mathsf{MF}_S(f)$. We check that

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \Phi = \Phi \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

and this defines a functor $\mathfrak{G} : \mathsf{MF}_S(f) \to \mathsf{MF}_{S_1}(f + y^2)$.

To sum up, we have the following diagram:



To study the relations between these functors, first observe that

$$\mathfrak{r} \circ \mathsf{Coker} \simeq \mathsf{Coker} \circ \mathfrak{R} : \mathsf{MF}_{S_1}(f + y^2) \to \mathsf{MCM}(R). \tag{3.4}$$

Then we claim that

$$\mathsf{Coker} \circ \mathfrak{A} \circ \mathfrak{E} \simeq \mathfrak{r} : \mathsf{MCM}(R_1) \to \mathsf{MCM}(R). \tag{3.5}$$

In fact, by (3.3), $\forall M \in \mathsf{MCM}(R_1)$,

$$\begin{aligned}\mathsf{Coker} \circ \mathfrak{A} \circ \mathfrak{E}(M) &\simeq \mathsf{Coker} \circ \mathfrak{A}(M \otimes_{R_1} R_1[\sigma]) \\ &\simeq \mathsf{Coker} \circ \mathfrak{A}(M \otimes_{R_1} R_1^+) \oplus \mathsf{Coker} \circ \mathfrak{A}(M \otimes_{R_1} R_1^-) \\ &\simeq M/yM \simeq \mathfrak{r}(M). \quad \square\end{aligned}$$

Another relation we are interested in is between $\mathsf{Coker} \circ \mathfrak{G}$ and $\mathfrak{U} \circ \mathfrak{A}^{-1}$ from $\mathsf{MF}_S(f)$ to $\mathsf{MCM}(R_1)$. Let $(\phi : S^r \rightleftarrows S^r : \psi) \in \mathsf{MF}_S(f)$, then the S-module $\mathfrak{U}\circ\mathfrak{A}^{-1}(\phi, \psi) = S^r \oplus S^r$ is equipped with a structure of R_1-module given by

$$y(m_1, m_2) = (-\psi(m_2), \phi(m_1)), \quad \forall (m_1, m_2) \in S^r \oplus S^r.$$

On the other hand, $\mathsf{Coker}\circ\mathfrak{G}(\phi, \psi) = (S_1^r \oplus S_1^r)/\mathsf{Im}(\Phi)$, where $\Phi = \begin{pmatrix} y\cdot 1 & \psi \\ \phi & -y\cdot 1\end{pmatrix}$. Since $S_1 \simeq S \oplus yS$, $\mathsf{Im}(\Phi)$ is generated by 4 kinds of elements $\Phi(s \cdot e_{ij})$, $i, j = 1, 2$, where e_{ij} is the standard matrix with a single entry 1 at the position (i, j) and s traverses S^r. Simple calculation gives

$$\mathsf{Coker} \circ \mathfrak{G}(\phi, \psi) \simeq S^r \oplus S^r \oplus S^r \oplus S^r/I,$$
$$I = \langle (0, s, \phi(s), 0), (-fs, 0, 0, \phi(s)), (\psi(s), 0, 0, -s), (0, \psi(s), fs, 0) \mid s \in S^r\rangle.$$

We similarly define the R_1-module structure on the R-module $\mathsf{Coker} \circ \mathfrak{G}(\phi, \psi)$ by

$$y(s_1, s_2, s_3, s_4) = (-fs_2, s_1, -fs_4, s_3), \quad \forall (s_1, s_2, s_3, s_4) \in S^r \oplus S^r \oplus S^r \oplus S^r.$$

Now we claim that

$$\mathsf{Coker} \circ \mathfrak{G} \simeq \mathfrak{U} \circ \mathfrak{A}^{-1} : \mathsf{MF}_S(f) \to \mathsf{MCM}(R_1). \tag{3.6}$$

This is yielded by the R_1-isomorphism $\alpha : \mathsf{Coker} \circ \mathfrak{G}(\phi, \psi) \to \mathfrak{U} \circ \mathfrak{A}^{-1}(\phi, \psi)$ given by $(s_1, s_2, s_3, s_4) \mapsto (s_3 - \psi(s_2), s_1 + \phi(s_4))$. We can simply check

$$\alpha(y(s_1, s_2, s_3, s_4)) \subset y\alpha(s_1, s_2, s_3, s_4). \quad \square$$

The involution σ acts on $\mathsf{MF}_{S_1}(f + y^2)$ by $(\phi, \psi) \mapsto (\phi \circ \sigma, \psi \circ \sigma)$, and clearly for all $(\phi, \psi) \in \mathsf{MF}_{S_1}(f + y^2)$ we have $\sigma^*(\mathsf{Coker}(\phi, \psi)) \circ \sigma \simeq \mathsf{Coker} \circ \sigma(\phi, \psi)$, where $\sigma^* : \mathsf{MCM}(R_1) \to \mathsf{MCM}(R_1)$ denotes the pullback by σ, characterised by the commutative diagram:

$$\begin{array}{ccc} \mathsf{MF}_{S_1}(f + y^2) & \xrightarrow{\sigma} & \mathsf{MF}_{S_1}(f + y^2) \\ \downarrow{\scriptstyle \mathsf{Coker}} & & \downarrow{\scriptstyle \mathsf{Coker}} \\ \mathsf{MCM}(R_1) & \xrightarrow{\sigma^*} & \mathsf{MCM}(R_1) \end{array}$$

Given $M = \mathsf{Coker}(\phi, \psi) \in \mathsf{MCM}(R_1)$, $\sigma^*(M) = \mathsf{Coker}(\phi\sigma, \psi\sigma) = M \otimes_{R_1} R_1\sigma$. Also notice that T acts on $\mathsf{MF}_S(f)$ and $\mathsf{MF}_{S_1}(f + y^2)$ as well, and $T \circ \mathfrak{R} = \mathfrak{R} \circ T$. Moreover we have

$$\mathfrak{G} \circ T \simeq T \circ \mathfrak{G} = \mathfrak{G} : \mathsf{MF}_S(f) \to \mathsf{MF}_{S^1}(f + y^2).$$

In fact, for any $(\phi, \psi) \in \mathsf{MF}_S(f)$, an isomorphism between $\mathfrak{G} \circ T(\phi, \psi)$ and $\mathfrak{G}(\phi, \psi)$ is given by

$$(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}). \quad \square$$

In fact, the adjunction $(\mathfrak{E}, \mathfrak{U})$ worths a closer look. We claim that:

$$\mathfrak{E} \circ \mathfrak{U} \simeq 1 \oplus (- \otimes R_1^-) : \qquad \mathsf{MCM}_\sigma(R_1) \to \mathsf{MCM}_\sigma(R_1); \tag{3.7}$$

$$\mathfrak{U} \circ \mathfrak{E} \simeq 1 \oplus \sigma^* : \qquad \mathsf{MCM}(R_1) \to \mathsf{MCM}(R_1). \tag{3.8}$$

To show (3.7), first notice that as R_1-module, for any $M \in \mathsf{MCM}_\sigma(R_1)$,

$$\begin{aligned} \mathfrak{E} \circ \mathfrak{U}(M) &= M \otimes_{R_1} R_1[\sigma] \\ &\simeq M \otimes_{R_1} R_1 \ \oplus \ M \otimes_{R_1} R_1 \cdot \sigma \\ &\simeq M \oplus \sigma^*(M), \end{aligned}$$

and the action of σ is given by $(m_1, m_2) \mapsto (m_2, m_1)$. Then the map

$$\begin{aligned} M \oplus (M \otimes R_1^-) &\to M \oplus \sigma^*(M) \\ (x, y) &\mapsto (x + y, \sigma(x) - \sigma(y)) \end{aligned}$$

defines an $R_1[\sigma]$-isomorphism. The construction is clearly functorial.

For (3.8), it is even simpler. Just notice that

$$\begin{aligned} \mathfrak{U} \circ \mathfrak{E} &= \mathfrak{U} \circ (- \otimes_{R_1} R_1[\sigma]) \\ &\simeq \mathfrak{U} \circ 1 \ \oplus \ \mathfrak{U} \circ (- \otimes_{R_1} R_1 \cdot \sigma) \\ &\simeq 1 \oplus \sigma^*. \quad \square \end{aligned}$$

Similarly, on the level of matrix factorisations, we have

$$\mathfrak{R} \circ \mathfrak{G} \simeq 1 \oplus T : \mathsf{MF}_S(f) \to \mathsf{MF}_S(f). \tag{3.9}$$

Indeed, for any $(\phi, \psi) \in \mathsf{MF}_S(f)$, $\mathfrak{R} \circ \mathfrak{G}(\phi, \psi) = (\begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix})$, and an isomorphism to $(\phi, \psi) \oplus (\psi, \phi)$ is given by $(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. $\square$

Moreover, for any non-trivial indecomposable $(\Phi, \Psi) \in \mathsf{MF}_{S_1}(f + y^2)$,

$$\mathfrak{G} \circ \mathfrak{R}(\Phi, \Psi) \simeq (\Phi, \Psi) \oplus \sigma(\Phi, \Psi). \tag{3.10}$$

In fact, by Eisenbud's matrix factorisation theorem, it suffices to check

$$\mathsf{Coker} \circ \mathfrak{G} \circ \mathfrak{R}(\Phi, \Psi) \simeq \mathsf{Coker}(\Phi, \Psi) \oplus \mathsf{Coker}(\sigma(\Phi, \Psi)).$$

On one hand, $\mathsf{Coker} \circ \mathfrak{G} \circ \mathfrak{R}(\Phi, \Psi) \simeq \mathfrak{U} \circ \mathfrak{A}^{-1} \circ \mathfrak{R}(\Phi, \Psi)$ by (3.6); on the other hand,

$$\begin{aligned} \mathsf{Coker}(\Phi, \Psi) \oplus \mathsf{Coker}(\sigma(\Phi, \Psi)) &\simeq \mathsf{Coker}(\sigma(\Phi, \Psi)) \oplus \sigma^*(\mathsf{Coker}(\Phi, \Psi)) \\ &\simeq (1 \oplus \sigma^*) \circ \mathsf{Coker}(\Phi, \Psi) \\ &\simeq \mathfrak{U} \circ \mathfrak{E} \circ \mathsf{Coker}(\Phi, \Psi) \end{aligned}$$

by (3.8). Now it suffices to show that

$$\mathfrak{A} \circ \mathfrak{E} \circ \mathsf{Coker}(\Phi, \Psi) \simeq \mathfrak{R}(\Phi, \Psi),$$

and by Eisenbud's matrix factorisation theorem again it suffices to check

$$\mathsf{Coker} \circ \mathfrak{A} \circ \mathfrak{E} \circ \mathsf{Coker}(\Phi, \Psi) \simeq \mathsf{Coker} \circ \mathfrak{R}(\Phi, \Psi).$$

But we know from (3.5) and (3.4) that

$$\begin{aligned}\mathsf{Coker} \circ \mathfrak{A} \circ \mathfrak{E} \circ \mathsf{Coker}(\Phi, \Psi) &\simeq \mathfrak{r} \circ \mathsf{Coker}(\Phi, \Psi) \\ &\simeq \mathsf{Coker} \circ \mathfrak{R}(\Phi, \Psi),\end{aligned}$$

which completes the proof. □

Let $Y := (\Phi, \Psi) \in \mathsf{MF}_{S_1}(f + y^2)$. We have:
- $\exists X \in \mathsf{MF}_S(f)$ with $Y \simeq \mathfrak{G}(X)$, if and only if $\sigma(Y) \simeq Y$;
- if Y is non-trivial and indecomposable, then $TY \simeq \sigma(Y)$.

For the first point, clearly $\sigma(\mathfrak{G}(X)) \simeq \mathfrak{G}(X)$ for any $X \in \mathsf{MF}_S(f)$. Indeed, $(\sigma, 1) : \mathfrak{G}(X) \to \sigma \circ \mathfrak{G}(X)$ is an isomorphism. Conversely suppose $\sigma(Y) \simeq Y$. Without loss of generality we may assume that Y is indecomposable. By Eisenbud's matrix factorisation theorem, it suffices to show that $\exists X \in \mathsf{MF}_S(f)$, $\mathsf{Coker}(Y) = \mathsf{Coker} \circ \mathfrak{G}(X)$. Now by (3.6), it suffices to show that $\exists X \in \mathsf{MF}_S(f)$, $\mathsf{Coker}(Y) = \mathfrak{U} \circ \mathfrak{A}^{-1}(X)$. Since $\mathfrak{A}$ is an equivalence, it suffices to show that $\mathsf{Coker}(Y) = \mathfrak{U}(M)$ for some $M \in \mathsf{MCM}_\sigma(R_1)$, or in other words, that $M := \mathsf{Coker}(Y)$ admits a structure of $R_1[\sigma]$-module.

Given an R_1-isomorphism $\alpha : M \xrightarrow{\sim} \sigma^*(M)$, we consider its pullback $\alpha^* : \sigma^*(M) \xrightarrow{\sim} M$. Since M is indecomposable, according to Swan [1968], (2.19), we can assume again that $\alpha^* \circ \alpha = 1 + \rho$ for some $\rho \in \mathsf{rad}(\mathsf{End}(M))$. Now we choose again a convergent power series $P(x) = (1 + x)^{-\frac{1}{2}}$ and let

$$\alpha' := \alpha \circ (1 + \rho)^{-\frac{1}{2}},$$

which induces again $\alpha'^* = (1 + \rho)^{-\frac{1}{2}} \circ \alpha^*$, and we have

$$\alpha'^* \circ \alpha' = \alpha \circ (1 + \rho)^{-\frac{1}{2}} \circ (1 + \rho)^{-\frac{1}{2}} \circ \alpha^* = 1.$$

So α' defines an $R_1[\sigma]$-module structure on M.

For the second point, let $M := \mathsf{Coker}(Y)$ be free of rank r over S. By fixing an S-basis on M, the multiplication by y is given by some $A \in \mathsf{GL}(r, S)$, such that $A^2 = -f \cdot 1$. It is not hard to see that $(y \cdot 1 + A, y \cdot 1 - A) \in \mathsf{MF}_S(f)$ is also a matrix factorisation for M. By Eisenbud's matrix factorisation theorem, this implies that Y and $(y \cdot 1 + A, y \cdot 1 - A)$ are isomorphic up to trivial summands, so without loss of generality we may assume $Y = (y \cdot 1 + A, y \cdot 1 - A)$ is non-trivial and indecomposable. Then the claim follows instantly:

$$T(Y) = (y \cdot 1 - A, y \cdot 1 + A) \simeq (A - y \cdot 1, -A - y \cdot 1) = \sigma(Y). \quad \square$$

- If $X \in \mathsf{MF}_S(f)$ is non-trivial and indecomposable, then $\mathfrak{G}(X)$ is indecomposable if and only if $X \not\simeq TX$. If $X \simeq TX$, then $\mathfrak{G}(X) \simeq Y \oplus TY$ for some indecomposable $Y \in \mathsf{MF}_{S^1}(f+y^2)$ such that $Y \not\simeq TY$.
- Similarly, if $Y \in \mathsf{MF}_{S^1}(f+y^2)$ is indecomposable, then $\mathfrak{R}(Y)$ is indecomposable if and only if $Y \not\simeq TY$. If $Y \simeq TY$, i.e. $Y \in \mathsf{Im}(\mathfrak{G})$, then $\mathfrak{R}(Y) \simeq X \oplus TX$ for some indecomposable $X \in \mathsf{MF}_S(f)$ such that $X \not\simeq TX$.

In fact, as we have shown before, if $X \simeq TX$, then we may assume that $X = (\phi_0, \phi_0)$ with $\phi_0^2 = f \cdot 1$. Then $\mathfrak{G}(X) = (\begin{pmatrix} y \cdot 1 & \phi_0 \\ \phi_0 & -y \end{pmatrix}, \begin{pmatrix} y \cdot 1 & \phi_0 \\ \phi_0 & -y \end{pmatrix})$ is isomorphic to

$$(\phi_0 + iy \cdot 1, \phi_0 - iy \cdot 1) \oplus (\phi_0 - iy \cdot 1, \phi_0 + iy \cdot 1),$$

with an isomorphim given by $(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix})$.

Conversely if $\mathfrak{G}(X) \simeq Y \oplus Y'$ where Y' is a non-trivial indecomposable direct summand, then $\mathfrak{R} \circ \mathfrak{G}(X) \simeq \mathfrak{R}(Y) \oplus \mathfrak{R}(Y')$. But by (3.9), $\mathfrak{R} \circ \mathfrak{G}(X) \simeq X \oplus TX$, so $\mathfrak{R}(Y) \simeq X$ or $\mathfrak{R}(Y) \simeq TX$. According to what we have just shown above,

$$T \circ \mathfrak{R}(Y) \simeq \mathfrak{R}(TY) \simeq \mathfrak{R}(\sigma Y) \simeq \mathfrak{R}(Y),$$

hence we get $X \simeq TX$.

To proceed with the proof we claim that, for a non-trivial indecomposable $Y \in \mathsf{MF}_{S_1}(f+y^2)$, $\mathfrak{R}(Y)$ is indecomposable if and only if $Y \not\simeq TY$.

In fact, if $Y \simeq TY \simeq \sigma(Y)$, then we have already shown that Y is in the image of $\mathfrak{G}$, so $\mathfrak{R}(Y)$ decomposes as (3.9) suggests. Conversely let $\mathfrak{R}(Y) \simeq X \oplus X'$ for X non-trivial and indecomposable, and we have $\mathfrak{G} \circ \mathfrak{R}(Y) \simeq \mathfrak{G}(X) \oplus \mathfrak{G}(X')$. Meanwhile $\mathfrak{G} \circ \mathfrak{R}(Y) \simeq Y \oplus \sigma(Y) \simeq Y \oplus TY$ by (3.10), and we get $\mathfrak{G}(X) \simeq Y$ or $\mathfrak{G}(X) \simeq TY$. But $T \circ \mathfrak{G}(X) = \mathfrak{G}(X)$, so $Y \simeq TY$. This proves the claim.

Now let $X \in \mathsf{MF}_S(f)$ be a non-trivial indecomposable matrix factorisation such that $X \simeq TX$. We decompose $\mathfrak{G}(X) \simeq Y \oplus TY$ as above, where $Y := (\phi_0 + iy \cdot 1, \phi_0 - iy \cdot 1)$. Now $\mathfrak{R} \circ \mathfrak{G}(X) \simeq X \oplus TX$ has only 2 indecomposable summands by (3.9), which forces Y to be indecomposable. In fact, if Y decomposes into more than 2 indecomposable summands, then $\mathfrak{G}(X)$ would have at least 4 indecomposable summands, and so does $\mathfrak{R} \circ \mathfrak{G}(X)$. Suppose that $Y \simeq TY$, then as we have just shown, $\mathfrak{R}(Y)$ and $\mathfrak{R}(TY)$ would both decompose, and $\mathfrak{R} \circ \mathfrak{G}(X)$ would have at least 4 indecomposable summands. Hence $Y \not\simeq TY$.

For the second point, we can determine the number of indecomposable summands in $\mathfrak{G} \circ \mathfrak{R}(Y)$, and the proof is similar to the first point. □

As we have previously mentioned, the number of indecomposable objects in $\mathsf{MCM}(R)$ and $\mathsf{MCM}_\sigma(R_1)$ only differs by 1. Meanwhile, (3.7) and (3.8) imply[3] that $\mathsf{MCM}_\sigma(R_1)$ has only finitely many pairwise non-isomorphic indecomposable

[3] One also needs to apply Krull-Schmidt theorem.

objects, if and only if so does $\mathsf{MCM}(R_1)$. Therefore, we get the following corollary:

> R is MCM-finite, if and only if R_1 is MCM-finite.

Regarding the morphisms, we observe that

> Let $\alpha : M \to N$ be a morphism in $\mathsf{MCM}(R_1)$, such that $\mathfrak{r}(\alpha) : \mathfrak{r}(M) \to \mathfrak{r}(N)$ factors through a projective MCM R-module, then α factors through a projective MCM R_1-module.

In fact, by (3.5), $\mathsf{Coker} \circ \mathfrak{A} \circ \mathfrak{E}(\alpha)$ factors through a projective object, and by Eisenbud's matrix factorisation theorem so does $\mathfrak{A} \circ \mathfrak{E}(\alpha)$ and consequently $\mathfrak{E}(\alpha)$. Now by (3.8), the morphism

$$\alpha \ \oplus \ \sigma \circ \alpha \circ \sigma : M \oplus \sigma^*(M) \to N \oplus \sigma^*(N)$$

factors through a projective object as well, which implies that α factors through a projective MCM R_1-module. □

According to Reiten and Riedtmann [1985] Thm. 3.8, (3.7) and (3.8) actually imply that:

> If $0 \to M \to E \to M' \to 0$ is an almost split sequence in $\mathsf{MCM}_\sigma(R_1)$ (resp. $\mathsf{MCM}(R_1)$), then the induced sequence $0 \to \mathfrak{U}(M) \to \mathfrak{U}(E) \to \mathfrak{U}(M') \to 0$ (resp. $0 \to \mathfrak{E}(M) \to \mathfrak{E}(E) \to \mathfrak{E}(M') \to 0$) is a direct sum of almost split sequences in $\mathsf{MCM}(R_1)$ (resp. $\mathsf{MCM}_\sigma(R_1)$).

These results allow us to approach Knörrer's periodicity. Let $S_2 := S_1\{x\} = S\{x, y\}$ and $R_2 := S_2/(f + x^2 + y^2)$, and we want to compare the MCM's over R and over R_2. For simplicity we change the variables: $u := x + iy$, $v := x - iy$, and define a functor $\mathfrak{H} : \mathsf{MF}_S(f) \to \mathsf{MF}_{S_2}(f + uv)$ by

$$\mathfrak{H}(\phi, \psi) = (\begin{pmatrix} u & \psi \\ \phi & -v \end{pmatrix}, \begin{pmatrix} v & \psi \\ \phi & -u \end{pmatrix}), \quad \mathfrak{H}(\alpha, \beta) = (\alpha \oplus \beta, \alpha \oplus \beta),$$

on objects and morphisms, respectively.

By Eisenbud's matrix factorisation theorem, $\mathfrak{H}$ induces a functor from $\underline{\mathsf{MCM}}(R)$ to $\underline{\mathsf{MCM}}(R_2)$ as well. In the rest of the section we focus on this functor.

To avoid confusion, from now on we denote by $\mathfrak{G}_1 : \mathsf{MF}_S(f) \rightleftarrows \mathsf{MF}_{S_1}(f+y^2) : \mathfrak{R}_1$, and $\mathfrak{G}_2 : \mathsf{MF}_{S_1}(f + y^2) \rightleftarrows \mathsf{MF}_{S_2}(f + y^2 + x^2) : \mathfrak{R}_2$. These functors are all defined before in the notation of $\mathfrak{G}$ and $\mathfrak{R}$.

We now claim that:

$$\mathfrak{G}_2 \circ \mathfrak{G}_1 \simeq \mathfrak{H} \oplus T \circ \mathfrak{H} : \mathsf{MF}_S(f) \to \mathsf{MF}_{S_2}(f + y^2 + x^2), \tag{3.11}$$

$$\mathfrak{R}_1 \circ \mathfrak{R}_2 \circ \mathfrak{H} \simeq 1 \oplus T : \mathsf{MF}_S(f) \to \mathsf{MF}_S(f), \tag{3.12}$$

$$T \circ \mathfrak{H} \simeq \mathfrak{H} \circ T : \mathsf{MF}_S(f) \to \mathsf{MF}_{S_2}(f + y^2 + x^2). \tag{3.13}$$

All of the 3 isomorphisms are proven by straightforward calculations. We only check (3.12) here. Given $(\phi,\psi) \in \mathsf{MF}_S(f)$, $\mathfrak{K}_1 \circ \mathfrak{K}_2 \circ \mathfrak{H}(\phi,\psi) = (\begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix})$ is isomorphic to $(\phi,\psi) \oplus T(\phi,\psi)$ by the isomorphism $(1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. □

We can then show the following result:

$\mathfrak{H}$ induces a bijection between the isomorphism classes of non-trivial indecomposable matrix factorisations of f and $f + y^2 + x^2$.

Let $X \in \mathsf{MF}_S(f)$ be non-trivial and indecomposable, then we have shown before that $\mathfrak{G}_2 \circ \mathfrak{G}_1$ decomposes into precisely 2 non-isomorphic indecomposable summands, and thus $\mathfrak{H}(X)$ is indecomposable by (3.11). Now suppose $X' \not\simeq X$ is another matrix factorisation of f such that $\mathfrak{H}(X) \simeq \mathfrak{H}(X')$. Then (3.12) implies $X' \simeq TX$. So $\mathfrak{G}_1(X)$ is indecomposable and consequently $\mathfrak{G}_2 \circ G_1(X)$ decomposes into 2 non-isomorphic summands. So by (3.11), we get $\mathfrak{H}(X) \not\simeq T \circ \mathfrak{H}(X)$. However by (3.13), $T \circ \mathfrak{H}(X) \simeq \mathfrak{H} \circ T(X) \simeq \mathfrak{H}(X') \simeq \mathfrak{H}(X)$, a contradiction!

To complete the proof, take any non-trivial indecomposable $Y \in \mathsf{MF}_{S_2}(f+y^2+x^2)$, and we need to show that Y is a direct summand of $\mathfrak{H}(X)$ for some $X \in \mathsf{MF}_S(f)$. But this is a immediate consequence of (3.11), (3.13) and (3.10):

$$\begin{aligned} \mathfrak{H} \circ \mathfrak{K}_1 \circ \mathfrak{K}_2(Y) \oplus \mathfrak{H} \circ T \circ \mathfrak{K}_1 \circ \mathfrak{K}_2(Y) &\simeq \mathfrak{H} \circ \mathfrak{K}_1 \circ \mathfrak{K}_2(Y) \oplus T \circ \mathfrak{H} \circ \mathfrak{K}_1 \circ \mathfrak{K}_2(Y) \\ &\simeq \mathfrak{G}_2 \circ \mathfrak{G}_1 \circ \mathfrak{K}_1 \circ \mathfrak{K}_2(Y) \\ &\simeq \mathfrak{G}_2 \circ \mathfrak{K}_2(Y) \oplus \mathfrak{G}_2 \circ \sigma \circ \mathfrak{K}_2(Y) \\ &\simeq Y \oplus \sigma(Y) \oplus \mathfrak{G}_2 \circ \sigma \circ \mathfrak{K}_2(Y). \quad \square \end{aligned}$$

We now state the Knörrer's periodicity result that we are to prove in this section:

Knörrer's periodicity. $\mathfrak{H} : \mathsf{MF}_S(f) \to \mathsf{MF}_{S_2}(f + uv)$ induces an equivalence

$$\underline{\mathsf{MCM}}(R) \simeq \underline{\mathsf{MCM}}(R_2). \tag{3.14}$$

For the proof, we still have to show that for any 2 matrix factorisations $X = (\phi : F_1 \rightleftarrows F_2 : \psi)$, $X' = (\phi' : F_1' \rightleftarrows F_2' : \psi') \in \mathsf{MF}_S(f)$, $\mathfrak{H}$ induces an isomorphism $\underline{\mathsf{Hom}}(X, X') \to \underline{\mathsf{Hom}}(\mathfrak{H}(X), \mathfrak{H}(X'))$. By (3.12), this map is clearly injective. So we are left to show the surjectivity.

Let $(\alpha, \beta) : \mathfrak{H}(X) \to \mathfrak{H}(X')$ be a morphism in $\mathsf{MF}_{S^2}(f + uv)$, then

$$\mathfrak{K}_1 \circ \mathfrak{K}_2(\alpha, \beta) =: (\bar{\alpha}, \bar{\beta}) : \mathfrak{K}_1 \circ \mathfrak{K}_2 \circ H(X) \to \mathfrak{K}_1 \circ \mathfrak{K}_2 \circ H(X')$$

is a morphism in $\mathsf{MF}_S(f)$, where $\mathfrak{K}_1 \circ \mathfrak{K}_2 \circ H(X) = (\begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi \\ \phi & 0 \end{pmatrix})$ and $\mathfrak{K}_1 \circ \mathfrak{K}_2 \circ H(X') = (\begin{pmatrix} 0 & \psi' \\ \phi' & 0 \end{pmatrix}, \begin{pmatrix} 0 & \psi' \\ \phi' & 0 \end{pmatrix})$.

Then we claim that

- the morphisms $(\bar{\alpha}_{21}, \bar{\beta}_{12}) : X \to TX'$ and $(\bar{\alpha}_{12}, \bar{\beta}_{21}) : TX \to X'$ factor through projectives;
- the morphisms $(\bar{\alpha}_{11}, \bar{\beta}_{22})$, $(\bar{\beta}_{11}, \bar{\alpha}_{22}) : X \to X'$ only differ by a morphism that factors through projectives.

These are proven by direct calculation. Since (α, β) is a morphism of matrix factorisations, we have

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} u & \psi \\ \phi & -v \end{pmatrix} = \begin{pmatrix} u & \psi' \\ \phi' & -v \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \tag{3.15}$$

from which we get

$$\begin{aligned} \phi' \circ \alpha_{11} - v\alpha_{21} &= u\beta_{21} + \beta_{22} \circ \phi; \\ u\alpha_{12} + \psi' \circ \alpha_{22} &= \beta_{11} \circ \psi - v\beta_{12}. \end{aligned}$$

Notice that $\alpha_{11}, \beta_{11}, \alpha_{22}, \beta_{22}$ are all dependent on u, v, so we can compare the coefficients of u, v on both sides of the equations and get

$$\begin{aligned} \bar{\beta}_{12} &= \psi' \circ \gamma_{12} + \delta_{12} \circ \psi, \\ \bar{\beta}_{21} &= \phi' \circ \gamma_{21} + \delta_{21} \circ \phi, \end{aligned}$$

where $\gamma_{12}, \gamma_{21}, \delta_{12}, \delta_{21}$ are certain maps. Clearly $\psi' \circ \gamma_{12}$ induces

$$\mathsf{Coker}(X) = \mathsf{Coker}(\phi : F_1 \to F_2) \xrightarrow{0} \mathsf{Coker}(TX') = \mathsf{Coker}(\psi' : F_2' \to F_1').$$

On the other hand $\psi \circ \phi = f \cdot 1 = 0$ over $R = S/(f)$, so $\delta_{12} \circ \psi : \mathsf{Coker}(\phi) \to \mathsf{Coker}(\psi')$ factors through the free R-module $R \otimes_P F_1$. This shows that the map $\mathsf{Coker}(X) \to \mathsf{Coker}(TX')$ induced by $\bar{\beta}_{12}$ factors through projectives. Similarly one can show that the one induced by $\bar{\beta}_{21}$ factors through projectives as well. Thus the first point follows.

For the second point, we again observe from (3.15) that

$$\phi' \circ \alpha_{12} - v\alpha_{22} = \beta_{21} \circ \psi - v\beta_{22}.$$

Then we similarly get

$$\bar{\alpha}_{22} - \bar{\beta}_{22} = \phi' \circ \gamma_{22} + \delta_{22} \circ \psi,$$

where γ_{22} and δ_{22} are certain maps. We then show that the morphism induced by $\bar{\alpha}_{22} - \bar{\beta}_{22}$ factors through projectives as in the proof of the first point. $\square$

Therefore, any morphism from $(\alpha, \beta) : \mathfrak{H}(X) \to \mathfrak{H}(X')$ can be altered by some morphism in $\mathfrak{H}(\mathsf{Hom}(X, X'))$ such that $\mathfrak{R}_1 \circ \mathfrak{R}_2(\alpha, \beta)$ factors through projectives. Now the Knörrer's periodicity (3.14) follows easily. $\square$

An immediate corollary of Knörrer's periodicity (3.14) is that:

The stable Auslander-Reiten quiver of $\mathsf{MCM}(R)$ and $\mathsf{MCM}(R_2)$ are isomorphic, i.e.

$$\underline{\Gamma}(R) \simeq \underline{\Gamma}(R_2).$$

Recall that the stable Auslander-Reiten quiver is obtained by deleting the vertex of indecomposable free module and all arrows adjacent in the original Auslander-Reiten quiver.

Moreover, to relate the Auslander-Reiten quiver of $\mathsf{MCM}(R)$ and $\mathsf{MCM}(R_2)$, we have:

The number of arrows ending in (or starting from) the vertex of free module is doubled when passing from $\mathsf{MCM}(R)$ to $\mathsf{MCM}(R_2)$.

See Solberg [1989], Prop. 4.6. $\square$

In conclusion, with Knörrer's periodicity, we can claim that simple singularities are MCM-finite, provided that we show it in 1-dimensional and 2-dimensional cases, which are well-known to be true and will be discussed in the next chapter. Combining that with Buchweitz-Greul-Schreyer's result, we get:

Any hypersurface is MCM-finite if and only if it is a simple singularity.

4. Auslander-Reiten quiver of simple singularities

4.1 Even-dimensional simple singularities

In this chapter we decide the (stable) Auslander-Reiten quiver of the category of MCM modules over any simple singularity. We refer to it for brevity as the (stable) Auslander-Reiten quiver of the simple singularity as long as no confusion is caused.

For even-dimensional simple singularities, the stable Auslander-Reiten quivers are rather easy to decide. We now briefly develop the isomorphism of the Auslander-Reiten quiver with the McKay graph in this case, discovered in Auslander [1986] and Auslander and Reiten [1987].

Let K be an algebraically closed field of characteristic 0 and $G \leq \mathsf{GL}_2(K)$ be a finite subgroup. Given a 2-dimensional K-vector space V, we can pick a K-basis $\{x, y\}$, on which G naturally acts:

$$\forall g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in G,\ g.(x, y) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_{11}x + g_{12}y \\ g_{21}x + g_{22}y \end{pmatrix}.$$

Then we can extend the action of G to the ring $S := K\{x, y\}$:

$$\forall f(x, y) \in S, \forall g \in G,\ g.f(x, y) = f(g.(x, y)).$$

Now let $R := \{s \in S \mid g.s = s, \forall g \in G\}$ be the invariant subring of S, and denote by $S[G] := \sum_{g \in G} Sg$ the skew group ring, considered as an S-module with the multiplication

$$s_1 g_1 \ \cdot s_2 g_2 := s_1 \ g_1.s_2 \ g_1 g_2, \quad s_i \in S, g_i \in G.$$

An $S[G]$-module M is exactly an S-module acted by G such that

$$g.sm = g.s \ g.m, \quad s \in S, m \in M, g \in G.$$

In addition, $f : M \to N$ is an $S[G]$-homomorphism if and only if it is simultaneously an S-homomorphism and a G-homomorphism, i.e. it satisfies

$$f(g.sm) = g.f(sm) = g.sf(m), \quad g \in G, s \in S, m \in M.$$

When $M, N \in \mathsf{mod}(S[G])$, $\mathsf{Hom}_S(M, N)$ is also an $S[G]$-module, with the G-action

$$(g.f)(m) = g.f(g^{-1}.m), \quad g \in G, m \in M, f \in \mathsf{Hom}_S(M, N).$$

Moreover, any $f \in \mathsf{Hom}_S(M, N)$ is G-invariant if and only if it is an $S[G]$-homomorphism. In fact, if f is an $S[G]$-homomorphism, $\forall g \in G$, $\forall m \in M$,

$$g.f(m) = g.f(g^{-1}.m) = f(gg^{-1}.m) = f(m),$$

i.e. f is G-invariant. Conversely, if f is G-invariant, $\forall g \in G$, $\forall s \in S$,

$$f(g.sm) = g.f(g^{-1}g.sm) = g.f(sm) = g.sf(m),$$

thus f is an $S[G]$-homomorphism.

So we get

$$\mathsf{Hom}_{S[G]}(M,N) = \mathsf{Hom}_S(M,N)^G,$$

where $\mathsf{Hom}_S(M,N)^G = \{f \in \mathsf{Hom}_S(M,N) \mid g.f = f, \forall g \in G\}$ denotes the G-invariant submodule as before.

Now we claim that:

$(-)^G : \mathsf{mod}(S[G]) \to \mathsf{mod}(R)$ is an exact functor.

In general, taking G-invariant is a left exact functor. Given any short exact sequence in $\mathsf{mod}(S[G])$ of the form

$$0 \to M \xrightarrow{\iota} N \xrightarrow{\pi} P \to 0,$$

there are induced maps $M^G \to N^G$ and $N^G \to P^G$, denoted by ι^G and π^G, respectively. Clearly, $\mathsf{Ker}(\iota^G) \subset \mathsf{Ker}(\iota) = 0$. For any $m \in M^G$, $\pi^G \circ \iota^G(m) = \pi \circ \iota(m) = 0$, so $\mathsf{Im}(\iota) \subset \mathsf{Ker}(\pi)$. Now let $n \in N^G$ and suppose $\pi(n) = 0$, then $n \in \mathsf{Ker}(\pi) = \mathsf{Im}(\iota)$, so $n = \iota(m')$ for some $m' \in M$. $\forall g \in G$, $n = g.n = g.\iota(m') = \iota(g.m')$, so $m' = g.m'$. Therefore, taking G-invariant is left exact.

Now we would like to show that $\pi^G : N^G \to P^G$ is an epimorphism, for which we need to use the assumption that G is a finite group. In this case, it is an elementary result from group theory that

$$N^G = \mathsf{Im}(n \mapsto \frac{1}{|G|}\sum_{g\in G} g.n).$$

Denote by $\phi_{G,N} : N \to N^G$ the surjective map $n \mapsto \frac{1}{|G|}\sum_{g\in G} g.n$, and we get $\pi^G \circ \phi_{G,N} = \phi_{G,P} \circ \pi$. Since $\phi_{G,N}$, $\phi_{G,P}$ and π are all surjective, so is π^G. Thus we have shown that $(-)^G$ is an exact functor. □

On the level of derived functors, we have

$$\mathsf{Ext}^i_{S[G]}(M,N) = \mathsf{Ext}^i_S(M,N)^G, \quad i \geq 0.$$

See Yoshino [1990], p. 86. □

Therefore, an $S[G]$-module is projective if and only if it is projective over S.

Let KG be the group ring in the usual sense. The assignment $W \mapsto S \otimes_K W$ and $f \mapsto 1_S \otimes_K f$ defines a functor $F : \mathsf{mod}(KG) \to \mathsf{mod}(S[G])$, where the action of $S[G]$ is given by

$$(sg)(t \otimes w) = sg.t \otimes g.w, \quad (sg)(1_S \otimes f) = s \otimes g.f, \quad s,t \in S, g \in G, w \in W.$$

Consider the full subcategory of $\mathsf{mod}(S[G])$ consisting of all projectives, denoted by $\mathsf{proj}.S[G]$. Given any KG-module W, $F(W) = S \otimes_K W$ is clearly free over S, and hence projective over $S[G]$. Therefore F gives rise to a functor $\mathsf{mod}(KG) \to \mathsf{proj}.S[G]$, also denoted by F. Moreover we have:

> The functor $F : \mathsf{mod}(KG) \to \mathsf{proj}.S[G]$ admits a left adjoint $F' = \kappa_S \otimes_S - : \mathsf{proj}.S[G] \to \mathsf{mod}(KG)$. Moreover, F gives a one-to-one correspondence between the isomorphism classes of objects.

Here $\kappa_S = S/\mathfrak{m}_S$ denotes the residue field as before, where $\mathfrak{m}_S = (x, y)S$ is the unique maximal ideal of S. It is clear that $F' \circ F = 1_{\mathsf{mod}(KG)}$. We now consider $F \circ F'$ on the objects. Let $M \in \mathsf{proj}.S[G]$, then M is free over S, and we get $F \circ F'(M) = S \otimes_K (M/\mathfrak{m}_S M)$. We claim that $\pi : M \to M/\mathfrak{m}_S M$ is the minimal projective cover of $M/\mathfrak{m}_S M \in \mathsf{mod}(S[G])$!

In fact, take any proper submodule $N \subsetneq M$, we need to show that $\pi(N) \neq M/\mathfrak{m}_S M$. Suppose the opposite, i.e. $\pi(N) = M/\mathfrak{m}_S M$, and then $M = N + \mathfrak{m}_S M$. Then one of the variants of Nakayama's lemma implies that $M = N$, a contradiction! So π is the minimal projective cover.

Meanwhile, the natural map $S \otimes_K M/\mathfrak{m}_S M \to M/\mathfrak{m}_S M$ is also a projective cover of $M/\mathfrak{m}_S M \in \mathsf{mod}(S[G])$, so it must factor through M, i.e. there must be a surjective map $S \otimes_K M/\mathfrak{m}_S M \to M$. Since M is projective, this surjective map splits, i.e. M is a direct summand of $S \otimes_K M/\mathfrak{m}_S M$. Since $\mathsf{rank}(M/\mathfrak{m}_S M) = \mathsf{rank}(M) - 1$, we consequently get $M \simeq S \otimes_K M/\mathfrak{m}_S M$. The rest of the proof is straightforward. □

Therefore, in another word, there is a one-to-one correspondence between the irreducible representations of G over K and the indecomposable projective $S[G]$-modules.

McKay graph. Let $V_0, \cdots, V_d$ be the isomorphism classes of irreducible representations of G. The *McKay graph* $\mathsf{Mc}(V, G)$ is an oriented graph of vertices $V_0, \cdots, V_d$, where, for any $0 \leq i, j \leq d$, there are $\mathsf{mult}_i(V \otimes_K V_j)$ arrows. Here $\mathsf{mult}_i(-) := \mathsf{dim}_K \mathsf{Hom}_{KG}(V_i, -)$.

Let $P_0, \cdots, P_d$ be the isomorphism classes of indecomposable projective $S[G]$-modules with $F(V_i) = S \otimes_K V_i = P_i$, $0 \leq i \leq d$. For any projective $S[G]$-module P, let $\nu_i(P)$ be the number of copies of P_i appearing in the direct decomposition of P, $0 \leq i \leq d$. Then these integers can be used to calculate the multiplicities in the McKay graph:

> $$\mathsf{mult}_i(V \otimes_K V_j) = \nu_i(F(V \otimes_K V_j)), \quad 0 \leq i, j \leq d.$$

In fact, $V \otimes_K V_j = \bigoplus_i V_i^{\mathsf{mult}_i(V \otimes_K V_j)}$ by definition. It follows that $F(V \otimes_K V_j) = \bigoplus_i P_i^{(\mathsf{mult}_i(V \otimes_K V_j))}$. □

Denote by $\mathsf{add}_R(S)$ the full subcategory of $\mathsf{mod}(R)$ consisting of R-modules that are isomorphic to direct R-summands of free S-modules. We now show a famous result by Auslander:

As subcategories of $\mathsf{mod}(R)$,

$$\mathsf{add}_R(S) = \mathsf{MCM}(R).$$

In particular, indecomposable MCM R-modules are identified with the indecomposable R-summands of S. Hence R is MCM-finite.

First notice that $R = S^G$ is a 2-dimensional normal domain, hence $\mathsf{MCM}(R)$ consists of reflexive R-modules. Since S is reflexive over R, clearly $\mathsf{add}_R(S) \subset \mathsf{MCM}(R)$. Now let $M \in \mathsf{MCM}(R)$. The inclusion map $R \hookrightarrow S$ is a split R-monomorphism, with a retraction given by

$$\phi : S \to R, \quad s \mapsto \frac{1}{|G|} \sum_{g \in G} g.s.$$

Applying $\mathsf{Hom}_R(\mathsf{Hom}_R(M, R), -)$, we get a split monomorphism

$$M \simeq \mathsf{Hom}_R(\mathsf{Hom}_R(M, R), R) \to \mathsf{Hom}_R(\mathsf{Hom}_R(M, R), S).$$

Now that $\mathsf{Hom}_R(\mathsf{Hom}_R(M, R), S)$ is reflexive as an S-module, it is even free over S! This famous result by Serre follows from the fact that S is a 2-dimensional regular local ring. Instead of citing the original proof, we can prove it even faster: since S is regular, $\mathsf{gl.dim}(S) = 2$. For brevity we denote by $M^* := \mathsf{Hom}_R(M, R)$. Now we take a free presentation of $\mathsf{Hom}_S(\mathsf{Hom}_R(M^*, S), S)$ of the following form:

$$F_1 \to F_0 \to \mathsf{Hom}_S(\mathsf{Hom}_R(M^*, S), S) \to 0,$$

and dualise it to get an exact sequence

$$0 \to \mathsf{Hom}_R(M^*, S) \to \mathsf{Hom}_S(F_0, S) \xrightarrow{\phi} \mathsf{Hom}_S(F_1, S) \to \mathsf{Coker}(\phi) \to 0.$$

Now that $\mathsf{proj.dim}(\mathsf{Coker}(\phi)) \leq 2$ and the dual of F_0 and F_1 are both free, we get that $\mathsf{Hom}_R(M^*, S)$ is free. Therefore, M is a direct summand of a free S-module, i.e. $M \in \mathsf{add}_R(S)$. This shows that $\mathsf{add}_R(S) = \mathsf{MCM}(R)$. The rest of the statement simply follows from Krull-Schmidt theorem. □

An element $\sigma \in \mathsf{GL}_2(K)$ is a *pseudo-reflection* if $\mathsf{rank}(\sigma - 1) \leq 1$. M. Auslander discovered the following striking relation between MCM R-modules and $S[G]$-modules:

Assume that G has no non-trivial pseudo-reflection, then there is an equivalence of categories

$$H : \mathsf{proj}.S[G] \xrightarrow{\sim} \mathsf{MCM}(R), \quad M \mapsto M^G, \; f \mapsto f|_{M^G}.$$

First notice that given $M \in \mathsf{proj}.S[G]$, M^G is a direct summand of M as an R-module. Since M is also projective over S, it is a free S-module, hence $M^G \in$

$\mathsf{add}_R(S)$, and it is easy to see that the functor H is well-defined. See Auslander [1986], p. 515 for the proof of the result. □

Combining the results above, we get that:

> The composition $H \circ F$ gives a one-to-one correspondence between the isomorphism classes of irreducible representations of G and the isomorphism classes of MCM R-modules.

Now we would like to construct Auslander-Reiten sequences. Let $V_0 = K$ be the trivial simple KG-module. We denote by

$$\tau(V_i) = \bigwedge^2 V \otimes_K V_i \ , 0 \leq i \leq d,$$

and also $\tau(P_i) = F(\tau(V_i))$. Let $L_0, \cdots, L_d$ be the indecomposable MCM R-modules, where $L_i = H(P_i)$. So $L_0 = H \circ F(V_0) = R$. We denote by $\tau(L_i) = H(\tau(P_i))$ for $0 \leq i \leq d$. Then it is clear that $\tau(V_i) \simeq \tau(V_j)$, if and only if $\tau(P_i) \simeq \tau(P_j)$, if and only if $\tau(L_i) \simeq \tau(L_j)$, if and only if $i = j$. Moreover, we have

> $$\tau(L_0) \simeq \omega_R.$$

See Yoshino [1990], p.92. □

Consider the Koszul complex associated to $S \to K$:

$$0 \to S \otimes_K \bigwedge^2 V \to S \otimes_K V \to S \to K \to 0,$$

and this is also exact as $S[G]$-modules. Tensoring V_i, we obtain

$$0 \to S \otimes_K (\bigwedge^2 V \otimes_K V_i) \to S \otimes_K (V \otimes_K V_i) \to S \otimes_K V_i \to V_i \to 0,$$

which gives the minimal projective $S[G]$-resolution of V_i which we rewrite as:

$$0 \to \tau(P_i) \to F(V \otimes_K V_i) \to P_i \to V_i \to 0.$$

Now we apply H to get

$$0 \to \tau(L_i) \to H \circ F(V \otimes_K V_i) \to L_i \to V_i^G \to 0,$$

where $V_i^G = \begin{cases} K, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases}$ This is because each V_i is a simple KG-module. For brevity we denote by $E_i := H \circ F(V \otimes_K V_i)$, and we get the sequences

$$0 \to \tau(L_0) \to E_0 \xrightarrow{\pi_0} L_0 \to K \to 0, \quad (4.1)$$
$$0 \to \tau(L_i) \to E_i \xrightarrow{\pi_i} L_i \to 0, \quad 0 < i \leq d. \quad (4.2)$$

We now show that:

- Given $L \in \mathsf{MCM}(R)$ and $f \in \mathsf{Hom}_R(L, L_i)$ which is not a split epimorphism, then there exists $g \in \mathsf{Hom}_R(L, E_i)$ with $f = \pi_i \circ g$, where $0 \leq i \leq d$.
- In particular, for $i \neq 0$, (4.2) is the Auslander-Reiten sequence ending in L_i.

Since f is not a split epimorphism, we have $\mathsf{Im}(f) \subset \mathsf{Im}(\pi_i)$, and hence we get $\mathsf{Im}(H^{-1}(f)) \subset \mathsf{Im}(H^{-1}(\pi_i))$, where $H^{-1}(\pi_i) : F(V \otimes_K V_i) \to P_i$ and $H^{-1}(f) : H^{-1}(L) \to P_i$. So there is $g' \in \mathsf{Hom}_{S[G]}(H^{-1}(L), F(V \otimes_K V_i))$, such that $H^{-1}(f) = H^{-1}(\pi_i) \circ g'$, for $H^{-1}(L)$ is projective over $S[G]$. Now we let $g = H(g') : L \to E_i$, and we get $f = \pi_i \circ g$. □

Finally we can decide the Auslander-Reiten quiver of R:

As before assume that $G \leq \mathsf{GL}_2(K)$ has no non-trivial pseudo-reflection, then $R = S^G$ is MCM-finite, and its Auslander-Reiten quiver $\Gamma(R)$ coincides with the McKay graph $\mathsf{Mc}(V, G)$, where $V = \langle x, y \rangle$, $S = K\{x, y\}$.

Clearly the vertices of $\Gamma(R)$ are $L_0, \cdots, L_d$. If $i \neq 0$, $i(L_j, L_i)$ is the number of copies of L_j appearing in the direct decomposition of E_i. When $i = 0$, the previous statement still applies, though the sequence is longer. So the same property holds. Therefore, $i(L_j, L_i) = \nu_j(F(V \otimes_K V_i)) = \mathsf{mult}_j(V \otimes_K V_i)$. So there is an isomorphism $\Gamma(r) \to \mathsf{Mc}(V, G)$ given by $[L_i] \mapsto [V_i]$. □

Klein groups. Now the Auslander-Reiten quiver of any 2-dimensional simple singularity is reduced to some classic result in Klein [1893]. Let ζ_k be the k-th primitive root of unity in K. Felix Klein classified all finite subgroups of $\mathsf{SL}(2, K)$ as conjugates to the following *Klein groups*:

- A_k: cyclic group of order $k + 1$:

$$C_k = \langle \begin{pmatrix} \zeta_{k+1} & 0 \\ 0 & \zeta_{k+1}^{-1} \end{pmatrix} \rangle;$$

- D_k: binary dihedral group of order $4(k - 2)$:

$$D_k = \langle \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}, C_{2k-5} \rangle;$$

- E_6: binary tetrahedral group of order 24:

$$T = \langle \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}, D_4 \rangle;$$

- E_7: binary octahedral group of order 48:

$$O = \langle \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}, T \rangle;$$

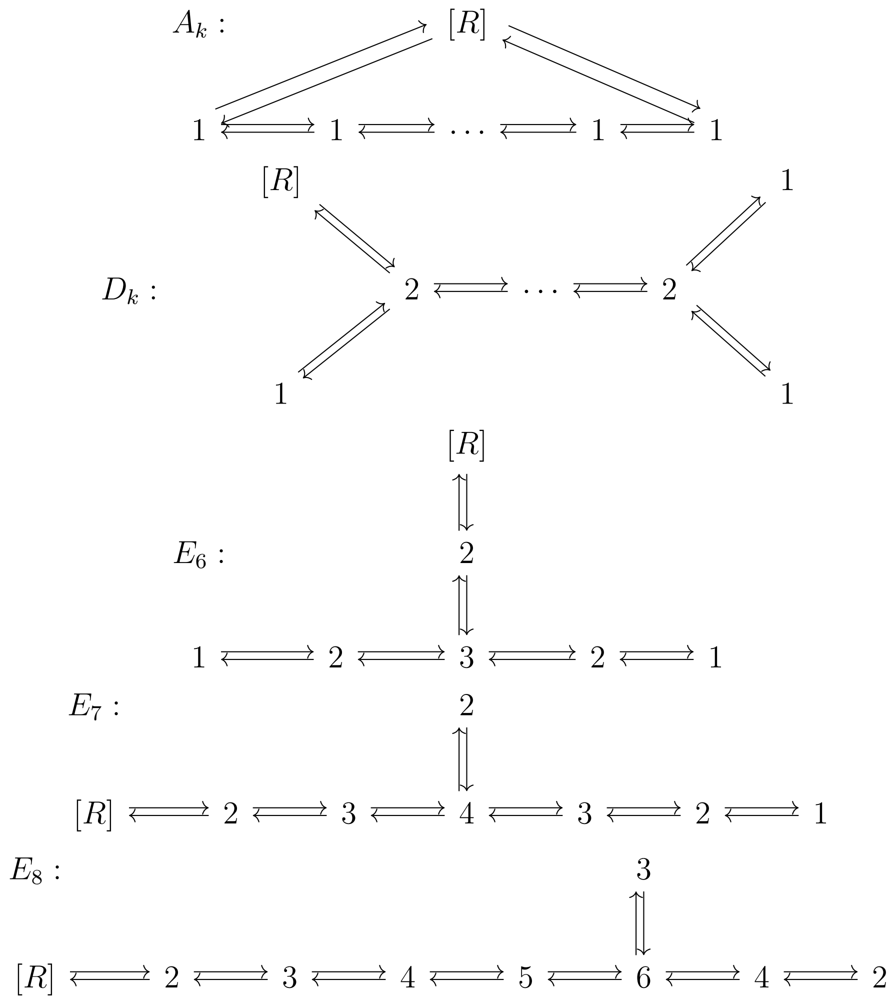
- E_8: binary icosahedral group of order 120:

$$I = \langle \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^2 - \zeta_5^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{pmatrix} \rangle.$$

He also proved that the invariant subrings by these groups are simple singularities, i.e. $S^G \simeq K\{x, y, z\}/(f)$ where f is one of the following polynomials respectively in each case:

- $A_k : x^2 + y^{k+1} + z^2$, $k \geq 1$;
- $D_k : x^2 y + y^{k-1} + z^2$, $k \geq 4$;
- $E_6 : x^3 + y^4 + z^2$;
- $E_7 : x^3 + xy^3 + z^2$;
- $E_8 : x^3 + y^5 + z^2$.

We now exhibit the McKay graphs of these rings, where the number attached to a vertex indicates the degree of the corresponding irreducible representation. When we consider them as Auslander-Reiten quivers, the numbers are the ranks of the corresponding MCM modules.



In conclusion, we have determined the Auslander-Reiten quiver of any simple singularity of dimension 2 using McKay graphs, and by Knörrer's periodicity, the stable Auslander-Reiten quiver of any even-dimensional simple singularity is the same as in the 2-dimensional case. For the Auslander-Reiten quiver itself, the only change appears in the number of arrows adjacent to $[R]$.

4.2 Odd-dimensional simple singularities

By Knörrer's periodicity for simplicity we need only consider the 1-dimensional case. Recall that the types of singularities we consider are:

- $A_k : x^2 + y^{k+1}, k \geq 1$;
- $D_k : x^2y + y^{k-1}, k \geq 4$;
- $E_6 : x^3 + y^4$;
- $E_7 : x^3 + xy^3$;
- $E_8 : x^3 + y^5$.

As always let $S = K\{x, y\}$, $R = S/(f)$ where f belongs to one of the types above.

First we show a useful statement.

Let R be a 1-dimensional analytic reduced local hypersurface ring. Then the Auslander-Reiten translation τ is given by

$$\tau(M) \simeq \mathsf{syz}^1_R M, \quad \forall M \in \mathsf{MCM}(R), \tag{4.3}$$

and $\tau^2 = 1$. Furthermore, if we can decompose $\mathfrak{m}_R = \bigoplus_i M_i$ into indecomposable modules M_i, then the natural inclusions $M_i \to R$ are the only irreducible morphisms from an indecomposable MCM R-module to R. Dually, the irreducible morphisms from R to an indecomposable MCM R-module are of the form $R \to \tau(M_i)$.

First recall that $R = S/(f)$ for some regular local ring S, and $0 \neq f \in \mathfrak{m}_S$. Thus R is a 1-dimensional Gorenstein ring, and in particular $\omega_R \simeq R$.

Take a free resolution of M of the form

$$\cdots \to F_1 \to F_0 \to M \to 0.$$

Then we have an exact sequence of the form

$$0 \to \mathsf{Hom}_R(M, R) \to \mathsf{Hom}_R(F_0, R) \to \mathsf{Hom}_R(F_1, R) \to \mathsf{Tr}(M) \to 0,$$

and hence by (2.1), we get a short exact sequence

$$0 \to \mathsf{Hom}_R(M, R) \to \mathsf{Hom}_R(F_0, R) \to \mathsf{Hom}_R(\tau(M), R) \to 0.$$

Taking the self-duality, we get another short exact sequence

$$0 \to \tau(M) \to F_0 \to M \to 0,$$

which shows that $\tau(M) \simeq \mathsf{syz}^1_R(M)$. In addition, recall that M has a 2-periodic free resolution, see (1.5). Therefore, $\mathsf{syz}^2_R(M) \simeq M$, i.e. $\tau^2 = 1$.

Now consider any irreducible morphism $\alpha : X \to R$, where $X \in \mathsf{MCM}(R)$ is indecomposable. Since α does not split, $1 \notin \mathsf{Im}(\alpha)$, thus $\mathsf{Im}(\alpha) \subset \mathfrak{m}_R$. In other words, α is decomposed as $X \to \mathfrak{m}_R \hookrightarrow R$. Since the inclusion $\mathfrak{m}_R \hookrightarrow R$ is not an epimorphism, we get that $X \to \mathfrak{m}_R$ is a split monomorphism, i.e. $X \simeq M_i$ for some i. Dually, any irreducible morphism from R end in $M_i^* := \mathsf{Hom}_R(M_i, R)$. Since R is an isolated singularity, there is an Auslander-Reiten sequence of the form

$$0 \to M_i \to N \oplus R \to M_i^* \to 0,$$

where N is an MCM R-module. Therefore $\tau(M_i^*) \simeq \mathsf{syz}^1_R(M) \simeq M_i$ as we have just shown, and hence $\tau(M_i) = \tau^2(M_i^*) = M_i^*$, which completes the proof. □

Now we shall calculate the stable Auslander-Reiten quivers by matrix factorisations and Auslander-Reiten sequences. The method is based on Yoshino [1990], Ch. 9.

4.2.1 Type A_k for even k

In this case $f = x^2 + y^{k+1}$, $k = 2, 4, 6, \cdots$. Letting

$$\phi_j := \begin{pmatrix} x & y^j \\ y^{k+1-j} & -x \end{pmatrix}, \quad 0 \le j \le k+1,$$

we see that $\phi_j^2 = \begin{pmatrix} x^2 + y^{k+1} & 0 \\ 0 & y^{k+1} + x^2 \end{pmatrix} = f1$, i.e. $(\phi_j, \phi_j) \in \mathsf{MF}_S(f)$. Denote by $M_j := \mathsf{Coker}(\phi_j, \phi_j) = (x, y^j)R$, and clearly $M_0 \simeq R$ and $M_j \simeq M_{k+1-j}$. Now we would like to show that the MCM R-modules M_j $(0 \le j \le \frac{k}{2})$ are all indecomposable. For $j = 0$ it is clear. Notice that f is irreducible since k is even, therefore, if for some $1 \le j \le \frac{k}{2}$ we have $M_j \simeq A \oplus B$, then A and B would be generated by 1 element, and thus isomorphic to $M_0 \simeq R$, which means M_j has a free direct summand, a contradiction to the fact that $\phi_j \in \mathsf{GL}_2(S)$ has no unit entry!

Now it follows from (1.9) and (4.3) that

$$\tau(M_j) = M_j, \quad \forall 1 \le j \le \frac{k}{2}.$$

Let $\epsilon_j := \begin{pmatrix} 0 & y^{j-1} \\ -y^{k-j} & 0 \end{pmatrix}$, and we see that

$$\epsilon_j \circ \phi_j = \begin{pmatrix} y^k & -xy^{j-1} \\ -xy^{k-j} & -y^k \end{pmatrix} = -\phi_j \circ \epsilon_j,$$

so $(\epsilon_j, -\epsilon_j)$ gives an element in $\mathsf{End}_R(M)$. Now that

$$\begin{pmatrix} \phi_j & \epsilon_j \\ 0 & \phi_j \end{pmatrix}^2 = \begin{pmatrix} \phi_j^2 & \phi_j\epsilon_j + \epsilon_j\phi_j \\ 0 & \phi_j^2 \end{pmatrix} = f1,$$

we get that $\left(\begin{pmatrix}\phi_j & \epsilon_j \\ 0 & \phi_j\end{pmatrix}, \begin{pmatrix}\phi_j & \epsilon_j \\ 0 & \phi_j\end{pmatrix}\right) \in \mathsf{MF}_S(f)$. Clearly

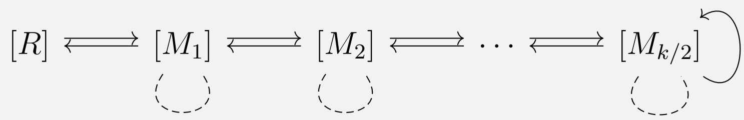
$$\mathsf{Coker}\left(\begin{pmatrix}\phi_j & \epsilon_j \\ 0 & \phi_j\end{pmatrix}, \begin{pmatrix}\phi_j & \epsilon_j \\ 0 & \phi_j\end{pmatrix}\right) \simeq M_{j-1} \oplus M_{j+1}.$$

Hence the Auslander-Reiten sequence ending in M_j is

$$0 \to M_j \to M_{j-1} \oplus M_{j+1} \to M_j \to 0.$$

Notice that when $j = \frac{k}{2}$, $M_{\frac{k}{2}+1} \simeq M_{\frac{k}{2}}$, and consequently we can draw a part of the Auslander-Reiten quiver for R:

Type A_k for even $k \geq 2$:



Now we would like to show that the graph above is a connected component of $\Gamma(R)$, for which it suffices to show that no other MCM module is connected to $[R]$ by arrows. But this is obvious from the statement we have shown at the beginning of this section, since $\mathfrak{m}_R = (x,y)R \simeq M_1$.

Finally, we can conclude that $\Gamma(R)$ is equal to its connected component we have drawn above by Dieterich-Yoshino's theorem, mentioned in Section 2.2.

4.2.2 Type A_k for odd k

In this case $f = x^2 + y^{k+1} = (y^{(k+1)/2} + ix)(y^{(k+1)/2} - ix)$, $k = 1, 3, 5, \cdots$. Above all, $N_\pm := R/(y^{(k+1)/2} \pm ix)$ are MCM R-modules, which are given by the matrix factorisations $(y^{(k+1)/2} + ix, y^{(k+1)/2} - ix)$ and $(y^{(k+1)/2} - ix, y^{(k+1)/2} + ix)$, respectively.

Now consider the same 2×2 matrices on $S = K\{x,y\}$ as in the previous case:

$$\phi_j = \begin{pmatrix} x & y^j \\ y^{k+1-j} & -x \end{pmatrix}, \quad 0 \leq j \leq k+1.$$

Then $(\phi_j, \phi_j) \in \mathsf{MF}_S(f)$ for each j, and we let $M_j := \mathsf{Coker}(\phi_j, \phi_j) \simeq (x, y^j)R$. Obviously $M_0 \simeq R$, $M_j \simeq M_{k+1-j}$ and $M_{(k+1)/2} \simeq N_+ \oplus N_-$. We would again like to show that $N_\pm$ and M_j $(0 \leq j \leq \frac{k-1}{2})$ are all indecomposable. For $N_\pm$ and M_0, it is clear since they are generated by a single element. For some $1 \leq j \leq \frac{k-1}{2}$, if we have $M_j \simeq A \oplus B$, then A and B must be generated by 1 element and non-free. Thus they are isomorphic to N_+ or N_-. However, $\mathfrak{I}(M_j) = (x, y^j)$ is not equal[1] to any of $\mathfrak{I}(N_+)$, $\mathfrak{I}(N_-)$ and $\mathfrak{I}(N_+) + \mathfrak{I}(N_-)$, a contradiction!

Now it follows again from (1.9) and (4.3) that

$$\tau(M_j) = M_j, \quad \tau(N_\pm) = N_\mp.$$

[1] Recall the definition of the ideal associated to a module in Section 3.1.

We then find the Auslander-Reiten sequence ending in these MCM R-modules in the same way as in the previous case. First consider the endomorphism on N_+ given by multiplication by $y^{\frac{k-1}{2}}$, which gives the Auslander-Reiten sequence ending in N_+:

$$0 \to N_- \to \mathsf{Coker}\begin{pmatrix} y^{(k+1)/2} - ix & y^{(k-1)/2} \\ 0 & y^{(k+1)/2} + ix \end{pmatrix} \to N_+ \to 0,$$

where the middle term is clearly isomorphic to $M_{\frac{k-1}{2}}$.

Similarly the Auslander-Reiten sequence ending in N_- is

$$0 \to N_+ \to M_{\frac{k-1}{2}} \to N_- \to 0.$$

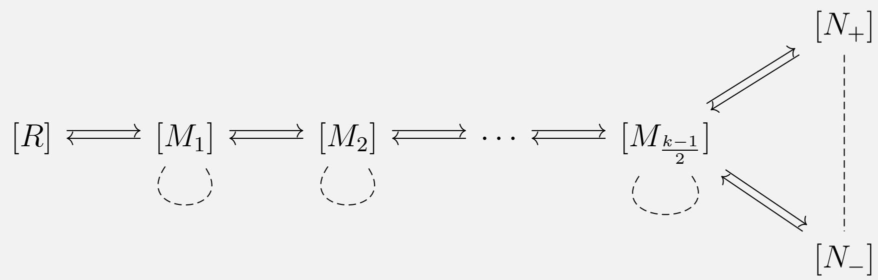
The Auslander-Reiten sequence ending in M_j does not change at all:

$$0 \to M_j \to M_{j-1} \oplus M_{j+1} \to M_j \to 0, \quad (1 \le j \le \frac{k-1}{2}),$$

which is given by $\begin{pmatrix} \phi_j & \epsilon_j \\ 0 & \phi_j \end{pmatrix}$, where $\epsilon = \begin{pmatrix} 0 & y^{j-1} \\ -y^{k-j} & 0 \end{pmatrix}$.

So we have obtained a part of $\Gamma(R)$:

Type A_k for odd $k \ge 1$:



We use the same argument to show that this is indeed a connected component of $\Gamma(R)$. In fact, when $k = 1$, $\mathfrak{m}_R \simeq N_+ \oplus N_-$; and when $k \ge 3$, $\mathfrak{m}_R \simeq M_1$. Finally, we apply Dieterich-Yoshino's theorem again to get that this is actually the whole quiver.

4.2.3 Type D_k for odd k

Now $R = K\{x, y\}/(x^2y + y^{k-1})$, where $k \ge 5$ is odd, and clearly we have matrix factorisations of $f = x^2y + y^{k-1}$ given by $(\alpha, \beta) := (y, x^2 + y^{k-2})$ and (β, α). Let $A := \mathsf{Coker}(\alpha, \beta)$ and $B := \mathsf{Coker}(\beta, \alpha)$.

For $0 \le j \le k-3$, we can find other matrix factorisations of f given by (ϕ_j, ψ_j) and (ξ_j, η_j), where

$$\phi_j = \begin{pmatrix} x & y^j \\ y^{k-j-2} & -x \end{pmatrix}, \quad \psi_j = \begin{pmatrix} xy & y^{j+1} \\ y^{k-j-1} & -xy \end{pmatrix},$$
$$\xi_j = \begin{pmatrix} x & y^j \\ y^{k-j-1} & -xy \end{pmatrix}, \quad \eta_j = \begin{pmatrix} xy & y^j \\ y^{k-j-1} & -x \end{pmatrix}.$$

Letting $M_j := \mathsf{Coker}(\phi_j, \psi_j)$, $N_j := \mathsf{Coker}(\psi_j, \phi_j)$, $X_j := \mathsf{Coker}(\xi_j, \eta_j)$ and $Y_j := \mathsf{Coker}(\eta_j, \xi_j)$, we see that

$$B \simeq M_0, \quad X_0 \simeq R \simeq Y_0, \quad N_0 \simeq A \oplus R, \quad X_{\frac{k-1}{2}} \simeq Y_{\frac{k-1}{2}},$$
$$M_j \simeq M_{k-j-2}, \quad N_j \simeq N_{k-j-2}, \quad X_j \simeq Y_{k-j-1}, \quad Y_j \simeq X_{k-j-1}, \ \forall 1 \leq j \leq k-3.$$

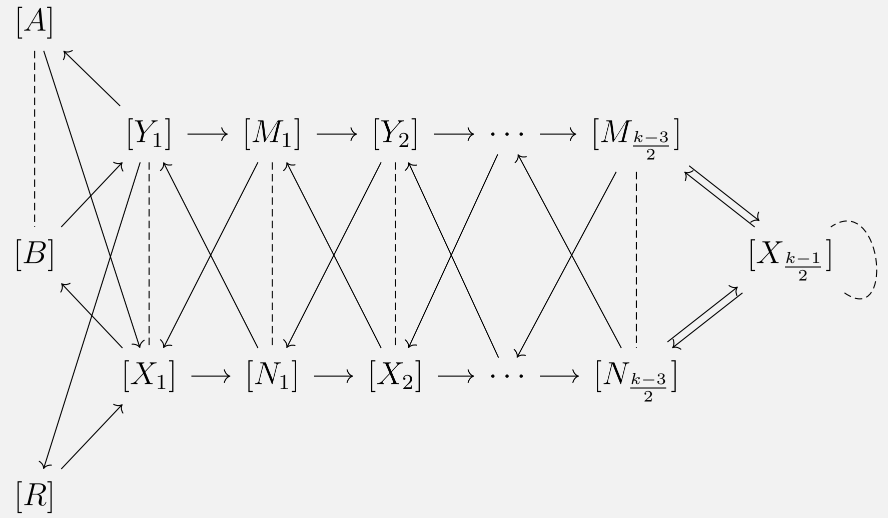
For $1 \leq j \leq k-3$, notice that M_j, N_j, X_j, Y_j are indecomposable MCM R-modules. In fact, $M_j \simeq (xy, y^{j+1})R$ and $Y_j \simeq (x, y^j)R$.

We thus obtain the Auslander-Reiten sequences, given by extensions below:

$$\begin{aligned}
\begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix} &: 0 \to A \to X_1 \to B \to 0, \\
\begin{pmatrix} \beta & x \\ 0 & \alpha \end{pmatrix} &: 0 \to B \to Y_1 \to A \to 0, \\
\begin{pmatrix} \phi_j & \epsilon_j \\ 0 & \psi_j \end{pmatrix} &: 0 \to M_j \to X_j \oplus Y_{j+1} \to N_j \to 0, \\
\begin{pmatrix} \psi_j & \epsilon_j \\ 0 & \phi_j \end{pmatrix} &: 0 \to N_j \to X_{j+1} \oplus Y_j \to M_j \to 0, \\
\begin{pmatrix} \xi_j & \epsilon_{j-1} \\ 0 & \eta_j \end{pmatrix} &: 0 \to X_j \to M_{j-1} \oplus N_j \to Y_j \to 0, \\
\begin{pmatrix} \eta_j & \epsilon_j \\ 0 & \xi_j \end{pmatrix} &: 0 \to Y_j \to M_j \oplus N_{j-1} \to X_j \to 0,
\end{aligned}$$

where $\epsilon_j = \begin{pmatrix} 0 & y^j \\ -y^{k-j-2} & 0 \end{pmatrix}$. Hence we get a connect component of the Auslander-Reiten quiver of R as follows:

Type D_k for odd $k \geq 5$:



This is indeed the whole quiver by Dieterich-Yoshino's theorem.

4.2.4 Type D_k for even k

Let $R = K\{x,y\}/(f)$ for $f = x^2y + y^{k-1}$ and $k \geq 4$ an even integer. We have seen in the type of A_k that the only change happens in the way we factorise f. The modules A, B, M_j, N_j, X_j, Y_j still well-defined, we furthermore let

$$C_{\pm} := \mathsf{Coker}(y(x \pm iy^{\frac{k-2}{2}}), x \mp iy^{\frac{k-2}{2}}), \quad D_{\pm} := \mathsf{Coker}(x \mp iy^{\frac{k-2}{2}}, y(x \pm iy^{\frac{k-2}{2}})).$$

We again have

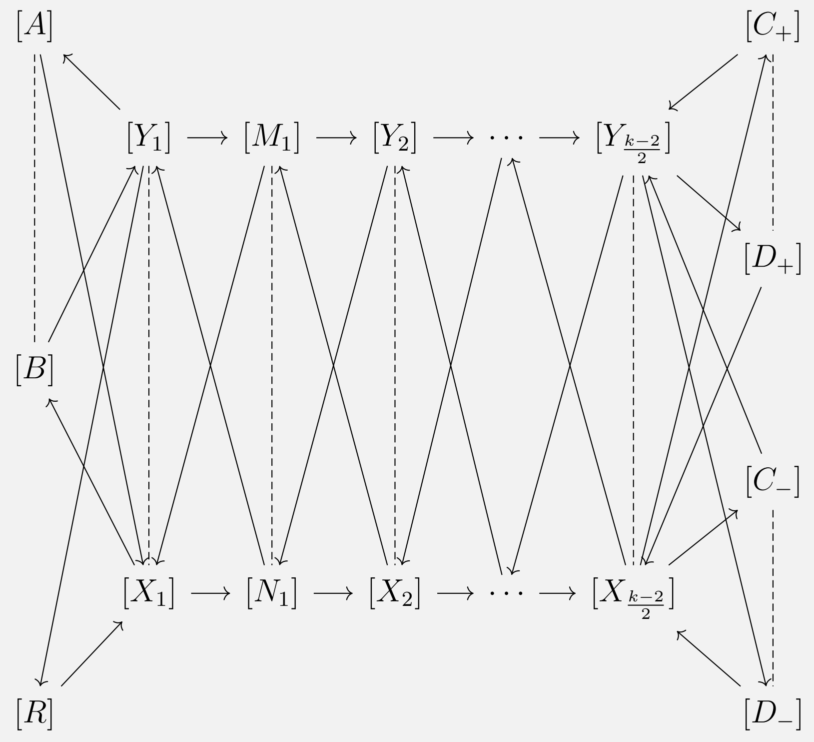
$$\begin{gathered} B \simeq M_0, \quad X_0 \simeq R \simeq Y_0, \quad N_0 \simeq A \oplus R, \quad X_{\frac{k-1}{2}} \simeq Y_{\frac{k-1}{2}}, \\ M_j \simeq M_{k-j-2}, \quad N_j \simeq N_{k-j-2}, \quad X_j \simeq Y_{k-j-1}, \quad Y_j \simeq X_{k-j-1}, \ \forall 1 \leq j \leq k-3. \end{gathered}$$

And moreover

$$M_{\frac{k-2}{2}} \simeq D_+ \oplus D_-, \quad N_{\frac{k-2}{2}} \simeq C_+ \oplus C_-.$$

The Auslander-Reiten sequences are obtained by the same extensions as when k is odd, and the Auslander-Reiten quiver turns out just slightly modified:

Type D_k for even $k \geq 4$:



4.2.5 Type E_6

In this case $R = K\{x, y\}/(f)$ with $f = x^3 + y^4$. The matrix factorisations of f are rather simple to calculate, as shown below row by row:

$$\phi_1 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix},$$
$$\phi_2 = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix},$$
$$\alpha = \begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}, \quad \beta = \begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}.$$

And we define the MCM R-modules as $A := \mathsf{Coker}(\alpha, \beta)$, $B := \mathsf{Coker}(\beta, \alpha)$, and $M_i := \mathsf{Coker}(\phi_i, \psi_i)$, $N_i := \mathsf{Coker}(\psi_i, \phi_i)$ for $i = 1, 2$, and there are following isomorphisms to ideals of R:

$$N_1 \simeq \mathfrak{m}_R, \quad M_1 \simeq (x^2, y)R, \quad N_2 \simeq (x^2, y^2)R \simeq M_2, \quad B \simeq (x^2, xy, y^2)R.$$

Further we can determine the action of τ since $\mathsf{rank}(A) = 2$:

$$\tau(M_1) = N_1, \tau(N_1) = M)1, \tau(M_2) = M_2, \tau(A) = B, \tau(B) = A.$$

We thus have the following Auslander-Reiten sequences:

$$\begin{pmatrix} \phi_1 & \begin{pmatrix} 0 & 1 \\ -xy^2 & 0 \end{pmatrix} \\ 0 & \psi_1 \end{pmatrix} : 0 \to M_1 \to A \to N_1 \to 0,$$
$$\begin{pmatrix} \psi_1 & \begin{pmatrix} 0 & x \\ -y^2 & 0 \end{pmatrix} \\ 0 & \phi_1 \end{pmatrix} : 0 \to N_1 \to B \oplus R \to M_1 \to 0.$$

On the other hand, observe that

$$\xi = \begin{pmatrix} \phi_2 & \begin{pmatrix} 0 & y \\ -xy & 0 \end{pmatrix} \\ 0 & \psi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \psi_2 & \begin{pmatrix} 0 & xy \\ y & 0 \end{pmatrix} \\ 0 & \phi_2 \end{pmatrix}$$

also gives an indecomposable matrix factorisation of f, and we let

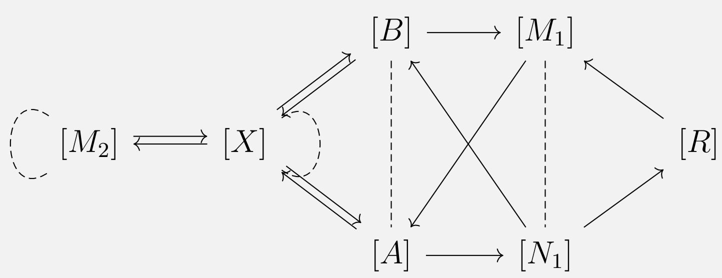
$$X := \mathsf{Coker}(\xi, \eta) \simeq \mathsf{Coker}(\eta, \xi)$$

to get the Auslander-Reiten sequence

$$0 \to M_2 \to X \to M_2 \to 0.$$

Thus we can easily get the Auslander-Reiten quiver as follows:

Type E_6:



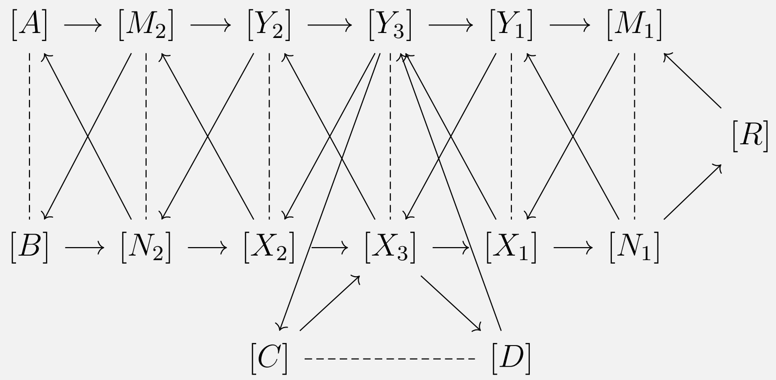
4.2.6 Type E_7

In this case $f = x^3 + xy^3$ can be factorised as follows:

$$\alpha = (x), \qquad \beta = (x^2 + y^3);$$
$$\gamma = \begin{pmatrix} x^2 & xy \\ xy^2 & -x^2 \end{pmatrix}, \qquad \delta = \begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix};$$
$$\phi_1 = \begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix};$$
$$\phi_2 = \begin{pmatrix} x & y^2 \\ xy & -x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ xy & -x \end{pmatrix};$$
$$\xi_1 = \begin{pmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{pmatrix}, \qquad \eta_1 = \begin{pmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{pmatrix};$$
$$\xi_2 = \begin{pmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{pmatrix}, \qquad \eta_2 = \begin{pmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{pmatrix};$$
$$\xi_3 = \begin{pmatrix} \gamma & \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \\ 0 & \delta \end{pmatrix}, \qquad \eta_3 = \begin{pmatrix} \delta & \begin{pmatrix} -y & 0 \\ 0 & -y \end{pmatrix} \\ 0 & \gamma \end{pmatrix}.$$

Now we define the MCM modules as $A := \mathsf{Coker}(\alpha, \beta)$, $B := \mathsf{Coker}(\beta, \alpha)$, $C := \mathsf{Coker}(\gamma, \delta)$, $D := \mathsf{Coker}(\delta, \gamma)$, and $M_i := \mathsf{Coker}(\phi_i, \psi_i)$, $N_i := \mathsf{Coker}(\psi_i, \phi_i)$, $X_i := \mathsf{Coker}(\xi_i, \eta_i)$, $Y_i := \mathsf{Coker}(\eta_i, \xi_i)$ for $i = 1, 2$, so that we can obtain the Auslander-Reiten quiver similarly as in previous type, shown below:

Type E_7:



4.2.7 Type E_8

Finally let $R = K\{x, y\}/(f)$ for $f = x^3 + y^5$. The higher degree makes matrix factorisation a bit harder, but the process to calculate the quiver does not change at all. We exhibit below all indecomposable matrix factorisations of f from Yoshino [1990], p. 82:

$$\phi_1 = \begin{pmatrix} x & y \\ y^4 & -x^2 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} x^2 & y \\ y^4 & -x \end{pmatrix};$$

$$\phi_2 = \begin{pmatrix} x & y^2 \\ y^3 & -x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} x^2 & y^2 \\ y^3 & -x \end{pmatrix};$$

$$\alpha_1 = \begin{pmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^3 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{pmatrix};$$

$$\alpha_2 = \begin{pmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} y^4 & xy^2 & x^2 \\ -x^2 & y^3 & xy \\ -xy^2 & -x^2 & y^3 \end{pmatrix};$$

$$\gamma_1 = \begin{pmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{pmatrix};$$

$$\gamma_2 = \begin{pmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{pmatrix};$$

$$\xi_1 = \begin{pmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{pmatrix},$$

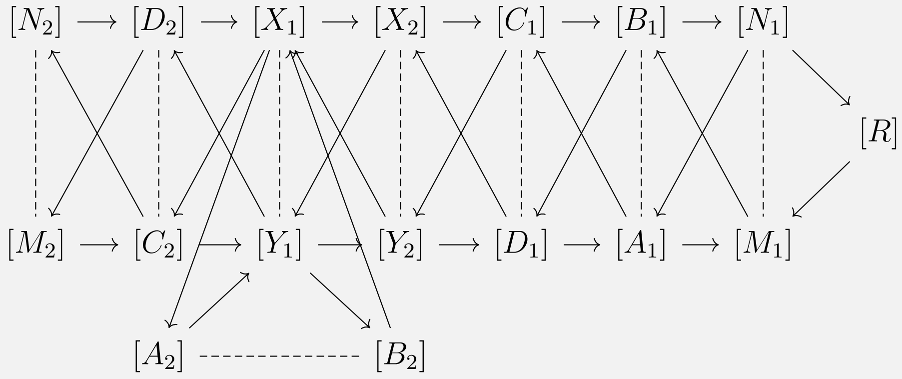
$$\eta_1 = \begin{pmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{pmatrix};$$

$$\xi_2 = \begin{pmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{pmatrix},$$

$$\eta_2 = \begin{pmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{pmatrix}.$$

Denote by $M_i := \mathsf{Coker}(\phi_i, \psi_i)$, $N_i := \mathsf{Coker}(\psi_i, \phi_i)$, $A_i := \mathsf{Coker}(\alpha_i, \beta_i)$, $B_i := \mathsf{Coker}(\beta_i, \alpha_i)$, $C_i := \mathsf{Coker}(\gamma_i, \delta_i)$, $D_i := \mathsf{Coker}(\delta_i, \gamma_i)$, $X_i := \mathsf{Coker}(\xi_i, \eta_i)$, $Y_i := \mathsf{Coker}(\eta_i, \xi_i)$, where $i = 1, 2$, and then the Auslander-Reiten quiver can be drawn as:

Type E_8:



So far we have calculated the Auslander-Reiten quiver of any 1-dimensional simple singularity, and in particular we see that any 1-dimensional simple singularity is MCM-finite, hence by Knörrer's periodicity the same holds for any odd-dimensional simple singularity.

Also by Knörrer's periodicity, the stable Auslander-Reiten quiver of any odd-dimensional simple singularity is isomorphic to the 1-dimensional case computed above, and the Auslander-Reiten quiver changes only in terms of the number of arrows going into and out of $[R]$, the vertex of free module.

Iyama's notation. Given a quiver Δ, we define the *extended quiver* of Δ as a new quiver $\mathbb{Z}\Delta$, whose vertices are points in $\mathbb{Z} \times \Delta_0$ and whose arrows are $(l, a) : (l, s(a)) \to (l, t(a))$ and $(l, a^*) : (l, t(a)) \to (l + 1, s(a))$ for each $l \in \mathbb{Z}$ and $a \in \Delta_1$.

This notation is widely used in Dieterich and Wiedemann [1986], and equips the following result presented in Iyama [2018], Prop. 2.24:

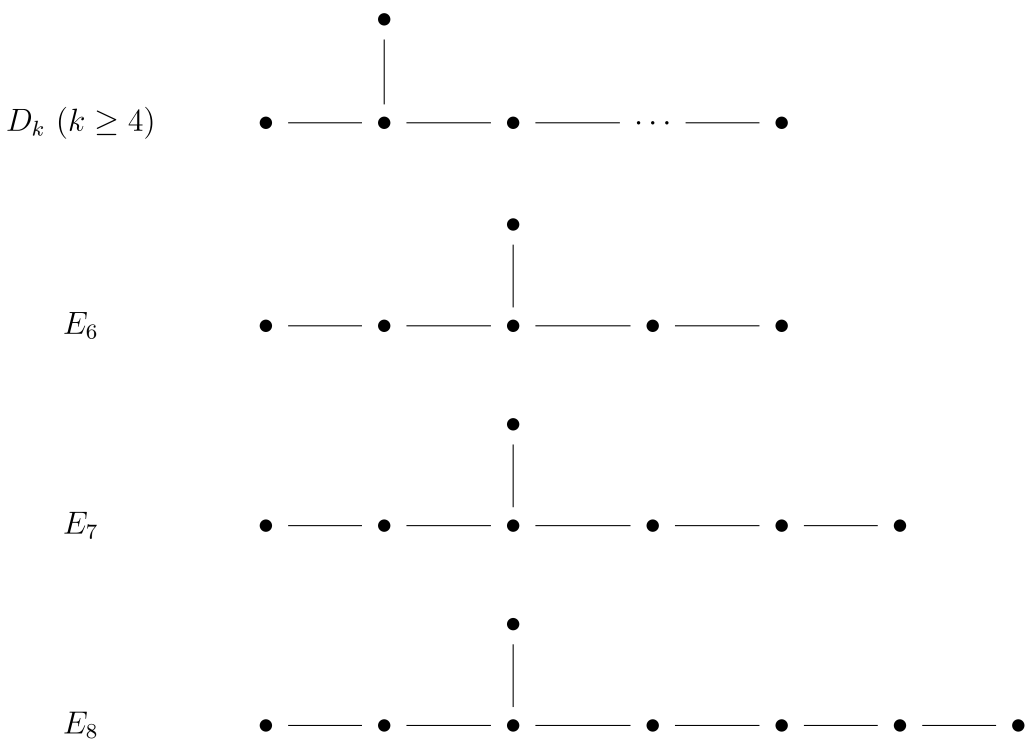
Let R be a simple singularity with $\mathsf{dim}(R) = d$. Then the Auslander-Reiten quiver of $\underline{\mathsf{MCM}}(R)$ is $\mathbb{Z}\Delta/\phi$, where Δ and ϕ are given as follows:

- if d is even, then Δ is the Dynkin diagram of the same type as R, and $\phi = \tau$ is the automorphism corresponding to the Auslander-Reiten translation;
- if d is odd, then

R	A_{2k-1}	A_{2k}	D_{2k}	D_{2k+1}	E_6	E_7	E_8
Δ	D_{k+1}	A_{2k}	D_{2k}	A_{4k-1}	E_6	E_7	E_8
ϕ	$\tau \circ \iota$	$\tau^{\frac{1}{2}}$	τ^2	$\tau \circ \iota$	$\tau \circ \iota$	τ^2	τ^2

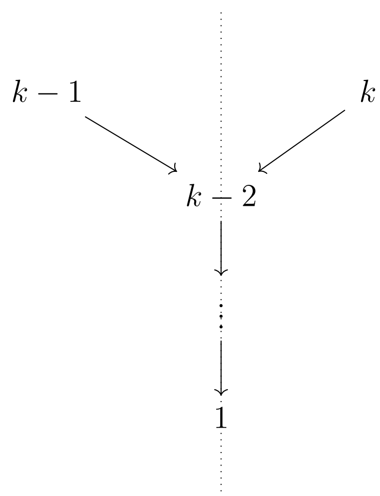
  where ι is the involution of $\mathbb{Z}\Delta$ induced by the non-trivial involution of Δ, and $\tau^{\frac{1}{2}}$ is the automorphism of $\mathbb{Z}A_{2k}$ such that $(\tau^{\frac{1}{2}})^2 = \tau$.

As a reminder, the Dynkin quivers concerned are listed below as well:



We shall briefly remark on this concise version of the result. For even-dimensional cases, we have seen in Section 4.1 that the stable Auslander-Reiten quiver equals $\overleftrightarrow{\Delta}$, where Δ is the Dynkin diagram of the same type as R. Meanwhile the automorphism τ on $\mathbb{Z}\Delta$ corresponding to the Auslander-Reiten translation is given by $(l, v) \to (l-1, v)$ for any $(l, v) \in \mathbb{Z} \times \Delta_0$. Therefore, the orbit quiver $\mathbb{Z}\Delta/\tau$ is nothing but $\overleftrightarrow{\Delta}$, obtained by adding an inverse arrow a^* for any $a \in \Delta_1$.

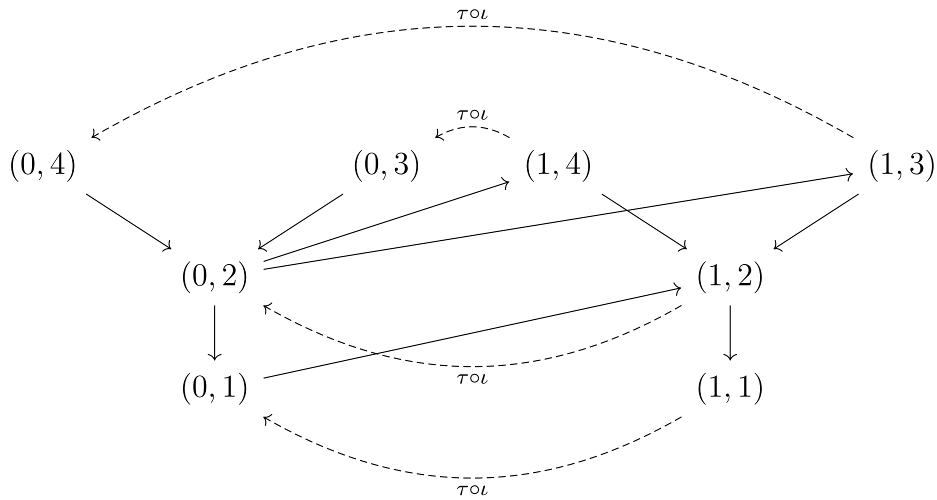
For odd-dimensional cases, we first describe the involution ι. For D_k, where $k \geq 4$, notice that the quiver is axisymmetric:



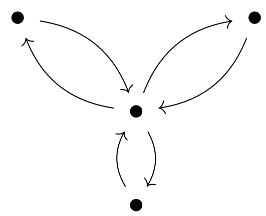
Therefore there is a natural involution ι on D_k given by the 2-cycle $(k-1\ k)$, which swaps the vertices $k-1$ and k while keeps the rest of the vertices fixed.

For A_k where k is odd, the quiver is also symmetric, and we can similarly define ι to be the reflection at the middle vertex of symmetry. For E_6, ι is defined as the reflection on the axis given by the 2 middle vertices. It is clear that the involution ι on any quiver Δ naturally induces an involution on $\mathbb{Z}\Delta$, still denoted by ι, which is given by $\iota(l, v) = (l, \iota(v))$, $\forall (l, v) \in \mathbb{Z} \times \Delta_0$.

For example, consider the type A_5 which has stable Auslander-Reiten quiver $\mathbb{Z}D_4/(\tau \circ \iota)$ according to Iyama's statement. The quiver $\mathbb{Z}D_4$ has the following form with solid arrows:



where the dashed arrows illustrate the action of $\tau \circ \iota$. Therefore we get a new quiver of the form



and one can verify that it coincides with the result calculated in Section 4.2.

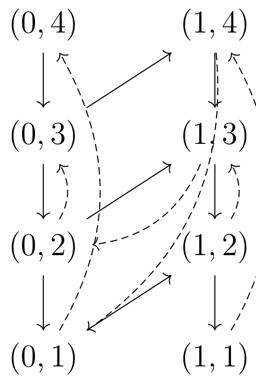
Then we would like to specify the automorphism $\tau^{\frac{1}{2}}$ of $\mathbb{Z}A_{2k}$. In fact, A_{2k} is also axisymmetric, which motivates us to define, for any $l \in \mathbb{Z}$,

$$\tau^{\frac{1}{2}}(l, i) = \begin{cases} (l, 2k+1-i), & \text{if } 1 \leq i \leq k; \\ (l-1, 2k+1-i), & \text{if } k+1 \leq i \leq 2k. \end{cases}$$

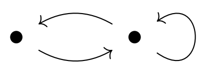
Here we number the vertices of A_{2k} as follows:

$$2k \longrightarrow 2k-1 \longrightarrow \cdots \longrightarrow k+1 \longrightarrow k \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

For example, consider the type A_4 which has stable Auslander-Reiten quiver according to Iyama's statement $\mathbb{Z}A_4/\tau^{\frac{1}{2}}$. We again draw the quiver as follows:



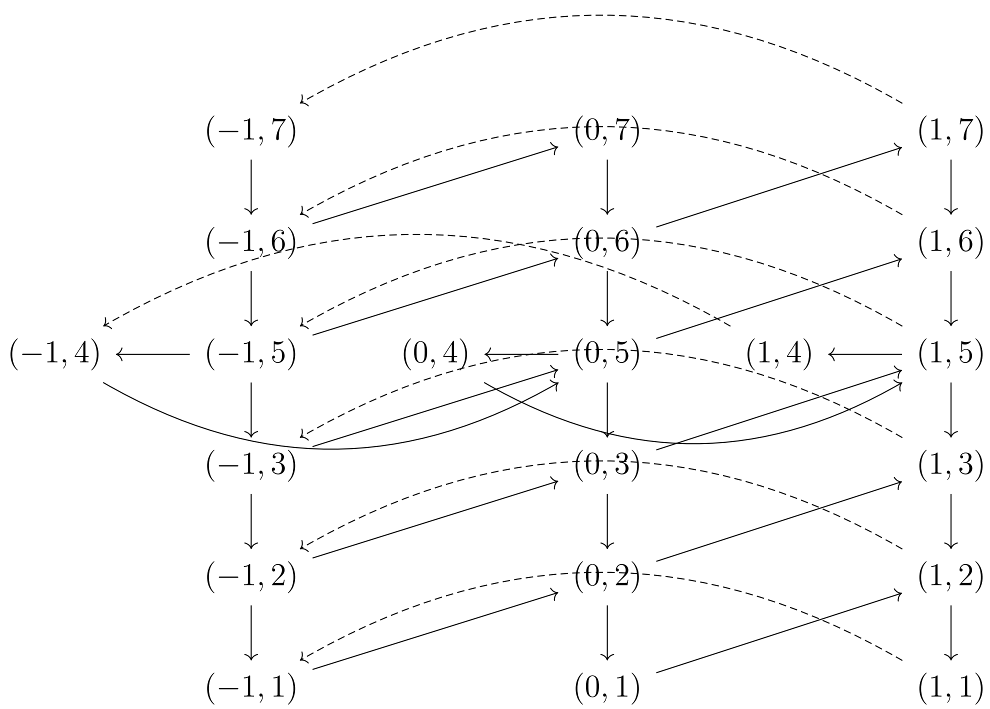
where the dashed arrows denote the action of $\tau^{\frac{1}{2}}$. Therefore we get the following quiver:



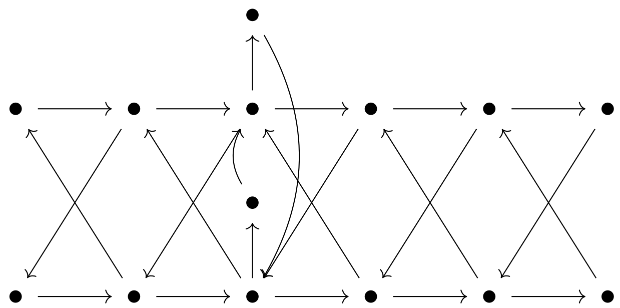
and one can verify that this coincides with the our calculation in Section 4.2.

Finally the automorphism τ^2 is clearly given by $\tau^2(l, v) = (l - 2, v)$ for any $(l, v) \in \mathbb{Z} \times \Delta_0$.

For example, consider the type E_7 which has stable Auslander-Reiten quiver $\mathbb{Z}E_7/\tau^2$ according to Iyama's statement. This quiver is drawn below:



where the dashed arrows denote the action of τ^2. Therefore we get the following quiver:



and one can verify that this coincides with the result we have presented in Section 4.2.

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List of Symbols

$\mathbb{N}$	set of natural numbers including 0
$\mathsf{mod}(R)$	category of finitely generated modules over ring R
$\mathsf{Mod}(R)$	category of all modules over ring R
$\mathfrak{m}_R$	maximal ideal of local ring R
κ_R	residue field of local ring R
$\mathsf{dim}(R)$	Krull dimension of ring R
$\mathsf{dim}_R(M)$	Krull dimension of R-module M
$\mathsf{Ann}(M)$	annihilator of R-module M
$[L : K]$	degree of field extension $K \subset L$
$\mathsf{gl.dim}(R)$	global dimension of ring R
$\mathsf{proj.dim}(M_R)$	projective dimension of R-module M
$\mathsf{inj.dim}(M_R)$	injective dimension of R-module M
$\mathsf{depth}(M)$	depth of R-module M
$\mathsf{rank}_R(M)$	rank of module M over domain R
$l(M)$	length of R-module M
$K\{x_1, \cdots, x_n\}$	convergent power series ring over valued field K
$K[[x_1, \cdots, x_n]]$	formal power series ring over field K
$\mathsf{MCM}(R)$	category of MCM R-modules
$\mathsf{syz}^n_R(M)$	reduced n-th syzygy of R-module M
ω_R	canonical module over R
$S^{m\times n}$	$m \times n$ matrix ring over ring S
$\mathsf{MF}_S(f)$	category of matrix factorisations of $f \in \mathfrak{m}_S$
$R^\times$	set of invertible elements in ring R
$\underline{\mathcal{C}}$	stable category of category $\mathcal{C}$
$\mathcal{C}/\mathcal{I}$	quotient category of category $\mathcal{C}$ by its ideal $\mathcal{I}$
$\mathsf{rad}(R)$	Jacobson radical of ring R
$\tau(M)$	Auslander-Reiten translation of M
M'	dual of R-module M by R
M^*	dual of R-module M by ω_R
$\mathsf{Tr}(M)$	Auslander transpose of M
$\mathsf{rad}_R(M, N)$	R-module of radical morphisms $M \to N$
$\mathsf{Irr}_R(M, N)$	R-module of irreducible morphisms $M \to N$
$i(M, N)$	dimension of $\mathsf{Irr}_R(M, N)$ over κ_R
$\Gamma(R)$	Auslander-Reiten quiver of $\mathsf{MCM}(R)$
$\underline{\Gamma}(R)$	stable Auslander-Reiten quiver of $\mathsf{MCM}(R)$
$c(f)$	set of ideals I of S such that $f \in I^2$
$\mathfrak{I}(M)$	ideal associated to module M, see Section 3.1
$\beta_n(\kappa_R)$	n-th Betti number of κ_R
A_k, D_k, E_6, E_7, E_8	ADE Dynkin quivers
$\mathsf{Mc}(V, G)$	McKay graph of group G
$\mathbb{Z}\Delta$	extended quiver of quiver Δ
$\overleftrightarrow{\Delta}$	quiver obtained from Δ by making all arrows invertible

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