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Toric Varieties and Their Applications

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I am very thankful to my supervisor who, even under stressful circumstances, was willing to provide useful critique and pointed out many mistakes and inconsistencies throughout earlier versions of my work.

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Abstract: The thesis provides an introduction into the theory of affine and abstract toric varieties. In the first chapter, tools from algebraic geometry indispensable for the comprehension of the topic are introduced. Many properties of convex polyhedral cones and affine toric varieties are proven and discussed in detail as is the deep connection between the two objects. The second chapter establishes the notion of an abstract variety and translates obtained results to this more general setting, giving birth to the theory of abstract toric varieties and the closely associated theory of fans. Finally, an algorithmic approach to the resolution of singularities on toric surfaces and its relation to continued fractions is revealed.

Keywords: Algebraic Geometry, Toric Varieties, Singularities, Convex Polyhedral Cones

Contents

Preface

The goal of the thesis is to present the intriguing theory of toric varieties in a way that stresses both their geometric and combinatorial qualities and their mutual inseparability.

The thesis is, in principle, a conjunction of selected sections from Cox et al. [2011], Fulton [1993] and Oda [1985]. These works certainly have a non-trivial intersection as they present the basics of the same theory. Nonetheless, the, often stark, differences in approach, employed methods and style of expression certainly do set them apart. I have often found myself struggling to understand a topic in one of them, only to found it nearly trivial in another.

With that said, the statements (and some proofs thereof) are largely borrowed from the first, and the most detailed, book by Cox et al. [2011]. This has two reasons. One is consistency. The other is the sheer number of exercises and purposefully excluded details in proofs making it into an ideal source material for a work such as this.

I always tend to write with more emphasis on didactics and literariness than is normal, and than is perhaps appropriate. This work of mine, being no exception to the rule, teems with passages meant to elucidate or furnish intuition about the discussed notion or to simply make it more linguistically aureate. These are never intended as rigorous proofs and I kindly ask they be treated as such. Of course, rigorous proofs are very much present and, unless explicitly specified, are original. Up to a small number of exceptions, all those that are cited have been augmented by inserting additional details omitted in the original text.

1. Toric Varieties

As custom dictates, the first chapter assumes the introductory rôle. Neither the story's protagonist nor its antagonist, it serves to present key general results from algebraic geometry employed heavily throughout the text, to define toric varieties and to showcase their essential properties.

1.1 Background in Algebraic Geometry

We consider most definitions and results contained in this section, standard, as they form contents of any introductory book on algebraic geometry. As such, they will not be cited separately and often left unproven, save findings of particular utility to presented theory. Contents of this section can be found in the same or slightly altered form in Fulton [2008], Chapters 1, 2 & 6.

We limit ourselves in our efforts to the field of complex numbers. To much chagrin, the 'visually appealing' real numbers are left to their fate for not being, quite unlike their complex counterparts, algebraically closed. This fact introduces very many technical difficulties and prevents usage of several fundamental theorems. With that being said, in this and the following sections, all constructs are made over the complex numbers.

1.1.1 Basic Notions and Hilbert's Nullstellensatz

Algebraic geometry studies polynomials and the sets of their roots. Readers should be already well-acquainted with the ensuing definition.

Definition 1.1.1 (Affine Algebraic Set)**.** Given an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$, we denote

$$\mathbf{V}(I) \coloneqq \{P \in \mathbb{C}^n \mid f(P) = 0 \ \forall f \in I\}.$$

We call it the *affine algebraic set generated by* I or, seldom, the set of common zeroes of I. For each affine algebraic set V, we also define

$$\mathbf{I}(V) \coloneqq \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(P) = 0 \ \forall P \in V\},$$

the set of polynomials that vanish on V. It is easily shown to be an ideal of $\mathbb{C}[x_1, \dots, x_n]$ and is simply called the *ideal of* V.

Note that in general $\mathbf{V}(\mathbf{I}(V)) = V$ but $\mathbf{I}(\mathbf{V}(I)) \neq I$. Take for instance $I = (x_1^k)$ for any $k > 1$. We get

$$\mathbf{V}(I) = \{P \in \mathbb{C}^n \mid x_1^k(P) = 0\} = \{(0, p_2, \dots, p_n) \in \mathbb{C}^n\}.$$

However, the polynomials vanishing at points $(0, p_2, \dots, p_n)$ are precisely the multiples of x_1, in other words, those lying in (x_1). It follows that $\mathbf{I}(\mathbf{V}(I)) =$

$(x_1) \neq (x_1^k) = I$. This particular ideal is illustrative in the sense that, intuitively, elevating a polynomial does not create any new roots. The formalization thereof, broadly speaking, is the content of the famous Hilbert's Nullstellensatz, which we now recall.

Theorem 1.1.2 (Nullstellensatz)**.** *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Then, $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$ where*

$$\sqrt{I} \coloneqq \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^k \in I \text{ for some } k \in \mathbb{N}\}.$$

In particular, it is easily shown that $\mathbf{I}(V) = \sqrt{\mathbf{I}(V)}$ for every affine algebraic set V, and thus $\mathbf{I}(\mathbf{V}(\mathbf{I}(V))) = \mathbf{I}(V)$.

Remark. The Nullstellensatz relies implicitly but heavily on the fact that the ring $\mathbb{C}[x_1, \ldots, x_n]$ is noetherian, its every ideal finitely generated. This has an especially interesting interpretation in the case of ideals generated by affine algebraic sets in the plane. No matter how many points I choose, each polynomial curve passing through all of them is a combination of an adequate finite set of polynomial curves.

It is often said that Hilbert's Nullstellensatz establishes a link between algebra and geometry. That link assumes the shape of a Galois correspondence (via $\mathbf{I}$ and $\mathbf{V}$) between affine algebraic sets $V \subseteq \mathbb{C}^n$ and ideals satisfying $I = \sqrt{I}$, so called radical ideals.

In our exploration of algebraic sets, we often find it easier to study their elementary constituents. This leads us to the idea of reducibility. We call an affine algebraic set *reducible* if V is non-empty and $V = V_1 \cup V_2$ for non-trivial affine algebraic sets $V_1, V_2 \subsetneq V$. The key insight here is that irreducible algebraic sets give prime ideals.

Proposition 1.1.3. *Let $V \subseteq \mathbb{C}^n$ be an affine algebraic set. Then, V is irreducible if and only if $\mathbf{I}(V) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal.*

We will, as is often done, call irreducible affine algebraic sets, *affine varieties.* The content of the last proposition quickly becomes crucial as we move on to coordinate rings.

Definition 1.1.4 (Coordinate Ring)**.** Let V be an affine algebraic set. We define

$$\mathbb{C}[V] \coloneqq \mathbb{C}[x_1, \ldots, x_n]/\mathbf{I}(V).$$

and call it the *coordinate ring* of V. If V is an affine variety, then $\mathbf{I}(V)$ is prime and $\mathbb{C}[V]$ is a domain.

The coordinate ring is often interpreted as the ring of polynomial functions from V to $\mathbb{C}$. This interpretation is justified by the fact that complex polynomials f and g define the same function on points of V if and only if $f - g \in \mathbf{I}(V)$. Hence, the coset $f + \mathbf{I}(V) \in \mathbb{C}[V]$ can be considered a map $\varphi : V \to \mathbb{C}$ with $\varphi(P) := f(P)$ for all $P \in V$.

Coordinate rings also offer new ways of looking at our favourite set V. Instead of keeping things simple and settle for points in space, we can, as we mathematicians love doing, complicate things a notch and identify V with the set of maximal ideals of $\mathbb{C}[V]$. How, you ask? For every $P \in V$, consider the set

$$M_P := \{f \in \mathbb{C}[V] \mid f(P) = 0\}.$$

It is clear that $M_P \subseteq \mathbb{C}[V]$. We will show that M_P is a maximal ideal. Since we have not proven anything yet and cheekily assumed potential readers do not expect us to lie, we might as well give it a shot and prove a result of a little farther reaching importance.

We need to make use of one additional statement, though, widely known as the Weak Nullstellensatz. It seems apt to recall it, as well.

Theorem 1.1.5 (Weak Nullstellensatz)**.** *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal satisfying $\mathbf{V}(I) = \emptyset$. Then, $I = \mathbb{C}[x_1, \ldots, x_n]$.*

Proposition 1.1.6. *Let V be an affine algebraic set. Then, for every point $P \in V$, the set M_P is a maximal ideal of $\mathbb{C}[V]$. Moreover, all maximal ideals of $\mathbb{C}[V]$ are of this form.*

Proof. Firstly, M_P is of course an ideal. Given $f, g \in M_P$, we get $(f + g)(P) = f(P) + g(P) = 0$ and also $(h \cdot f)(P) = h(P) \cdot f(P) = 0$ for $h \in \mathbb{C}[V]$.

Moving further, we find an isomorphism $\mathbb{C}[V]/M_P \simeq \mathbb{C}$. Since $\mathbb{C}$ is a field, existence of such an isomorphism implies that M_P is maximal. Let us thus define a map $\varphi : \mathbb{C}[V] \to \mathbb{C}$ by $\varphi(f + \mathbf{I}(V)) := f(P)$. Clearly, $\ker \varphi = M_P$ since $f(P) = 0$ exactly when $f + \mathbf{I}(V) \in M_P$. Also, $\operatorname{im} \varphi = \mathbb{C}$ trivially because the cosets of all constant polynomials lie in $\mathbb{C}[V]$. First Isomorphism Theorem now ascertains that indeed $\mathbb{C}[V]/M_P \simeq \mathbb{C}$. Finally, this construction is valid for any choice of P, so M_P is a maximal ideal of $\mathbb{C}[V]$ for every $P \in V$.

To prove the third and final statement, we employ the Weak Nullstellensatz. Let $\overline{I} \subsetneq \mathbb{C}[V]$ be a maximal ideal. Let I be its pre-image via the canonical projection $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[V]$. Suppose for contradiction that $\overline{I} \neq M_P$ for all $P \in V$. From maximality of $\overline{I}$, we have that also $\overline{I} \nsubseteq M_P$ for all $P \in V$. Translating this into a more common language, for each $P \in V$, there is some function $f \in \overline{I}$ which is non-zero at P. This is because M_P is the set of all polynomial functions $V \to \mathbb{C}$ which vanish at P. Since $\overline{I}$ is not contained in any of them, there can be no $P \in V$ which is a common zero of all polynomial functions in $\overline{I}$. This remains true for its pre-image I as well. If we denote $J := \mathbf{I}(V)$, by the last statement, we get $\mathbf{V}(I) \cap \mathbf{V}(J) = \emptyset$. Note that $\mathbf{V}(I) \cap \mathbf{V}(J) = \mathbf{V}(I + J)$. By the Weak Nullstellensatz, $I + J = \mathbb{C}[x_1, \ldots, x_n]$. Projecting back to $\mathbb{C}[V]$, this equality becomes $\overline{I} + \overline{J} = \mathbb{C}[V]$. However, $\overline{J}$ is 0 in $\mathbb{C}[V]$. So, we get $\overline{I} = \mathbb{C}[V]$, a contradiction with the choice $\overline{I}$ as a maximal ideal of $\mathbb{C}[V]$. □

Remark. For the sake of rigorousness, it should be mentioned that we regarded $\mathbb{C}[V]$ as the ring of functions $V \to \mathbb{C}$ without warning. We will henceforth make use of this identification whenever suitable and will not mention it explicitly again.

The relation between V and $\mathbb{C}[V]$ is most succinctly expressed by

$$V = \operatorname{Specm}(\mathbb{C}[V]),$$

where $\operatorname{Specm}(\mathbb{C}[V])$ denotes the set of maximal ideals of $\mathbb{C}[V]$, the so-called *maximal spectrum* of $\mathbb{C}[V]$.

The introduction into affine algebraic sets would not be complete without our being able to move between them. As affine algebraic are defined through polynomials, it is hopefully not too astonishing a revelation that 'nice maps' between them are exactly polynomial maps. To adhere to the nomenclature of category theory, we call a polynomial map between affine algebraic sets, a *morphism*.

Given that coordinate rings are really complex functions defined on affine algebraic sets, any morphism $f : V \to W$ induces a homomorphism of $\mathbb{C}$-algebras $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ given by $\varphi \mapsto \varphi \circ f$.

In contrast to the same notion in the theory of groups or rings, an *isomorphism* between affine algebraic sets is not simply a bijective morphism. Actually, it is a stronger notion. We call a morphism $f : V \to W$ an *isomorphism* if we can find a morphism $g : W \to V$ such that $f \circ g = 1_W$ and $g \circ f = 1_V$ where 1_V denotes the identity map on V. Expressed differently, an isomorphism between affine algebraic sets is a bijective polynomial map whose inverse is also a bijective polynomial map.

The reason for this definition is partially given by the following result.

Proposition 1.1.7. *Two affine algebraic sets V and W are isomorphic if and only if their coordinate rings $\mathbb{C}[V]$ and $\mathbb{C}[W]$ are isomorphic as $\mathbb{C}$-algebras.*

See **Example 1.1.19** for a setting where a bijective morphism which is not an isomorphism would assimilate two affine varieties that we wish not consider similar in a strong sense.

1.1.2 Zariski Topology

In later chapters, we will be engrossed in smoothing out wrinkled varieties. To even start defining the necessary tools, we would very much like to have a topology on affine varieties and, somewhat surprisingly, everyone's favourite pick, the classical topology on $\mathbb{C}^n$, does not quite cut it.

In comes the *Zariski* topology, the new kid on the block. It is defined specifically to serve the purposes of algebraic geometry. By definition, closed sets are precisely affine algebraic sets. As such, the Zariski topology covers the entirety of $\mathbb{C}^n$ and is then inherited by affine algebraic sets where closed sets are their subsets which are themselves affine algebraic sets.

It is easy to check that $\mathbf{V}(I) \cup \mathbf{V}(J) = \mathbf{V}(IJ)$ and $\mathbf{V}(I) \cap \mathbf{V}(J) = \mathbf{V}(I+J)$. This confirms that affine algebraic sets are closed under finite unions and arbitrary intersections, and can thus be used to define a topology. Also observe that we have $\mathbf{V}(0) = \mathbb{C}^n$ and that, by the Weak Nullstellensatz, $\mathbf{V}(\mathbb{C}[x_1, \ldots, x_n]) = \emptyset$, so $\emptyset$ and $\mathbb{C}^n$ are closed.

As a matter of fact, it holds that $\mathbf{V}(I) = \mathbb{C}^n$ *if and only if* $I = 0$. With the leftward implication being obvious, we expand on the rightward one as we need make use of it before long.

Lemma 1.1.8. *If $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then $\mathbf{V}(I) = \mathbb{C}^n$ if and only of $I = 0$.*

Proof. We already observed that $\mathbf{V}(0) = \mathbb{C}^n$.

The converse will be proven by induction on n. We will show that only the zero polynomial can vanish on the entirety of $\mathbb{C}^n$. If $n = 1$, then any non-zero $f \in \mathbb{C}[x]$ has only as many roots as is its degree, in particular, finitely many. Hence $\mathbf{V}(I) = \mathbb{C}$ enforces $I = 0$.

For $n > 1$, choose $f \in \mathbb{C}[x_1, \ldots, x_n]$ and write $f = \sum_{i=1}^{k} f_i x_n^i$ for polynomials $f_i \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. Since f is a polynomial in one variable over $\mathbb{C}[x_1, \ldots, x_{n-1}]$, then under the assumption that $f_i(p_1, \ldots, p_{n-1}) \neq 0$ for at least one i, there are only finitely many $p_n \in \mathbb{C}$ satisfying $f(p_1, \ldots, p_n) = 0$. So, the only way for every point P to be a root of f is to require that $f_i(p_1, \ldots, p_{n-1}) = 0$ for all $(p_1, \ldots, p_{n-1}) \in \mathbb{C}^{n-1}$, in other words, to require that $\mathbf{V}(f_1, \ldots, f_k) = \mathbb{C}^{n-1}$. From the induction hypothesis, we get $f_1 = \ldots = f_k = 0$, hence also $f = 0$ which completes the proof. □

It is beneficial to note that the Zariski topology indeed differs from the classical one. To list just one of those differences, it is not Hausdorff. This can be seen with fairly little work. Let us first label $\mathbf{D}(I)$ the complement of $\mathbf{V}(I)$ for a given ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. That is, if

$$\mathbf{V}(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \; \forall f \in I\},$$

then $\mathbf{D}(I)$ can be described as

$$\mathbf{D}(I) = \{x \in \mathbb{C}^n \mid \exists f \in I \; f(x) \neq 0\}.$$

These sets, which do not have any particular denomination as far as we are aware, are of course open in the Zariski topology. It is quite easy to see that any two non-empty Zariski open subsets of $\mathbb{C}^n$ always intersect. Let $I, J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be any ideals. Notice that $\mathbf{D}(I) \cap \mathbf{D}(J) = \mathbf{D}(IJ)$. By **Lemma 1.1.8**, $\mathbf{V}(H) = \mathbb{C}^n$ if and only if $H = 0$. By definition, $\mathbf{D}(IJ) = \emptyset$ when $\mathbf{V}(IJ) = \mathbb{C}^n$ and so $IJ = 0$. From this, it follows that also either $I = 0$ or $J = 0$. The conclusion is that

$\mathbf{D}(I) \cap \mathbf{D}(J) = \emptyset$ only in case either open set is empty. So, the Zariski topology on $\mathbb{C}^n$ cannot be Hausdorff.

In the rare case that 'not being Hausdorff' is not quite enough to drown any topological intuition dear readers possess, the following behaviour open sets in the Zariski topology exhibit and those in the classical topology, of course, do not, will surely finish the job. Despite being counter-intuitive, the fact we are about to reveal is indispensable for the theory of toric varieties as becomes transparent in the very start of **Section 1.3**.

Proposition 1.1.9. *Let* $V \coloneqq \mathbf{V}(f_1, \ldots, f_k)$ *and for a non-zero* $g \in \mathbb{C}[V]$ *denote* $V_g \coloneqq V \setminus \mathbf{V}(g) = \{P \in V \mid g(P) \neq 0\}$*. Then, there is a homeomorphism with respect to the Zariski topology*

$$V_g \leftrightarrow W \coloneqq \mathbf{V}(f_1, \ldots, f_k, 1 - x_{n+1}g).$$

Proof. The natural map to consider is

$$\begin{aligned} \pi : W &\to V_g, \\ (p_1, \ldots, p_{n+1}) &\mapsto (p_1, \ldots, p_n). \end{aligned}$$

We show that it is bijective. If $(p_1, \ldots, p_n) = (r_1, \ldots, r_n) \in V_g$, then also $p_{n+1} = r_{n+1}$ because

$$p_{n+1} = 1/g(p_1, \ldots, p_n) = 1/g(r_1, \ldots, r_n) = r_{n+1}$$

so π is injective. Also, the pre-image of every $P \coloneqq (p_1, \ldots, p_n) \in V_g$ is the point $(p_1, \ldots, p_n, 1/g(P)) \in W$ because $g(P) \neq 0$ by the definition of V_g. Thus, π is surjective.

To show that π is a homeomorphism, we first recall one more classical result – polynomial maps are continuous in the Zariski topology. Since π is a continuous map which is bijective, it remains to show that π is open. Let $U \subseteq W$ be open, we want that $\pi(U) \subseteq V_g$ is open. This is equivalent to $\pi(U)$ being open in V as V_g is also open in V.

Since U is open, there exist polynomials $g_1, \ldots, g_l$ such that $U = W \setminus \mathbf{V}(g_1, \ldots, g_l)$. Define

$$\tilde{g}_i(x_1, \ldots, x_n) \coloneqq g_i(x_1, \ldots, x_n, 1/g(x_1, \ldots, x_n)).$$

Let $R \in \mathbb{N}$ be large enough so that $g^R \tilde{g}_i \in \mathbb{C}[x_1, \ldots, x_n]$ for all $i \leq l$.

We are going to prove that $\pi(U) = V \setminus \mathbf{V}(g^R \tilde{g}_1, \ldots, g^R \tilde{g}_l) =: X$.

If $P \in \pi(U)$, then $(P, 1/g(P)) \in U$ so $g(P) \neq 0$, thus $g^R(P) \neq 0$, $\tilde{g}_i(P)$ is well-defined and $\tilde{g}_i(P) = g_i(P, 1/g(P)) \neq 0$ for all $i = 1, \ldots, l$. Since $U \subseteq W$, also $f_i(P) = 0$ for all $i = 1, \ldots, k$. In words, all f_i's vanish at P and g plus all $\tilde{g}_i$'s do not vanish at P. Hence, $P \in X$ and $\pi(U) \subseteq X$.

The argument for the inverse direction is very similar. If $P \in X$, then $g^R(P) \neq 0$, implying $g(P) \neq 0$, and $\tilde{g}_i(P) \neq 0$ for all i. Combination of these two inequalities ensures that also $g_i(P) \neq 0$ for all $i \leq l$. Finally, all the f_i's vanish at P. Put together, the immediate conclusion is that $(P, 1/g(P)) \in U$ and $P \in \pi(U)$.

This finishes the proof as $X = V \setminus \mathbf{V}(g^R\tilde{g}_1, \dots, g^R\tilde{g}_l)$ is open in V. □

The construction of V_g can be extended to any finite number of polynomials. The proof that there is a homeomorphism between

$$\{P \in \mathbb{C}^n \mid f_i(P) = 0 \wedge g_j(P) \neq 0 \; \forall i \leq k, j \leq l\}$$

and

$$\mathbf{V}(f_1, \dots, f_k, 1 - x_{n+1}g_1, \dots, 1 - x_{n+l}g_l)$$

follows immediately by induction on l from **Proposition 1.1.9**.

1.1.3 Local Rings And Tangent Spaces

As is mostly the case in geometry, smooth points on objects are defined using tangent spaces at those points. In case of affine algebraic sets, it is beneficial to define these purely algebraically using a construction styled *local ring at a point.* To that end, let us recall a few more results and definitions.

Definition 1.1.10 (Multiplicative Set)**.** A subset S of a commutative ring R is called *multiplicative* if $0 \notin S$, $1 \in S$ and $a, b \in S \implies a \cdot b \in S$.

Multiplicative sets are useful to us for two reasons. First, they serve well for localization, as they do not contain 0 and are closed under multiplication. Second, given an algebraic set V and a point $P \in V$, the set

$$S_P \coloneqq \{f \in \mathbb{C}[V] \mid f(P) \neq 0\}$$

is multiplicative. Indeed, $1(P) \neq 0$ and if $f, g \in S_P$ do not vanish at P, neither does their product. What shall we use this set for? You have guessed it. We shall localize $\mathbb{C}[V]$ at it.

Definition 1.1.11 (Local Ring at a Point)**.** Let V be an affine algebraic set and $P \in V$. With set S_P defined as above, we denote $\mathcal{O}_{V,P} \coloneqq \mathbb{C}[V]_{S_P}$, the localization of $\mathbb{C}[V]$ at S_P, and call it the *local ring of V at P.*

It follows from results in commutative algebra that if V is irreducible, and $\mathbb{C}[V]$ ergo an integral domain, $\mathbb{C}[V]_{S_P}$ is a subring of the field of fractions of $\mathbb{C}[V]$, which is often denoted $\mathbb{C}(V)$. In other words, using the definition of S_P for a particular point $P \in V$, $\mathcal{O}_{V,P} = \mathbb{C}[V]_{S_P}$ consists of equivalence classes of expressions f/s where $f, s \in \mathbb{C}[V]$ and $s(P) \neq 0$ modulo the relation $f/s \sim g/t \iff ft - gs = 0$. For a general algebraic set, relation $\sim$ must be subtly redefined. We will not go into more detail here because toric varieties are by definition irreducible and reducible algebraic sets thus lose all relevance to our purpose. When talking about local rings in the future, we shall limit ourselves to affine varieties.

You would be right to call us out on the unjustified naming used in the last definition. It is not immediately obvious that $\mathcal{O}_{V,P}$ is a *local ring* for all affine varieties V and all points $P \in V$. We are going to make certain it is.

Proposition 1.1.12. *Let V be an affine variety and $P \in V$. Then, $\mathcal{O}_{V,P}$ is a local ring, that is, it has a unique maximal ideal. Moreover, if we label the maximal ideal $\mathfrak{m}_{V,P}$, then*

$$\mathfrak{m}_{V,P} = \{q \in \mathcal{O}_{V,P} \mid q(P) = 0\}.$$

Proof. Let us first show that $\mathfrak{m}_{V,P}$ is a maximal ideal. Clearly, $\mathfrak{m}_{V,P} \subsetneq \mathcal{O}_{V,P}$. As in the proof of **Proposition 1.1.6**, we can show what $\mathcal{O}_{V,P}/\mathfrak{m}_{V,P} \simeq \mathbb{C}$. The projection $\mathcal{O}_{V,P} \twoheadrightarrow \mathbb{C}$ is canonically given by $f/g \mapsto f(P)/g(P)$ since $g(P) \neq 0$ by definition of $\mathcal{O}_{V,P}$. Its kernel is $\mathfrak{m}_{V,P}$ and its image the entirety of $\mathbb{C}$. First Isomorphism Theorem takes care of the rest.

For contradiction, suppose that $I \subsetneq \mathcal{O}_{V,P}$ is a maximal ideal different from $\mathfrak{m}_{V,P}$. In particular, there exists an element $r \in I$ such that $r(P) \neq 0$. We write $r = f/s$ for some $f, s \in \mathbb{C}[V]$. We have $f(P) \neq 0$ since $r(P) \neq 0$, and $s(P) \neq 0$ by definition. This means that r is invertible in $\mathcal{O}_{V,P}$, its inverse being s/f which is defined at P as $f(P) \neq 0$. It follows that $1 \in I$, a contradiction. $\square$

Before we move on, we mention one specific localization which we feel closer to more so than to the others. Remember **Proposition 1.1.9**? We bet you do. If V is an affine algebraic set and $f \in \mathbb{C}[V]$, then the set

$$S_f \coloneqq \{1, f, f^2, \ldots\}$$

is multiplicative. We will denote the localization $\mathbb{C}[V]_{S_f}$ by $\mathbb{C}[V]_f$ and call it the *localization* of $\mathbb{C}[V]$ *at* f. If V is in addition irreducible, then

$$\mathbb{C}[V]_f = \{g/f^k \mid g \in \mathbb{C}[V] \wedge k \in \mathbb{N}_0\}.$$

Even better, it actually turns out irreducibility also entails the convenient relation $\operatorname{Specm}(\mathbb{C}[V]_f) = V_f$ as we proceed to formally prove.

Lemma 1.1.13. *If $V \coloneqq \mathbf{V}(f_1, \ldots, f_k)$ is an affine variety, $f \in \mathbb{C}[V]$ and $W \coloneqq \mathbf{V}(f_1, \ldots, f_k, 1 - x_{n+1} f)$, then $\mathbb{C}[W] \simeq \mathbb{C}[V]_f$. Consequently, $\operatorname{Specm}(\mathbb{C}[V]_f) = V_f$.*

Proof. We already know that W is homeomorphic to V_f via the projection map $\pi : W \to V_f$ sending $(p_1, \ldots, p_{n+1})$ to $(p_1, \ldots, p_n)$. We show that there exists an isomorphism $\mathbb{C}[V]_f \simeq \mathbb{C}[W]$ of $\mathbb{C}$-algebras.

Because we have '$x_{n+1} = 1/f$' in W, the natural candidate is $\varphi : g/f^l \mapsto g x_{n+1}^l$. Scalars in $\mathbb{C}[V]_f$ are fractions $c/1$ for $c \in \mathbb{C}$ and $\varphi(c/1) = c$, hence φ sends scalars to scalars. We compute

$$\varphi\left(\frac{g}{f^l} + \frac{h}{f^m}\right) = \varphi\left(\frac{gf^m + hf^l}{f^{m+l}}\right) = gf^m x_{n+1}^{m+l} + hf^l x_{n+1}^{m+l} = \varphi\left(\frac{g}{f^l}\right) + \varphi\left(\frac{h}{f^m}\right).$$

Compatibility of φ with multiplication is proved similarly.

We check φ is well-defined. Suppose $g/f^l = h/f^k$ and, without loss of generality, $l \geq k$. Then, $gf^k = hf^l$ and so $g = hf^{l-k}$. We obtain

$$g(\tilde{P})p_{n+1}^l = h(\tilde{P})f^{l-k}(\tilde{P})p_{n+1}^l = h(\tilde{P})p_{n+1}^{k-l}p_{n+1}^l = h(\tilde{P})p_{n+1}^k$$

for all $P = (\tilde{P}, p_{n+1}) \in W$ as we wanted.

It remains to see that φ is injective and surjective. If $g \in \mathbb{C}[W]$, write g as

$$g(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{d} g_i(x_1, \ldots, x_n)x_{n+1}^i$$

where $d \coloneqq \deg_{x_{n+1}} g$. Then, we have

$$\varphi\left(\sum_{i=1}^{d} \frac{g_i}{f^i}\right) = \sum_{i=1}^{d} \varphi\left(\frac{g_i}{f^i}\right) = \sum_{i=1}^{d} g_i x_{n+1}^i = g.$$

As for injectivity, it is probably easiest to prove $\ker\varphi = 0$. Suppose it is not so. Then, we have $\varphi(g/f^l) = 0$ for some non-zero $g \in \mathbb{C}[V]$. Meaning, $gx_{n+1}^l(P) = 0$ for all points in W. However, every point $P \in W$ by definition satisfies $p_{n+1} = 1/f(p_1, \ldots, p_n) \neq 0$ so we must have $g(p_1, \ldots, p_n) = 0$ for all points $(p_1, \ldots, p_n) \in V_f$. This means that $gf(p_1, \ldots, p_n) = 0$ for all $(p_1, \ldots, p_n) \in V$. V is an affine variety and $f \not\equiv 0$ on V but $gf \equiv 0$ on V which means that $g \equiv 0$ on V. In other words, $g \in \mathbf{I}(V)$, hence g is 0 in $\mathbb{C}[V]$, which is a contradiction.

Now that we know that $\mathbb{C}[W] \simeq \mathbb{C}[V]_f$, we also know

$$\operatorname{Specm}(\mathbb{C}[V]_f) = \operatorname{Specm}(\mathbb{C}[W]) = W \simeq V_f. \qquad \square$$

With local rings at points and their maximal ideals at our disposal, we are nearing the end of the section. We finish strong by defining smooth points and dimensions of affine varieties.

Definition 1.1.14 (Zariski Tangent Space)**.** Given an affine variety V and a point $P \in V$, we define the *cotangent space* of V at P as $\mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2$. It is a vector space over $\mathcal{O}_{V,P}/\mathfrak{m}_{V,P}$, which is isomorphic to $\mathbb{C}$ by the proof of **Proposition 1.1.12**. Its dual, the vector space $(\mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2)^* = \operatorname{Hom}(\mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2, \mathbb{C})$ is called the *tangent space* of V at P and denoted T_PV.

This purely algebraic definition is elegant but not very useful for calculation of tangent spaces of particular affine varieties. The following lemma provides a better way.

Lemma 1.1.15. *Let V be an affine variety, $V = \mathbf{V}(I)$ for some prime ideal $I \subsetneq \mathbb{C}[x_1, \ldots, x_n]$, $P \in V$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$. Define*

$$d_P(f) \coloneqq \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)x_i$$

and also

$$T \coloneqq \{X \in \mathbb{C}^n \mid d_P(f)(X) = 0\ \forall f \in I\}.$$

Then,

$$T^* \simeq \mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2.$$

Proof. See Hulek [2003], Theorem 3.14. □

This lemma formalizes an intuitive idea that a tangent space should linearly approximate a manifold on a neighbourhood of a chosen point. In case of affine varieties, a function defined near a point P means a rational function regular at P, an element of $\mathcal{O}_{V,P}$. Since linear forms vanish at the origin, we require our rational functions to vanish at P, to lie in $\mathfrak{m}_{V,P}$. We quotient out all parts that are not linear, leaving us with $\mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2$.

The last tool we need to define a smooth affine variety is the concept of dimension. Since smoothness at a point is a local property, the dimension of the local ring seems like a good candidate. Let us recall how we can define dimensions of rings.

Definition 1.1.16 (Krull Dimension)**.** For a commutative ring R, we define its *Krull dimension* to be the supremum across the lengths of all chains of prime ideals in R. More specifically, if

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_k$$

is a strictly ascending chain of prime ideals, we define its *length* to be k. Now, for a specific prime ideal $P \subsetneq R$, its *height* is the supremum of the lengths of all chains of prime ideals contained in P. The Krull dimension of R is then the supremum of the heights of all prime ideals in R. We denote it simply, $\dim R$.

Even though the construct of Krull dimension might seem overly dissociated from the 'typical' definition of dimension of a vector space or a manifold, it is actually a natural extension. It is carefully defined so that polynomial rings have dimension equal to the number of variables. Indeed, the longest chain of prime ideals one can get in a polynomial ring $K[x_1, \ldots, x_n]$, where K is an arbitrary field, is for instance

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \ldots \subsetneq (x_1, \ldots, x_n).$$

The choice of the generating variables is of course arbitrary. This construction works as longs as the k-th ideal in the chain is generated by precisely $k-1$ variables. The fact that no longer chain can be produced requires some work to prove and distrustful readers should consult for instance Thierry and Lombardi [2005] for a relatively simple proof. Do notice that the way we have created the chain above is akin to how one can produce a chain of length n of subspaces of a vector space with basis $\{x_1, \ldots, x_n\}$.

Time is ripe to a define a smooth point of V and the dimension of V at any point. We are going to accomplish it in one breath.

Definition 1.1.17 (Smooth Affine Variety)**.** Let V be an affine variety. The *dimension of* V at P, denoted simply $\dim_P V$, is the Krull dimension of $\mathcal{O}_{V,P}$. A point $P \in V$ is *smooth* if

$$\dim T_P V = \dim_P V,$$

where $\dim T_P V$ denotes the dimension of $T_P V$ as a vector space over $\mathbb{C}$. A point which is not smooth is called *singular.* Finally, V is also dubbed *smooth* if its every point is smooth and *singular* otherwise.

Now, that we know what a smooth point is, we would like some convenient way to check smoothness if we know the dimension of the affine variety. The following statement is an almost immediate corollary of **Lemma 1.1.15**.

Corollary 1.1.18 (Exercise 1.0.9 in Cox et al. [2011])**.** *Let* $V := \mathbf{V}(I)$ *where* $I = (f_1, \ldots, f_k)$ *for adequate* $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$*. A point* $P \in V$ *is smooth if and only if the Jacobian matrix*

$$J_P(f_1, \ldots, f_k) := (\partial_i f_j(P))_{i=1,\ldots,n;j=1,\ldots,k}$$

has rank $n - \dim_P V$*.*

Proof. First of all, note that, thanks to **Lemma 1.1.15**,

$$\begin{aligned} T_P V &= \{X \in \mathbb{C}^n \mid d_P(f)(X) = 0 \; \forall f \in I\} \\ &= \{X \in \mathbb{C}^n \mid d_P(f_1)(X) = d_P(f_2)(X) = \ldots = d_P(f_k)(X) = 0\}. \end{aligned}$$

The proof is based upon the fact that

$$T_P V = \ker J_P(f_1, \ldots, f_k).$$

Indeed, we have at $d_P(f_i)(X) = \nabla f_i(P) X^T$ where ∇f_i is the row vector of partial derivatives $(\partial_1 f_i, \ldots, \partial_n f_i)$ and $X \in \mathbb{C}^n$ is a point in $\mathbb{C}^n$ identified with the corresponding row vector. The space $T_P V$ is then defined by the set of equations $\nabla f_i(P) X^T = 0$. However, that is precisely the space determined by $J_P(f_1, \ldots, f_k) X^T = 0$.

From linear algebra, we know that

$$\dim(\ker J_P(f_1, \ldots, f_k)) = n - \operatorname{rank} J_P(f_1, \ldots, f_k).$$

Denote $d := \dim_P V$. If $\operatorname{rank} J_P(f_1, \ldots, f_k) = n - d$, then

$$\dim T_P V = \dim(\ker J_P(f_1, \ldots, f_k)) = n - (n - d) = d.$$

Conversely, if $\dim T_P V = d$, then the same calculation gives $\operatorname{rank} J_P(f_1, \ldots, f_k) = n - d$. □

Remark. We should mention that the inequality $\dim_P V \leq \dim T_P V$ always holds. To give a not absolutely rigorous overview of the reason, do note that, by the preceding corollary, $\dim_P V = n - \operatorname{rank} J_P$ where J_P is the corresponding Jacobian matrix if P is a smooth point of V. If partial derivatives of the defining polynomials of V vanish, the rank of J_P decreases and so the dimension of $T_P V$ increases because $\dim T_P V = \dim(\ker J_P) = n - \operatorname{rank} J_P$. Hence, $\dim_P V$ is always *at most* $\dim T_P V$.

It is not always easy to determine the dimension of an affine variety. Fortunately, the irreducibility of an affine algebraic set equates the dimension of local rings at a point across all points in the set. For a proof and a careful explanation, see Atiyah [2015], Chapter 11. Hence, to compute the dimension of an affine variety, it is enough to compute the Krull dimension of its coordinate ring, which is often a little bit easier to do.

As will become apparent especially in the second chapter, we care a considerable bunch about smoothness. In light of the last definition, we can give an elucidating example as to why we do not want to consider bijective morphisms as isomorphisms of affine varieties. It just so happens that there exist bijective morphisms between smooth and singular varieties.

Example 1.1.19. Let $V \coloneqq \mathbf{V}(y - x)$ and $W \coloneqq \mathbf{V}(y^2 - x^3)$. Firstly, note that both $y - x$ and $y^2 - x^3$ are irreducible so $\mathbf{I}(\mathbf{V}(y-x)) = (y-x)$ and $\mathbf{I}(\mathbf{V}(y^2-x^3)) = (y^2 - x^3)$. Also, both of these ideals are prime because they are generated by a single irreducible polynomial. This means that V and W are affine varieties.

The map given by $(t, t) \mapsto (t^2, t^3)$ for $z \in \mathbb{C}$ is a bijective morphism $V \to W$. Clearly, $(t, t) \in V$ and $(t^2, t^3) \in W$ for all $t \in \mathbb{C}$. It is injective, because $s^2 = t^2$ implies $s = \pm t$ and applying $s^3 = t^3$ gives us $s = t$. For $(x, y) \in W$, we have $y^2/x^2 = x^3/x^2 = x$. It follows that $(t, t) \mapsto (t^2, t^3)$ is also surjective because the pre-image of every non-zero point $(x, y) \in W$ is the point $(y^2/x^2, y^2/x^2) \in V$ and the pre-image of $(0, 0) \in W$ is of course $(0, 0) \in V$.

We have $\mathbb{C}[V] = \mathbb{C}[x, y]/(y - x) = \mathbb{C}[x]$ and $\mathbb{C}[W] = \mathbb{C}[x, y]/(y^2 - x^3)$. We will show that these $\mathbb{C}$-algebras are not isomorphic. For instance, we know from basic algebra course that $\mathbb{C}[x]$ is a principal ideal domain. However, $\mathbb{C}[W]$ is not because for instance the ideal $(\overline{x}, \overline{y})$ where $\overline{x}$ and $\overline{y}$ denote the cosets of x and y is easily seen not to be principal. Hence, we have $\mathbb{C}[V] \not\cong \mathbb{C}[W]$ so $V \not\cong W$.

To give a more geometric argument as to why we do not want to consider V and W to be 'the same' in any meaningful sense, we talk about smoothness. You see, V is smooth but W is not. The problematic point is $(0, 0)$. If we denote $f(x, y) \coloneqq y - x$ and $g(x, y) \coloneqq y^2 - x^3$, we have $(\partial_x f(0, 0), \partial_y f(0, 0)) \neq 0$ but $\partial_x g(0, 0) = \partial_y g(0, 0) = 0$. Ergo, $\dim T_{(0,0)} V = 1 = \dim_{(0,0)} V$ as $\dim \mathbb{C}[V] = \dim \mathbb{C}[x] = 1$ so V is smooth at $(0, 0)$. We could prove that $\dim_{(0,0)} W = 1$ as well but we do not need to because it is definitely not zero. Then, **Corollary 1.1.18** can be used to conclude that $(0, 0)$ is not a smooth point of W because $\operatorname{rank} J_{(0,0)}(g) = 0$.

Since the entire second chapter of the thesis deals with the problem of singular points on toric varieties, our notion of isomorphism must preserve the smoothness of points. We have not yet proven that it does but we shall, further down the line.

1.2 Cones

Actually, the title of this section should sound ‘Strongly Convex Rational Polyhedral Cones’ but advanced marketing techniques suggest that titles should be short and punchy. We are not sure about the punchy part, so we have at least kept it short. This section loosely follows Cox et al. [2011], §1.2. and Fulton [1993], Section 1.2. Many omitted proofs were added.

We believe you wonder what *exactly* are we doing here? How do cones connect to toric varieties, or even affine algebraic sets in general? The connection indeed exists but is not at all obvious. We bid you hold your breath for a while longer as this section only deals with the necessary technical details and offers a glassful of geometric intuition in order to make the transition to toric varieties as painless as possible.

We shall start with a few definitions. Not exactly of the cone just yet since they say you should save the best for last.

Definition 1.2.1 (Lattice)**.** A free commutative group of a finite rank $r \in \mathbb{N}$, hence isomorphic to $\mathbb{Z}^r$, is called a *lattice*.

Each lattice N and its dual $M \coloneqq \mathrm{Hom}(N, \mathbb{Z})$ give naturally rise to a mapping

$$\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z},$$

where $\langle m, n \rangle \coloneqq m(n)$. This mapping also canonically identifies the lattice N with $\mathrm{Hom}(M, \mathbb{Z})$ via $\langle n, - \rangle : M \to \mathbb{Z}$ and is thus, by definition of a homomorphism, bilinear. We shall make use of this duality and regard $\langle \cdot, \cdot \rangle$ as either a mapping $M \times N \to \mathbb{Z}$ or $N \times M \to \mathbb{Z}$. It follows from the identification $N = \mathrm{Hom}(M, \mathbb{Z})$ that $\langle m, n \rangle = \langle n, m \rangle$ for all $n \in N, m \in M$.

By scalar extension to the field of real numbers, a lattice N gives rise to a real vector space with basis formed by its generators. In symbols, if N is generated by $n_1, \dots, n_r$, then we define

$$N_{\mathbb{R}} \coloneqq \{a_1 n_1 + \ldots + a_r n_r \mid a_i \in \mathbb{R}\}.$$

If M is the lattice dual to N, then $M_{\mathbb{R}}$ is the real vector space dual to $N_{\mathbb{R}}$, hence $\langle \cdot, \cdot \rangle$ is naturally extended to a mapping from $M_{\mathbb{R}} \times N_{\mathbb{R}}$ to $\mathbb{R}$.

You might wonder why we bother with lattices and scalar extensions and not simply talk about $\mathbb{Z}^r$ and $\mathbb{R}^r$. Whilst this is not really an answer, the reason is strictly formal and will become apparent when we meet toric varieties with their respective groups of characters and one-parameter subgroups.

To make our definition of a polyhedral cone very geometric and natural, we shall make use of *polytopes*. Polytopes are generally defined as convex hulls of finite sets of points in $\mathbb{R}^n$. Most prominent examples are probably two- and three-dimensional polytopes, typically styled convex polygons and convex polyhedra, respectively. For formal reasons, we extend the traditional definition to the real vector space $N_{\mathbb{R}}$ and formulate our definitions accordingly. Hence, to us a *polytope* denotes a set

$$P = \text{Conv}(S) \coloneqq \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{R}, \lambda_s \geq 0 \text{ and } \sum_{s \in S} \lambda_s = 1 \right\}$$

where $S \subseteq N_{\mathbb{R}}$ is **finite**.

Each polytope then defines a *convex polyhedral cone* in the following way. For simplicity, suppose there is a hexagon sitting somewhere in $\mathbb{R}^3$. Choose an arbitrary point (typically the origin) which does not lie in it and draw rays starting in this chosen point through each of the points of the hexagon. This process yields a sort of unbounded prism in $\mathbb{R}^3$ tipped at the chosen point. These prisms are exactly what we shall call *convex polyhedral cones* in this text. Do note that convex polyhedral cones are never cones in the widespread meaning of the word since no polytope with a finite number of vertices is ever circular. Despite this fact, we will refer to *convex polyhedral cones* simply as *cones*, owing to a need for brevity.

Now, for the formalities. As the previous paragraph was meant to make intuitive, we replace each point of a given polytope P with a ray starting at the origin and passing through that point. This is as simple as replacing a point $p \in P$ with the line $\{t \cdot (p, 1) \mid t \geq 0\}$. Hence, we define

$$C(P) \coloneqq \{t \cdot (p, 1) \mid p \in P, t \geq 0\}$$

and call the set a *convex polyhedral cone* of P.

Before we move on to proving some important properties of cones (meaning convex polyhedral cones), we require a simpler description. Notice that if P is a subset of $N_{\mathbb{R}}$, then $C(P)$ is a subset of $N_{\mathbb{R}} \times \mathbb{R}$ which, however, is also a real vector space of dimension one higher. If $P = \text{Conv}(S)$ for a finite $S \subseteq N_{\mathbb{R}}$, then

$$\begin{aligned} C(P) &= \left\{ \left(\sum_{s \in S} t \lambda_s s, t \right) \mid t \geq 0, \lambda_s \geq 0 \text{ and } \sum_{s \in S} \lambda_s = 1 \right\} \\ &= \left\{ \sum_{(s,1) \in S \times \{1\}} \lambda'_s (s, 1) \mid \lambda'_s \geq 0 \right\} \end{aligned} \tag{1.2.1}$$

because the condition $\sum_{s \in S} \lambda_s = 1$ becomes irrelevant when the sum is multiplied by a scalar t spanning $[0, \infty)$.

Indeed, the description (1.2.1) not only provides a more succinct definition of a cone but also summarizes a geometric notion. Instead of defining cones through polytopes, we could have decided to do so by taking all the 'positive' linear combinations of a finite set of rays starting at the origin.

In the light of (1.2.1), we generally define a *convex polyhedral cone* as follows.

Definition 1.2.2 (Convex Polyhedral Cone)**.** For a finite set $S \subseteq N_{\mathbb{R}}$, we define

$$\text{Cone}(S) \coloneqq \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \geq 0 \right\} \subseteq N_{\mathbb{R}}.$$

and call it the *convex polyhedral cone*, or the *cone* for short, generated by S.

The underlying polytope is then obtained by simply 'forgetting' one coordinate, with respect to a chosen basis of $N_\mathbb{R}$, and allowing only convex combinations.

When studying geometric objects, the notion of *duality* is often of great import. In the case of cones, their duals, which we are about to define and dedicate the next few pages to, are key components in the construction of toric varieties.

Definition 1.2.3 (Dual Cone)**.** Let $S \subseteq N_\mathbb{R}$ be finite and $\sigma \coloneqq \text{Cone}(S)$. The *dual cone* to σ is defined as

$$\sigma^* \coloneqq \{m \in M_\mathbb{R} \mid \langle m, s \rangle \geq 0 \; \forall s \in \sigma\}.$$

We know state and proof a specific version of a classical result in convex geometry called the *hyperplane separation theorem*, which says that any two disjoint closed convex sets can be separated by a hyperplane. As we will promptly explain, its version for convex polyhedral cones can be stated as such.

Lemma 1.2.4 (Separation Theorem for Cones)**.** *Let σ be a cone and $v \in N_\mathbb{R}$ a point lying outside of σ. Then, there exists $m \in \sigma^*$ such that $\langle m, v \rangle < 0$.*

Before the proof, we elucidate how elements of dual cones actually define hyperplanes which delimit the original cone in the space. Formally, by a *hyperplane* in $N_\mathbb{R}$, we mean a subset

$$\{x \in N_\mathbb{R} \mid \langle m, x \rangle = c_m\} \subseteq \mathbb{R}^n$$

for some fixed $m \in M_\mathbb{R}$ and $c_m \in \mathbb{R}$. If we identify $N_\mathbb{R}$ and $M_\mathbb{R}$ with $\mathbb{R}^r$, then $\langle \cdot, \cdot \rangle$ becomes the standard dot product and a hyperplane is indeed a subspace of codimension one. In particular, for a chosen $m \in M_\mathbb{R}$, we denote

$$H_m \coloneqq \{x \in N_\mathbb{R} \mid \langle m, x \rangle = 0\}.$$

Given a cone σ, we call H_m a *supporting hyperplane* of σ if $\langle m, s \rangle \geq 0 \; \forall s \in \sigma$. Notice that, by definition, H_m is a supporting hyperplane if and only if $m \in \sigma^*$. After this short venture into hyperplanes, we need a well-known result from linear algebra known as Farkas' lemma.

Lemma 1.2.5 (Farkas' lemma)**.** *Let $\mathbf{A} \in \mathbb{R}^{n \times s}$ and $\mathbf{b} \in \mathbb{R}^n$. Then exactly one of the following assertions holds:*

1. *There exists an $\mathbf{x} \in \mathbb{R}^s$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.*
2. *There exists a $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{A}^T\mathbf{y} \geq 0$ and $\mathbf{b}^T\mathbf{y} < 0$.*

Here, $\mathbf{x} \geq 0$ means $x_i \geq 0$ for all $i \leq s$.

Proof. For a detailed proof and an interesting account of the usage of this lemma in linear programming, consult Matoušek and Gärtner [2007], pages 81-104. □

We trust you think “Wait, what?”. If you do, it is a very natural response. We take upon ourselves the educational challenge of translating this lemma to the theory of cones. The translation will itself be the proof of **Lemma 1.2.4** as it will become apparent that the two lemmata actually say exactly the same thing.

Fix some $S \coloneqq \{u_1, \dots, u_s\} \subseteq N_{\mathbb{R}}$ and let $\sigma \coloneqq \mathrm{Cone}(S)$. To shake hands with linear algebra, we identify $M_{\mathbb{R}} \simeq N_{\mathbb{R}} \simeq \mathbb{R}^r$ for some $r \in \mathbb{N}$. Let

$$\mathbf{A} \coloneqq (u_1 \mid u_2 \mid \cdots \mid u_s) \in \mathbb{R}^{n \times s},$$

that is, the i-th column of $\mathbf{A}$ is u_i. Notice that by definition

$$\mathrm{Cone}(u_1, \dots, u_s) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \geq 0\}.$$

Now, choose some point $v \notin \sigma$. In the notation of Farkas' lemma, $\mathbf{b} \coloneqq v$. Since $\mathbf{b} \notin \sigma$ and $\sigma = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \geq 0\}$, there is no $\mathbf{x} \geq 0$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$. So, the assertion (2) in Farkas' lemma must hold. We get an existence of a point $\mathbf{y} \in \mathbb{R}^n$ satisfying $\mathbf{A}^T\mathbf{y} \geq 0$ and $\mathbf{b}^T\mathbf{y} < 0$. What does it mean in the cone tongue? Well, $\mathbf{A}^T\mathbf{y}$ means that $\mathbf{y} \in \sigma^*$ because

$$\mathbf{A}^T\mathbf{y} = \begin{pmatrix} u_1^T\mathbf{y} \\ \vdots \\ u_s^T\mathbf{y} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{y}, u_1 \rangle \\ \vdots \\ \langle \mathbf{y}, u_s \rangle \end{pmatrix}.$$

If $\langle \mathbf{y}, u_i \rangle \geq 0$ for all i and u_i generate σ, then $\langle \mathbf{y}, u \rangle \geq 0$ for all $u \in \sigma$ as u is a non-negative linear combination of the u_i. Finally, $\mathbf{b}^T\mathbf{y} = \langle \mathbf{y}, \mathbf{b} \rangle < 0$ so $\mathbf{y} \in \sigma^*$ is the sought-after element m from **Lemma 1.2.4**.

We now derive consequences of **Lemma 1.2.4** which will help us greatly when we use cones to build toric varieties. The following statements are, up to some exceptions, due to Cox et al. [2011], Section 1.2. Missing proofs were added.

Lemma 1.2.6 (Dualing 'Round And 'Round)**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then, $(\sigma^*)^* = \sigma$.*

Proof. Let $u \in \sigma$. Then, $\langle m, u \rangle \geq 0 \ \forall m \in \sigma^*$. Since $\langle m, u \rangle = \langle u, m \rangle$, this implies $u \in (\sigma^*)^*$.

For the other direction, suppose $u \notin \sigma$. Then, by **Lemma 1.2.4**, we find an $m \in \sigma^*$ such that $\langle m, u \rangle < 0$. This, however, means that $u \notin (\sigma^*)^*$ because $\langle m, u \rangle \geq 0$ does not hold for every one $m \in \sigma^*$. □

The next important step for us is to prove that σ^* is also a convex polyhedral cone. For that we need the concept of a *face.*

Definition 1.2.7 (Face of a Cone)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Its *face* is any subset $H \cap \sigma \subseteq \sigma$ where H is a supporting hyperplane of σ.

Lemma 1.2.8 (Properties of Faces)**.** *Let σ be a cone. Then,*

1. *A face of σ is a cone.*

2. *Intersection of faces is a face.*

3. *A face of a face is a face.*

Proof.

1. Indeed, let $H_m, m \in M_{\mathbb{R}}$ be a supporting hyperplane of $\sigma := \text{Cone}(u_1, \ldots, u_s)$. The face $H_m \cap \sigma$ is then a cone generated by those u_i which lie in H_m. This is due to linearity of $\langle \cdot, \cdot \rangle$ as

$$\langle m, u \rangle = \left\langle m, \sum_{i=1}^{s} \lambda_i u_i \right\rangle = \sum_{i=1}^{s} \lambda_i \langle m, u_i \rangle$$

for $u \in \sigma, \lambda_i \in \mathbb{R}$. In particular, σ has only finitely many faces.

2. If $m, m' \in \sigma^*$, then $(H_m \cap H_{m'}) \cap \sigma = H_{m+m'} \cap \sigma$. Indeed, by linearity of $\langle \cdot, \cdot \rangle$, we have $m, m' \in \sigma^* \implies m + m' \in \sigma^*$. Since $\langle m, u \rangle \geq 0$ and $\langle m', u \rangle \geq 0$ for every $u \in \sigma$, also

$$\langle m + m', u \rangle = \langle m, u \rangle + \langle m', u \rangle = 0 \iff \langle m, u \rangle = 0 \wedge \langle m', u \rangle = 0,$$

so indeed $(H_m \cap H_{m'}) \cap \sigma = H_{m+m'} \cap \sigma$. Because $m + m' \in \sigma^*$, $H_{m+m'}$ is supporting, so we are done.

3. We note that if H_m is a supporting hyperplane of σ, then H_m is also supporting of every $H'_m \cap \sigma$ where $H_{m'}$ is another supporting hyperplane of σ. The rest follows similarly as in (b), given that a face of $H'_m \cap \sigma$ assumes the shape $H_m \cap (H_{m'} \cap \sigma) = H_{m+m'} \cap \sigma$. □

The *span* of a cone σ is the linear subspace $\mathbb{R}\sigma = \sigma + (-\sigma) \subseteq N_{\mathbb{R}}$. We call the dimension of $\mathbb{R}\sigma$, the dimension of the cone σ itself. Faces of codimension one are called *facets*. The following three propositions and significant parts of their proofs are due to Fulton [1993], Section 1.2.

Proposition 1.2.9. *Each proper face τ of σ, that is, $\tau \neq \sigma$, is contained in a facet.*

Proof. If we prove that each face of codimension at least two is contained in a face of higher dimension, the result follows.

Without loss of generality, suppose $\sigma = \text{Cone}(u_1, \ldots, u_s)$ spans a vector space V of dimension s. Otherwise consider a subset $\{v_1, \ldots, v_n\} \subseteq \{u_1, \ldots, u_s\}$ which spans V. If H_m is a supporting hyperplane, then the face $\tau := H_m \cap \sigma$ spans a subspace W of V. Denote by $\pi : V \twoheadrightarrow V/W$ the canonical projection. If $W = \text{span}(u_{i_1}, \ldots, u_{i_k})$ for adequate $\{i_1, \ldots, i_k\} \subsetneq \{1, \ldots, s\}$ and $k \leq s - 2$, we view V/W as the span of $\{u_{j_1}, \ldots, u_{j_{s-k}}\}$ for $\{j_1, \ldots, j_{s-k}\}$ the complement of $\{i_1, \ldots, i_k\}$. The projection π then simply 'nullifies' the coordinates whose indices are not in $\{j_1, \ldots, j_{s-k}\}$.

Thus, the hyperplane $\overline{H}_m := \{x \in V/W \mid \langle m, x\rangle = 0\}$ is a supporting hyperplane for $\pi(\sigma)$. So, all the $\pi(u_i)$ lie on one side of $\overline{H}_m$. We rotate $\overline{H}_m$ inside V/W so that all $\pi(u_i)$ still lie on one side but $\pi(u_{j_l})$ lies directly on it for some $l \leq s-k$. Since the rotated hyperplane is still supporting, this gives us the existence of an $m' \in \sigma^*$ such that $\tau \subseteq H_{m'}$ but also $u_{j_l} \in H_{m'}$. This means that $\sigma \cap H_{m'}$ is a face one dimension higher than τ which contains τ. □

Remark. Notice that if τ is a face of codimension two, then V/W is a plane and there are exactly two lines H' from the proof of the previous proposition. Hence a face of codimension two lies in the intersection of two facets. This observation drives the proof of the following corollary.

Corollary 1.2.10. *Each face is the intersection of facets containing it.*

Proof. If τ is a face of codimension larger than two, then by **Proposition 1.2.9** it lies in some facet ρ. By induction, τ is an intersection of facets of ρ, which are themselves an intersection of two facets of σ by the previous remark. So, τ is an intersection of facets of σ. □

Proposition 1.2.11. *The boundary of a cone σ is the union of its facets.*

Proof. Let $\tau := H_m \cap \sigma$ be a facet of σ. Since $\langle m, u\rangle < 0$ for each point $u \in N_{\mathbb{R}}$ lying across from σ with respect to H_m, there are points lying outside of σ which arbitrarily close to τ. Interior points of σ satisfy $\langle m, u\rangle > 0$, so, for the same reason, there are interior points arbitrarily close to τ. Hence, τ lies in the boundary of σ.

In the other direction, suppose $u \in \partial\sigma$ and take a sequence $v_i \to u$ with $v_i \notin \sigma$. By **Lemma 1.2.4**, there are points $m_i \in \sigma^*$ such that $\langle m_i, v_i\rangle < 0$. Note that the m_i can be chosen so that $\|m_i\| = 1$ because $\langle m_i, v_i\rangle < 0$ implies $\langle m_i/\|m_i\|, v_i\rangle = \|m_i\|^{-1}\langle m_i, v_i\rangle < 0$. Because a sphere of radius 1 is a compact set, the sequence $(m_i)_{i=1}^{\infty}$ has a converging subsequence. Without loss of generality, we assume that $m_i \to m$. Since σ^* is closed, $m \in \sigma^*$, $\langle m, u\rangle = 0$ and u lies in the face $\sigma \cap H_m$. □

The last proposition allows us to describe a convex polyhedral cone σ as the intersection of the half-spaces determined by its facets. This result connects to a traditional result from convex analysis concerning the expression of polytopes given as convex hulls of finite sets of points in terms of intersections of finitely many half-spaces. This result does not serve us well for two reasons: cones are not polytopes as we defined them and we do not need the generality provided. It would require arguably more work to translate this result into our setting than to prove it directly.

Notice that if τ is a facet of σ, then there is a vector $m \in \sigma^*$ unique up to scalar multiplication such that $H_m \cap \sigma = \tau$. This is true because τ spans a subspace of V, the space spanned by σ, of codimension one – a hyperplane. For each facet τ, we denote this unique vector m_τ. Then, the following and hopefully not too surprising result holds.

Proposition 1.2.12. *If σ spans $N_{\mathbb{R}}$ but $\sigma \neq N_{\mathbb{R}}$, then σ is the intersection of half-spaces determined by $H_\tau := H_{m_\tau}$, that is*

$$\sigma = \bigcap_{\tau \text{ facet of } \sigma} \{v \in V \mid \langle m_\tau, v\rangle \geq 0\}.$$

There is one little trick we must use in the proof of the above proposition. For two subsets $A, B \subseteq \mathbb{R}^n$, their distance is defined as

$$\mathrm{d}(A, B) := \inf\{\mathrm{d}(a, b) = \|a - b\| \mid a \in A, b \in B\}.$$

Now, the case interesting for us is $A = \{a\}$ and $B = \sigma$, a convex polyhedral cone. We claim that there exists a point $b \in \partial\sigma$ which is closest to a. Let $C := \partial\sigma \cap B$ be the intersection of $\partial\sigma$ with a ball centred at the origin. If B is chosen large enough, then $\mathrm{d}(a, C) = \mathrm{d}(a, \partial\sigma)$. Since both $\{a\}$ and C are compact, the continuous function $(a, u) \mapsto \mathrm{d}(a, u) : \{a\} \times C \to \mathbb{R}_+$ attains minimum for some point $b \in C$.

Proof of Proposition 1.2.12. Clearly, σ lies in the intersection above as each of the hyperplanes H_τ is supporting.

Suppose a point u lies in the intersection of half-spaces but not in the cone σ. Choose the point $v \in \partial\sigma$ closest to u. By **Proposition 1.2.11**, v lies in some facet τ. Then, $\langle m_\tau, v\rangle = 0$ and $\langle m_\tau, u\rangle < 0$ as u lies on the opposite side of σ with respect to H_τ. This is a contradiction. □

All this work bears fruit, as we reach a result about cones which is of utmost importance in their connection to toric varieties.

Theorem 1.2.13 (Dual Cone Is a Cone)**.** *If σ is a convex polyhedral cone, then σ^* is a convex polyhedral cone. Moreover, if σ spans $N_{\mathbb{R}}$, then $\sigma^* = \mathrm{Cone}(\{m_\tau \mid \tau$ a facet of $\sigma\})$.*

Proof. First of all, recall that σ has only finitely many faces so the number of facets is also finite.

Suppose first that σ spans $N_{\mathbb{R}}$. Then, each of the vectors m_τ lies in σ^* trivially. If an element $m \in \sigma^*$ were not a linear combination of m_τ's with non-negative coefficients, then we would, by **Lemma 1.2.4** applied to σ^*, find $u \in N_{\mathbb{R}}$ such that $\langle m_\tau, u\rangle \geq 0$ but $\langle m, u\rangle < 0$. This contradicts **Proposition 1.2.12**.

If σ spans a linear subspace $\mathbb{R}\sigma$ of $N_{\mathbb{R}}$, then it is straightforward to check that σ^* is generated by the preimages (via the canonical projection) of the generators of the dual cone to σ inside $(\mathbb{R}\sigma)^*$ together with all the vectors and their opposites in the basis of $(\mathbb{R}\sigma)^\perp := \{m \in M_{\mathbb{R}} \mid \langle m, w\rangle = 0 \; \forall w \in W\}$. □

We are reaching the climax of the section, with only two concepts remaining to explore. In the next section where we talk, among other things, about algebraic tori, there are two important lattices which give rise to characters and one-parameter subgroups. The property of being integral (or rational for that matter) of lattices is their defining property as geometric objects inside real vector spaces.

For everything to work out smoothly, we demand cones be rational as well. Quite naturally, a cone is called *rational* if its generators belong to N (as opposed to belonging to the real extension $N_{\mathbb{R}}$). The important observation here is, coming directly from the construction of the generators m_τ, that if σ is rational, so is σ^*. Indeed, if $\tau = H_\tau \cap \sigma$ is a facet of a rational cone σ, then its generators as a cone are also rational. The vector m_τ must then either be rational or a real scalar multiple of a rational vector, otherwise the equations $\langle m_\tau, u \rangle = 0$ as u ranges over the generators of τ would not hold. In the latter case, simply exchange m_τ for cm_τ where $c \in \mathbb{R}$ is chosen so that cm_τ is rational. Since we do not concern ourselves with norms of generating vectors, we thus assume that m_τ are always rational.

For those affiliated, the structure of a cone looks suspiciously close to the structure of a semigroup – having a neutral element, being closed under addition but having no inverses. This hunch turns out to be right on target as the rational points of cones actually do form a semigroup. Notice that the rational part of a cone, that is $\sigma \cap N$, is not the same as σ even if σ is itself rational. The quality of being rational only requires the cone to be *spanned* by rational vectors whereas $\sigma \cap N$ are points of the lattice N lying inside σ.

Actually, the semigroups stemming from lattice points inside rational cones exhibit other geometrically nice properties and are thus called *affine*. The defining properties of an *affine* semigroup are

- commutativity,
- finite number of generators,
- ability to be embedded in some lattice $N \simeq \mathbb{Z}^r$.

The very last result of the section formalizes the preceding paragraphs, albeit for dual cones, and establishes a connection which will be in the heart of the rest of the thesis.

Proposition 1.2.14 (Gordan's Lemma)**.** *If σ is a rational convex polyhedral cone, then $\sigma^* \cap M$ is an affine semigroup.*

Proof. The proof is taken from Fulton [1993], Section 1.2, Proposition 1.

Since $\sigma^* \cap M$ is embedded in M and addition of vectors is commutative, it is enough to show that it is finitely generated.

Let $m_1, \ldots, m_s$ denote the generators of σ^*. Denote

$$K := \left\{ \sum_{i=1}^{s} t_i m_i \mid t_i \in [0, 1] \right\}.$$

Then, K is compact and $K \cap M$ is finite as M is discrete. We show that $K \cap M$ generates $\sigma^* \cap M$ as a semigroup, in other words, every element $m \in \sigma^* \cap M$ is a $\mathbb{Z}$-linear combination of elements in $K \cap M$. If $m \in \sigma^* \cap M$, then, $m = \sum_{i=1}^{s} r_i m_i$ for some $r_i \geq 0$. Factor r_i as $r_i = z_i + t_i$ where z_i are non-negative integers and $t_i \in [0, 1]$. Then, $m = \sum_{i=1}^{s} z_i m_i + m'$ where $m_i \in K \cap M$ and also $m' = \sum_{i=1}^{s} t_i m_i \in K \cap M$. □

Before we march off this and on to the next section, we define one last property our cones *may* have which will make them all the nicer to work with.

Definition 1.2.15 (Smooth Cone)**.** A cone σ is called *smooth* if its generators form a $\mathbb{Z}$-basis of the lattice N.

1.3 Affine Toric Varieties

The time is ripe to introduce the crux of the thesis – toric varieties. In vague terms, toric varieties are affine varieties which contain an algebraic torus. This description is in fact not even *all* that vague as we will soon see. The sole definition of affine toric varieties is really not much more complicated than this.

The section is a compilation of Cox et al. [2011], Section 1.1 and Oda [1985], Section 1.2 spiced with a small number of short excerpts from Fulton [1993], Section 1.3. Proofs are original unless stated otherwise.

1.3.1 Algebraic Torus

We start off with the definition of an algebraic torus. The name *torus* here is used in the same sense as in Lie group theory. Algebraic torus of any dimension is not homeomorphic, in Zariski or classical topology, to the 'topological' torus – $\mathbb{S}^1 \times \mathbb{S}^1$.

Definition 1.3.1 (Algebraic Torus)**.** By an *algebraic torus*, we mean any affine variety T isomorphic to $(\mathbb{C}^*)^n$ which inherits the structure of a multiplicative group from $(\mathbb{C}^*)^n$.

Remark. It might have struck you that we claim $(\mathbb{C}^*)^n$ to be an affine *variety* in the preceding definition. To justify this claim, notice that $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1 x_2 \cdots x_n)$. Hence, backed up by **Proposition 1.1.9**, we identify $(\mathbb{C}^*)^n$ with $\mathbf{V}(1 - x_1 \cdots x_n x_{n+1})$. For a few lines, we concern ourselves with proving that this algebraic set is irreducible and its coordinate ring are precisely the Laurent polynomials.

Lemma 1.3.2 (Algebraic Torus Is Irreducible)**.** *The affine algebraic set* $(\mathbb{C}^*)^n = \mathbf{V}(1 - x_1 \cdots x_n x_{n+1})$ *is irreducible and its coordinate ring is the ring*

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

of Laurent polynomials.

Proof. First of all, notice that $1-x_1\cdots x_{n+1}$ is irreducible, so $I \coloneqq (1-x_1\cdots x_{n+1})$ is prime and $\mathbf{V}(I)$ is indeed an affine variety. Thanks to **Lemma 1.1.13**, we can calculate the coordinate ring of $(\mathbb{C}^*)^n$ as $R \coloneqq \mathbb{C}[x_1,\dots,x_n]_{x_1\cdots x_n}$. If $f/x_1\cdots x_n \in R$, then we can write f as a sum of Laurent monomials by simplifying each term. So, indeed $f \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. In the other direction, we have $\mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}] \subseteq R$ since $x_i = x_i/1$ and $x_i^{-1} = x_1\cdots x_{i-1}x_{i+1}\cdots x_n/x_1\cdots x_n$. □

We now introduce two important lattices tied to the concept of algebraic torus – characters and one-parameter subgroups.

Definition 1.3.3 (Character)**.** A *character* of an algebraic torus T is a morphism $\lambda : T \to \mathbb{C}^*$ which is also a group homomorphism.

It turns out that the quality of being both a morphism of affine varieties *and* a homomorphism of groups is severely restricting. We need to prove the following characterization of characters.

Lemma 1.3.4 (Characterizing Characters)**.** *Let $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ be a character of $(\mathbb{C}^*)^n$. Then,*

$$\chi(P) = p_1^{m_1}\cdots p_n^{m_n}$$

for all $P = (p_1,\dots,p_n) \in (\mathbb{C}^*)^n$ and adequate $(m_1,\dots,m_n) \in \mathbb{Z}^n$.

Proof. Since χ is a morphism $(\mathbb{C}^*)^n \to \mathbb{C}^*$ and we know that the coordinate ring of $(\mathbb{C}^*)^n$ are the Laurent polynomials, we identify χ with its defining polynomial $\chi \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. Let

$$\chi(x_1,\dots,x_n) = \sum_{i=-d}^{d} f_i x_n^i$$

for adequate $f_i \in \mathbb{C}[x_1^{\pm 1},\dots,x_{n-1}^{\pm 1}]$. Due to being a group homomorphism, χ satisfies $\chi(zw) = \chi(z)\chi(w)$ for $z, w \in (\mathbb{C}^*)^n$. Writing out the sums, we get

$$\sum_{i=-d}^{d} f_i(\tilde{z}\tilde{w})z_n^i w_n^i = \left(\sum_{i=-d}^{d} f_i(\tilde{z})z_n^i\right)\left(\sum_{i=-d}^{d} f_i(\tilde{w})w_n^i\right)$$

where $z = (z_1,\dots,z_n), w = (w_1,\dots,w_n)$ and $\tilde{z},\tilde{w}$ denote projections to the first $n-1$ coordinates. Comparing coefficients, we get

$$f_i(\tilde{z}\tilde{w}) = f_i(\tilde{z})f_i(\tilde{w})\ \forall -d \le i \le d \quad \text{and} \quad f_i(\tilde{z})f_j(\tilde{w}) = 0\ \forall i \neq j. \tag{1.3.1}$$

We argue that if $f_i \not\equiv 0$, then f_i cannot vanish at all points of $(\mathbb{C}^*)^{n-1}$. Suppose $f_i(P) = 0$ for some $P \in (\mathbb{C}^*)^{n-1}$. Choose $K \in \mathbb{N}$ large enough so that $g \coloneqq x_1^K\cdots x_{n-1}^K f_i \in \mathbb{C}[x_1,\dots,x_{n-1}]$. Then, g also vanishes at P. By **Lemma 1.1.8**, this cannot be true for all $P \in (\mathbb{C}^*)^{n-1}$, as $(\mathbb{C}^*)^{n-1}$ is Zariski dense in $\mathbb{C}^{n-1}$, so f_i does not vanish everywhere on $(\mathbb{C}^*)^{n-1}$. For this reason, the second equality in (1.3.1) necessarily means that f has only one term and can thus be written as $f(z) = f_m(\tilde{z})z_n^m$ where $f_m \in \mathbb{C}[x_1^{\pm 1},\dots,x_{n-1}^{\pm 1}]$ is also a group homomorphism $(\mathbb{C}^*)^{n-1} \to \mathbb{C}^*$ by the first equality in (1.3.1).

In comes induction. If $n = 1$, then f_m is some constant $a_m \in \mathbb{C}^*$, so $f_m(\tilde{z}\tilde{w}) = f_m(\tilde{z})f_k(\tilde{w})$ becomes just $a_m = a_m^2$ whence we get $a_m = 1$. In the case of $n = 1$, we thus arrive at the conclusion that $f(z) = z^m$. Applying the inductive hypothesis, we get $f_m(\tilde{z}) = z_1^{m_1} \cdots z_{n-1}^{m_{n-1}}$. Finally, we see that f is of the shape $f(z) = z_1^{m_1} \cdots z_n^{m_n}$ as desired. □

For a chosen $m \in \mathbb{Z}^n$, we denote by χ^m the corresponding character. The previous lemma implies that characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$ where $\chi^m \chi^{m'} \coloneqq \chi^{m+m'}$. If T is an arbitrary algebraic torus, its characters instead form a lattice which we denote by M.

Definition 1.3.5 (One-parameter Subgroup)**.** A *one-parameter subgroup* of an algebraic torus T is a morphism $\lambda : \mathbb{C}^* \to T$ which is also a group homomorphism.

Just as a character, each one-parameter subgroup is wholly determined by a choice of some $t \in \mathbb{Z}^n$. One-parameter subgroups are in a sense a *dual* concept to the one of characters. Hopefully, the characterization of one-parameter subgroups of $(\mathbb{C}^*)^n$ does not come unforeseen.

Lemma 1.3.6 (Characterizing One-parameter Subgroups)**.** *If $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$ is a one-parameter subgroup of $(\mathbb{C}^*)^n$, then*

$$\lambda(z) = (z^{t_1}, \dots, z^{t_n})$$

for every $z \in \mathbb{C}^*$ and adequate $t = (t_1, \dots, t_n) \in \mathbb{Z}^n$.

Proof. Notice that $\lambda(z) = (\lambda_1(z), \dots, \lambda_n(z))$ where $\lambda_i(z) : \mathbb{C}^* \to \mathbb{C}^*$ is a one-parameter subgroup of $\mathbb{C}^*$. Since characters and one-parameter subgroups of $\mathbb{C}^*$ coincide, we call out to the proof of **Lemma 1.3.4** and unveil the true form of λ_i, being $\lambda_i(z) = z^{t_i}$ for some $t_i \in \mathbb{Z}$. The argument is over because

$$\lambda(z) = (\lambda_1(z), \dots, \lambda_n(z)) = (z^{t_1}, \dots, z^{t_n})$$

for $t_1, \dots, t_n \in \mathbb{Z}$. □

We denote λ^t the one-parameter subgroup given by $t \in \mathbb{Z}^n$. If we define $\lambda^t \lambda^s = \lambda^{t+s}$ we see that one-parameter subgroups of an algebraic torus T also form a lattice. We shall name it N.

The names M and N for the lattices of characters and one-parameter subgroups are chosen to coincide with the terminology established in the beginning of **Section 1.2**. Thanks to the description provided by **Lemma 1.3.4** and **Lemma 1.3.6**, we also get a reasonably explicit formula for the bilinear pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$.

Take $m \in M$ and $u \in N$. The composition $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$ because it is a morphism of affine varieties and also a group homomorphism. We know that it is given by $t \mapsto t^l$ for some $l \in \mathbb{Z}$. The reasonable choice is $\langle m, u \rangle \coloneqq l$.

For correctness, we should check that $\langle \cdot, \cdot \rangle$ coincides with the standard dot product if $T = (\mathbb{C}^*)^n$. In this case, we have

$$(\chi^m \circ \lambda^u)(t) = \chi^m(t^{u_1}, \ldots, t^{u_n}) = \prod_{i=1}^{n} t^{m_i u_i} = t^{\sum_{i=1}^{n} m_i u_i}.$$

so indeed $\langle m, u \rangle = \sum_{i=1}^{n} m_i u_i$.

We need two more results about tori before we finally define an affine toric variety. The first concerns images of tori under morphisms.

Proposition 1.3.7. *Let T_1, T_2 be algebraic tori. Then,*

1. *If $\Phi : T_1 \to T_2$ is a morphism which is also a group homomorphism, then $\Phi(T_1)$ is an algebraic torus which is closed in T_2.*
2. *If H is both a subvariety and a subgroup of T_1, then H is an algebraic torus.*

Proof. See Humphreys [1975], Section 16. □

The second result, which becomes of importance later on when we discuss constructions of affine toric varieties states that linear maps given by the action of an algebraic torus T can be diagonalized simultaneously for all $t \in T$.

Before we proceed, we elucidate what we mean by 'linear action'.

Definition 1.3.8 (Algebraic Action)**.** Let V be an affine variety and T an algebraic torus. We call a map $\varphi : T \times V \to V$ an *algebraic action* of T on V if the following is satisfied.

- φ is polynomial.
- $\varphi(1_T, P) = P$ for all $P \in V$ where $1_T = (1, \ldots, 1) \in T$ is the identity element of T.
- $\varphi(t_1, \varphi(t_2, P)) = \varphi(t_1 t_2, P)$ for all $t_1, t_2 \in T, P \in V$.

In the special case when V is a finite-dimensional complex vector space, we call φ linear if $\varphi(t, -) : V \to V$ is linear for all $t \in T$. We typically denote $\varphi(t, P)$ by $t \cdot P$.

Now that we understand one another, consider the linear action of T on a finite-dimensional complex vector space W.

Given an element m from the character lattice M, we denote

$$W_m := \{w \in W \mid t \cdot w = \chi^m(t) w \ \forall t \in T\}.$$

Really, W_m is the space of common eigenvectors of linear maps $t \cdot - : W \to W$ with eigenvalue $\chi^m(t)$. We need the following result.

Proposition 1.3.9. *If T is an algebraic torus, M its character lattice and W a complex vector space, then*

$$W = \bigoplus_{m \in M} W_m.$$

Proof. See Springer [2008], Theorem 3.2.3. □

The lattice N of one-parameter subgroups of an algebraic torus T provides a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$ by $u \otimes t \mapsto \lambda^u(t)$. For this reason, it is often the case that T is written T_N, specifying the underlying lattice of one-parameter subgroups. We adhere to this norm.

1.3.2 Affine Toric Variety

Without further ado, let us dig in to the main course.

Definition 1.3.10 (Affine Toric Variety)**.** An affine variety $V \subseteq \mathbb{C}^n$ is called an *affine toric variety* if it contains an algebraic torus T_N as an open subset (with respect to the Zariski topology) and the action of T_N on itself extends to an action of T_N on V.

Remark. The required existence of an extension of every morphism $T_N \times T_N \to T_N$ to a morphism $T_N \times V \to V$ might seem unnatural at first. It is in fact quite difficult to come up with an affine variety that contains a torus whose action on itself *does not* extend to an action on the variety.

That is why we beg you hold your breath a bit longer, until the beginning of **Section 1.4** where we present a *projective* toric variety which does not satisfy this criterion and is all the uglier for it.

Just as promised, in this section we elucidate the mystery behind the link between polyhedral cones and toric varieties. We start by showing how we can construct toric varieties directly from algebraic tori by delimiting finite subsets of their character lattices.

The following constructions and most of their proofs are due to Cox et al. [2011]. Explanatory remarks and some missing details in proofs were added.

Suppose T_N is an algebraic torus with character lattice M. For a finite subset $\mathcal{A} := \{m_1, \dots, m_s\} \subseteq M$ define the map

$$\begin{aligned} \Phi_{\mathcal{A}} : T_N &\to (\mathbb{C}^*)^s, \\ P &\mapsto (\chi^{m_1}(P), \dots, \chi^{m_s}(P)). \end{aligned}$$

Proposition 1.3.11. *Denote by $V_{\mathcal{A}}$ the Zariski closure of $\Phi_{\mathcal{A}}(T_N)$ in $\mathbb{C}^s$ where $\Phi_{\mathcal{A}}$ is defined as above. Then, $V_{\mathcal{A}}$ is an affine toric variety containing the torus $\Phi_{\mathcal{A}}(T_N)$ with character lattice $\mathbb{Z}\mathcal{A} = \{\sum_{i=1}^s z_i m_i \mid z_i \in \mathbb{Z}\}$. Moreover, $\Phi_{\mathcal{A}}(T_N) \simeq (\mathbb{C}^*)^r$ where $r := \operatorname{rank} \mathbb{Z}\mathcal{A}$.*

Before we proceed with the proof, there is a tiny technical detail we need to take care of.

Lemma 1.3.12.

(a) Let X be a topological space and Y an irreducible subspace of X. Then, $\overline{Y}$ is also irreducible, where $\overline{Y}$ denotes the closure of Y in X.

(b) Let V, W be affine algebraic sets and $\varphi : V \to W$ a surjective morphism. If V is irreducible, then so is W.

Proof.

(a) If $\overline{Y} = Y_1 \cup Y_2$ for some proper closed subsets $Y_1, Y_2 \subsetneq \overline{Y}$, both of them non-empty, then $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$ which makes Y into a union of proper closed subsets, into a reducible subspace.

(b) Suppose $W = \varphi(V)$ is reducible and write $W = W_1 \cup W_2$ for two proper closed subsets $W_1, W_2 \subsetneq W$. By continuity of φ, $\varphi^{-1}(W_1)$ and $\varphi^{-1}(W_2)$ are proper closed subsets of V and $V = \varphi^{-1}(W_1) \cup \varphi^{-1}(W_2)$. □

Proof of Proposition 1.3.11. Note that $\Phi_{\mathcal{A}}$, being defined through characters, is a morphism and a group homomorphism. By **Proposition 1.3.7**, $\Phi_{\mathcal{A}}(T_N)$ is torus which is closed in $(\mathbb{C}^*)^s$. From this, it follows that $V_{\mathcal{A}} \cap (\mathbb{C}^*)^s = \Phi_{\mathcal{A}}(T_N)$ because $V_{\mathcal{A}} = \overline{\Phi_{\mathcal{A}}(T_N)}$ and is thus closed in $\mathbb{C}^s$. So $\Phi_{\mathcal{A}}(T_N)$ is open in $V_{\mathcal{A}}$. By **Lemma 1.3.12**, $\Phi_{\mathcal{A}}(T_N)$ is irreducible and consequently so is $V_{\mathcal{A}}$. Thus $V_{\mathcal{A}}$ is an affine variety.

To show that it is indeed toric, we extend an action $\Phi_{\mathcal{A}}(T_N) \times \Phi_{\mathcal{A}}(T_N) \to \Phi_{\mathcal{A}}(T_N)$ to $V_{\mathcal{A}}$. Since $T \subseteq (\mathbb{C}^*)^s$, T acts on $\mathbb{C}^s$ and takes affine varieties to affine varieties. Indeed, if $V = \mathbf{V}(I) \subseteq \mathbb{C}^s$ is an affine variety, then from the definition of an algebraic action, we get $f(t \cdot P) = t \cdot f(P)$ for all $f \in I, P \in V$. So, $t \cdot V$ is an affine variety.

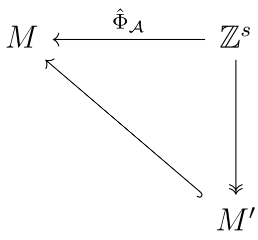
In our case, we care about the fact that $t \cdot V_{\mathcal{A}}$ is an affine variety for every $t \in \Phi_{\mathcal{A}}(T_N)$. Because $V_{\mathcal{A}} = \overline{\Phi_{\mathcal{A}}(T_N)}$ and $\Phi_{\mathcal{A}}(T_N) \subseteq t \cdot \Phi_{\mathcal{A}}(T_N)$, also $\Phi_{\mathcal{A}}(T_N) \subseteq t \cdot V_{\mathcal{A}}$ so $t \cdot V_{\mathcal{A}}$ contains the torus $\Phi_{\mathcal{A}}(T_N)$. However, its closure is $V_{\mathcal{A}}$, hence also $V_{\mathcal{A}} \subseteq t \cdot V_{\mathcal{A}}$. Repeating the same argument with t^{-1} gives $t \cdot V_{\mathcal{A}} \subseteq V_{\mathcal{A}}$. The action of $\Phi_{\mathcal{A}}(T_N)$ on itself thus extends to action on $V_{\mathcal{A}}$, proving that $V_{\mathcal{A}}$ is an affine toric variety.

We compute the character lattice of $\Phi_{\mathcal{A}}(T_N)$ which we for now denote by M'. We first observe that each morphism of algebraic tori $\varphi : T_1 \to T_2$ induces a map $\hat{\varphi} : M_2 \to M_1$ of their character lattices defined by the equality

$$\chi^{m_2} \circ \varphi = \chi^{\hat{\varphi}(m_2)}$$

for $m_2 \in M_2$. The induced map $\hat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \to M$ from the character lattice of $(\mathbb{C}^*)^s$ to the character lattice of T_N thus sends the canonical basis $\{e_1, \ldots, e_s\}$ to $\mathcal{A} = \{m_1, \ldots, m_s\}$. It follows that $\hat{\Phi}_{\mathcal{A}}(\mathbb{Z}^s) = \mathbb{Z}\mathcal{A}$. However, $\Phi_{\mathcal{A}}$ can also be seen as the surjective map $T_N \twoheadrightarrow \Phi_{\mathcal{A}}(T_N)$ and so $\hat{\Phi}_{\mathcal{A}|M'}$ is also an injective map

$M' \hookrightarrow M$. Finally, we also have the natural surjective map $\mathbb{Z}^s \twoheadrightarrow M'$ induced by the inclusion $\Phi_{\mathcal{A}}(T_N) \hookrightarrow (\mathbb{C}^*)^s$. Hence, we have this following commutative diagram



of character lattices. From this, it follows that $M' \simeq \mathbb{Z}\mathcal{A}$.

If $r \coloneqq \operatorname{rank} \mathbb{Z}\mathcal{A}$, then $\Phi_{\mathcal{A}}(T_N) \simeq (\mathbb{C}^*)^r$ by definition of the algebraic torus. $\square$

The preliminary step of choosing a finite subset of the character lattice of an algebraic torus might remind perceptive readers of a similar construction we did only a section ago. Indeed, recall that the generators of a rational polyhedral cone are always elements of a finite subset of a lattice. We have shown in **Section 1.2** that the points of the dual cone which are integral form an affine semigroup. We shall soon define *semigroup algebras* which quite remarkably happen to also be coordinate rings of affine toric varieties. Hence, every rational polyhedral cone directly defines an affine semigroup which again directly defines an affine toric variety.

Even more remarkably, the converse is also true. The coordinate ring of every affine toric variety is a semigroup algebra. To elucidate why, we need go astray momentarily and have a serious debate about *toric ideals*.

We would like to understand the structure of $\mathbf{I}(V_{\mathcal{A}})$ where $\mathcal{A} \subseteq M$ is a finite subset of the lattice of characters of an algebraic torus T_N. Fortunately, with a little bit of work, a relatively simple explicit description will make itself conspicuous.

We have shown that the character lattice of $\Phi_{\mathcal{A}}(T_N)$ is $\mathbb{Z}\mathcal{A}$. In light of that we define a map of lattices $\mathbb{Z}^s \to \mathbb{Z}\mathcal{A}$ which sends e_i to m_i, where $\mathcal{A} = \{m_1, \ldots, m_s\}$ and $\{e_1, \ldots, e_s\}$ is the canonical basis of $\mathbb{Z}^s$. We denote L the kernel of this map. For an s-tuple $l = (l_1, \ldots, l_s) \in L$, we denote

$$l_+ \coloneqq \sum_{l_i > 0} l_i e_i \quad \text{and} \quad l_- \coloneqq \sum_{l_i < 0} -l_i e_i.$$

Then, $l = l_+ - l_-$ and $l_+, l_- \in \mathbb{N}^s$. Moving on, for $z = (z_1, \ldots, z_s) \in \mathbb{Z}^s$, we simplify the notation somewhat and write

$$x^z \coloneqq \prod_{i=1}^{s} x_i^{z_i}.$$

We will show that for every $l \in L$, the binomial $x^{l_+} - x^{l_-}$ vanishes on $\Phi_{\mathcal{A}}(T_N)$ and consequently also on $V_{\mathcal{A}}$. From this, we will get the inclusion $(x^{l_+} - x^{l_-} \mid l \in L) \subseteq \mathbf{I}(V_{\mathcal{A}})$. As it turns out, the opposite inclusion is also true.

Let us first ascertain that $x^{l_+} - x^{l_-}$ is 0 on $\Phi_{\mathcal{A}}(T_N)$. Recall the definition $\Phi_{\mathcal{A}} = (\chi^{m_1}, \dots, \chi^{m_s})$. Then,

$$x^{l_+} \circ (\chi^{m_1}, \dots, \chi^{m_s}) = \prod_{i, l_i > 0} (\chi^{m_i})^{l_i} = \prod_{i, l_i > 0} \chi^{m_i l_i} = \chi^{\sum_{i, l_i > 0} m_i l_i}$$

and analogously for x^{l_-}. Since $\sum_{i=1}^{s} l_i m_i = 0$ by definition of L, we have that $\sum_{l_i > 0} l_i m_i = \sum_{l_i < 0} -l_i m_i$ because $l = l_+ - l_-$. By the computation above, this precisely means that $x^{l_+} \circ \Phi_{\mathcal{A}} = x^{l_-} \circ \Phi_{\mathcal{A}}$, as desired.

The other inclusion, concretely $\mathbf{I}(V_{\mathcal{A}}) \subseteq (x^{l_+} - x^{l_-} \mid l \in L)$, is harder to prove. We start by finding another useful description of the ideal on the right.

Lemma 1.3.13 (Exercise 1.1.2 in Cox et al. [2011]). *Let $\mathcal{A} \coloneqq \{m_1, \dots, m_s\} \subseteq M$ and L be the kernel of the lattice map $\mathbb{Z}^s \to \mathbb{Z}\mathcal{A}$ which sends $\{e_1, \dots, e_s\}$ to $\mathcal{A}$. Then,*

$$(x^{l_+} - x^{l_-} \mid l \in L) = (x^{\alpha} - x^{\beta} \mid \alpha, \beta \in \mathbb{N}^s \wedge \alpha - \beta \in L).$$

Proof. The rightward inclusion is evident. Both l_+ and l_- lie in $\mathbb{N}^s$ and $l_+ - l_- = l \in L$.

Label I_L the ideal on the left. Write $\alpha = (a_1, \dots, a_s)$, $\beta = (b_1, \dots, b_s)$. There exists some $l \in L$ such that $\alpha - \beta = l$. Without loss of generality assume that the coordinates of $l = (l_1, \dots, l_s)$ are ordered in such a way that there exists some $j \leq s$ satisfying $l_i \geq 0$ for $i \leq j$ and $l_i \leq 0$ for $i \geq j+1$. If this were not the case, we could just define indexing sets $I, J \subseteq \{1, \dots, s\}$ such that $l_i \geq 0$ for $i \in I$ and $l_j \leq 0$ for $j \in J$. The ensuing argument remains unaffected, only indices get uglier.

Notice that we have $a_i - b_i = l_i$ for each $i \leq s$, hence necessarily $a_i \geq b_i$ for $i \leq j$ and $a_i \leq b_i$ for $i \geq j+1$ because a_i, b_i are natural numbers. We get the following factorization of $x^{\alpha} - x^{\beta}$.

$$x^{\alpha} - x^{\beta} = \prod_{i=1}^{s} x_i^{a_i} - \prod_{i=1}^{s} x_i^{b_i} = \prod_{i=1}^{j} x_i^{b_i} \prod_{i=j+1}^{s} x_i^{a_i} \left(\prod_{i=1}^{j} x_i^{a_i - b_i} - \prod_{i=j+1}^{s} x_i^{b_i - a_i} \right).$$

However, the binomial in the parentheses is exactly $x^{l_+} - x^{l_-}$ because $(l_1, \dots, l_s) = (a_1 - b_1, \dots, a_s - b_s)$. We have $x^{\alpha} - x^{\beta} \in I_L$ as was required to show. □

We keep the established notation $I_L \coloneqq (x^{l_+} - x^{l_-} \mid l \in L)$. We now proceed to show the inclusion $\mathbf{I}(V_{\mathcal{A}}) \subseteq I_L$. The main idea behind the proof is taken from Sturmfels [1996], Lemma 4.1.

Proposition 1.3.14 (Ideal of An Affine Toric Variety). *The ideal of an affine toric variety $V_{\mathcal{A}} \subseteq \mathbb{C}^s$ is exactly $\mathbf{I}(V_{\mathcal{A}}) = I_L$.*

Proof. We already know that $I_L \subseteq \mathbf{I}(V_{\mathcal{A}})$.

We proceed to show the opposite inclusion. Let $<$ denote the lexicographical order on monomials of $\mathbb{C}[x_1, \dots, x_s]$, that is, $x^{\alpha} < x^{\beta}$ if there exists an index

$j \leq s$ such that $a_j < b_j$ and $a_i = b_i$ for all $i < j$. Also, fix an isomorphism $T_N \simeq (\mathbb{C}^*)^r$. We may thus assume $M = \mathbb{Z}^r$. The map $\Phi_{\mathcal{A}} : T_N = (\mathbb{C}^*)^r \to \mathbb{C}^s$ is then given Laurent monomials t^{m_i} in variables $t_1, \ldots, t_r$ for $m_i \in \mathbb{Z}^r$. If we suppose $I_L \subsetneq \mathbf{I}(V_{\mathcal{A}})$, then we can pick $f \in \mathbf{I}(V_{\mathcal{A}}) \setminus I_L$ with minimal leading monomial x^α. After potential rescaling of f by a suitable constant, x^α becomes its leading term.

Because $f \in \mathbf{I}(V_{\mathcal{A}})$, $f(t^{m_1}, \ldots, t^{m_s}) \equiv 0$ as a Laurent polynomial in variables $t_1, \ldots, t_r$ for every $m_1, \ldots, m_s \in \mathbb{Z}^r$. In particular, f must contain a monomial x^β which cancels out x^α. By choice of x^α, we have $x^\beta < x^\alpha$. Also, from

$$x^\alpha(t^{m_1}, \ldots, t^{m_s}) - x^\beta(t^{m_1}, \ldots, t^{m_s}) = 0$$

we get

$$\prod_{i=1}^{s} t_i^{a_i m_i} = \prod_{i=1}^{s} t_i^{b_i m_i},$$

hence

$$\sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i,$$

which implies $\alpha - \beta \in L$. By **Lemma 1.3.13**, $x^\alpha - x^\beta \in I_L$. This means that $f - (x^\alpha - x^\beta)$ lies in $\mathbf{I}(V_{\mathcal{A}}) \setminus I_L$ and has smaller leading term than f. A contradiction. □

If $L \subseteq \mathbb{Z}^s$ is now any sublattice, we call the ideal

$$I_L \coloneqq (x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \wedge \alpha - \beta \in L)$$

a *lattice* ideal. A prime lattice ideal is styled *toric*. This nomenclature is justified by the previous proposition as $\mathbf{I}(V_{\mathcal{A}})$ is a lattice ideal and also a prime ideal due to $V_{\mathcal{A}}$ being irreducible. The fact that given a toric ideal I, the affine variety $\mathbf{V}(I)$ is also toric is subject to be proven later in the section. For now, we are satisfied with proving a useful characterization of toric ideals.

Proposition 1.3.15 (Characterizing Toric Ideals). *An ideal of* $\mathbb{C}[x_1, \ldots, x_s]$ *is toric if and only if it is prime and generated by binomials.*

Proof. Toric ideals are prime and generated by binomials by definition.

If $I \subseteq \mathbb{C}[x_1, \ldots, x_s]$ is a prime ideal generated by binomials, we claim that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is a torus. Firstly, it is non-empty because $(1, \ldots, 1) \in \mathbf{V}(I) \cap (\mathbb{C}^*)^s$. Secondly, it is a subgroup of $(\mathbb{C}^*)^s$. This is true because monomials satisfy $x^\alpha(PQ) = x^\alpha(P)x^\alpha(Q)$ for all points $P, Q \in \mathbb{C}^s$. Hence, if $x^\alpha - x^\beta \in I$ and $P, Q \in \mathbf{V}(I)$, then

$$x^\alpha(PQ) = x^\alpha(P)x^\alpha(Q) = x^\beta(P)x^\beta(Q) = x^\beta(PQ),$$

so $(x^\alpha - x^\beta)(PQ) = 0$. Finally, $\mathbf{V}(I)$ is irreducible, so $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is irreducible and thus a subvariety of $(\mathbb{C}^*)^s$. By **Proposition 1.3.7**, $T \coloneqq \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is a torus.

Projection to the i-th coordinate defines a character $\chi^{m_i} : T \hookrightarrow (\mathbb{C}^*)^s \twoheadrightarrow \mathbb{C}^*$ of T for some $m_i \in M$. If we let $\mathcal{A} := \{m_1, \ldots, m_s\}$, then $\Phi_{\mathcal{A}}((\mathbb{C}^*)^s) = T$ and $V_{\mathcal{A}} = \overline{T} = \overline{\mathbf{V}(I) \cap (\mathbb{C}^*)^s} = \mathbf{V}(I)$ where the last equality holds because $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is Zariski dense in $\mathbf{V}(I)$. By the Nullstellensatz, $I = \mathbf{I}(V_{\mathcal{A}})$ for I is prime, therefore radical. By **Proposition 1.3.14**, $\mathbf{I}(V_{\mathcal{A}})$ is toric. $\square$

Now that we know what a toric ideal is and that it is safe for kids, we make a U-turn and land right back on affine semigroups.

Two of the three defining properties of affine semigroups were finite number of generators and ability to be embedded into a lattice. Buying a doughnut T_N with character lattice M, each finite set $\mathcal{A} \subseteq M$ gives the affine semigroup $\mathbb{N}\mathcal{A} \subseteq M$. Conversely, for every affine semigroup $\mathsf{S} \subseteq M$ there is some finite generating set $\mathcal{A} \subseteq M$ such that $\mathsf{S} = \mathbb{N}\mathcal{A}$.

Definition 1.3.16 (Semigroup Algebra)**.** Let T_N be an algebraic torus with character lattice M and $\mathsf{S} \subseteq M$ an affine semigroup. Each element $m \in M$ gives a character χ^m. We define the *semigroup algebra* $\mathbb{C}[\mathsf{S}]$ as

$$\mathbb{C}[\mathsf{S}] := \left\{ \sum_{m \in \mathsf{S}} c_m \chi^m \mid c_m \in \mathbb{C} \wedge c_m \neq 0 \text{ for only finitely many } m \in \mathsf{S} \right\}$$

with multiplication given by $\chi^m \chi^{m'} = \chi^{m+m'}$.

In particular, if $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$ is such that $\mathsf{S} = \mathbb{N}\mathcal{A}$, then

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}].$$

Proposition 1.3.17 (Affine Toric Variety From Affine Semigroup)**.** *Let $\mathcal{A} \subseteq M$ be a finite subset. Let $\mathsf{S} := \mathbb{N}\mathcal{A}$, then*

1. $\mathbb{C}[\mathsf{S}]$ *is an integral domain and finitely generated as a $\mathbb{C}$-algebra.*
2. $\mathrm{Specm}(\mathbb{C}[\mathsf{S}])$ *is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathcal{A}$. Moreover,* $\mathrm{Specm}(\mathbb{C}[\mathsf{S}]) \simeq V_{\mathcal{A}}$.

There are two preparatory steps to be made before the proof of this key proposition.

The first step is to observe that the lattice M can itself be seen as an affine semigroup. If $\{e_1, \ldots, e_r\}$ is the basis of M as a lattice, then $\{\pm e_1, \ldots, \pm e_r\}$ are its generators as an affine semigroup. One gets a canonical isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[M] \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ which sends $\chi^{\pm e_i}$ to $x_i^{\pm 1}$.

The second step is to analyse the kernel of the $\mathbb{C}$-algebra homomorphism $\Phi_{\mathcal{A}}^* : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}[M]$ given by $x_i \mapsto \chi^{m_i}$ where $\mathcal{A} = \{m_1, \ldots, m_s\}$. To justify this notation, we view the map $\Phi_{\mathcal{A}}$ as a morphism $T_N \to \mathbb{C}^s$. Then, the induced map $\Phi_{\mathcal{A}}^*$ is a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[x_1, \ldots, x_s]$ to $\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \simeq \mathbb{C}[M]$. We will show that $\ker \Phi_{\mathcal{A}}^* = \mathbf{I}(V_{\mathcal{A}})$.

The veracity of this claim is straightforward to verify. If $f \in \mathbf{I}(V_{\mathcal{A}})$, then f vanishes on the image of $\Phi_{\mathcal{A}}$ thus $\Phi_{\mathcal{A}}^*(f) \equiv 0$. Symmetrically, if $\Phi_{\mathcal{A}}^*(f) = f \circ \Phi$ is identically 0, then f vanishes on the image of $\Phi_{\mathcal{A}}$. Hence, f vanishes also on the closure of the image of $\Phi_{\mathcal{A}}$ due to continuity of polynomial maps in Zariski topology. So $f \in \mathbf{I}(V_{\mathcal{A}})$.

Proof of Proposition 1.3.17.

1. If $\mathcal{A} = \{m_1, \dots, m_s\}$, then

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}],$$

thus finitely generated. Because $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[M] \simeq \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$, $\mathbb{C}[\mathsf{S}]$ is an integral domain.

2. Observe that the image of $\Phi_{\mathcal{A}}^* : \mathbb{C}[x_1, \dots, x_s] \to \mathbb{C}[M], x_i \mapsto \chi^{m_i}$ is precisely $\mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}] = \mathbb{C}[\mathsf{S}]$. We know that $\ker \Phi_{\mathcal{A}}^* = \mathbf{I}(V_{\mathcal{A}})$, hence

$$\mathbb{C}[V_{\mathcal{A}}] = \mathbb{C}[x_1, \dots, x_s]/\mathbf{I}(V_{\mathcal{A}}) = \mathbb{C}[x_1, \dots, x_s]/\ker \Phi_{\mathcal{A}}^* \simeq \operatorname{im} \Phi_{\mathcal{A}}^* = \mathbb{C}[\mathsf{S}].$$

This shows that $\operatorname{Specm}(\mathbb{C}[\mathsf{S}]) \simeq V_{\mathcal{A}}$. The character lattice of the torus of $\operatorname{Specm}(\mathbb{C}[\mathsf{S}])$ is $\mathbb{Z}\mathsf{S}$ by **Proposition 1.3.11**. Because $\mathsf{S} = \mathbb{N}\mathcal{A}$, we have $\mathbb{Z}\mathsf{S} = \mathbb{Z}\mathcal{A}$, which completes the proof. □

Corollary 1.3.18 (Affine Variety From A Cone). *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational convex polyhedral cone. Then* $\operatorname{Specm}(\mathbb{C}[\sigma^* \cap M])$*, where M is the dual lattice to N, is an affine toric variety.*

Proof. The set $\sigma^* \cap M$ is an affine semigroup by Gordan's Lemma. Hence, $\mathbb{C}[\sigma^* \cap M]$ is a semigroup algebra and $\operatorname{Specm}(\mathbb{C}[\sigma^* \cap M])$ an affine toric variety by **Proposition 1.3.17**. □

We promised to demonstrate that there is also a convex polyhedral cone hidden in the heart of an affine toric variety. To keep our promise, we need to argue that the coordinate ring of every affine toric variety is a semigroup algebra and that every semigroup algebra is of the shape $\mathbb{C}[\sigma^* \cap M]$ for a rational cone σ. Proof of the second fact is straightforward. If S is an affine semigroup generated by a finite set $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$, then choosing $\sigma \coloneqq (\operatorname{Cone}(m_1, \dots, m_s))^*$, that is, σ is the dual to the cone generated by $\mathcal{A}$, we immediately have $\mathsf{S} = \sigma^* \cap M$ because $\sigma^* = \operatorname{Cone}(m_1, \dots, m_s)$.

The proof of the first fact takes more work and to do it we need to pull the strange factorization of a vector space into a sum of eigenspaces of a torus action in **Proposition 1.3.9** out of the back of your mind as it is the main ingredient.

Our vector space W will be the queen herself, the semigroup algebra $\mathbb{C}[M]$. We define the action of T_N on $\mathbb{C}[M]$ by the formula $(t \cdot f)(P) = f(t^{-1} \cdot P)$ where $P \in T_N$ and $f \in \mathbb{C}[M]$. This might seem like it has fallen straight from the sky but to make things clearer we would need to dive into the theory of linear algebraic groups which we do not intend to do. We ask dear readers to cut us some slack on this one.

The following technical lemma is the last rung on the ladder.

Lemma 1.3.19. *If $A \subseteq \mathbb{C}[M]$ is a subspace of $\mathbb{C}[M]$ which is invariant under the action of T_N on $\mathbb{C}[M]$, then*

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. The leftward inclusion is clear. Pick a non-zero $f \in A$. Since $f \in \mathbb{C}[M]$, we can write

$$f = \sum_{m \in \mathcal{B}} c_m \chi^m$$

for some finite $\mathcal{B} \subseteq M$. Let $B \coloneqq \operatorname{span}(\chi^m \mid m \in \mathcal{B})$. We have

$$(t \cdot \chi^m)(P) = \chi^m(t^{-1} \cdot P) = \chi^m(t^{-1})\chi^m(P)$$

for all $t, P \in T_N$ because characters are group homomorphisms. It follows that B and in addition $B \cap A$ are invariant under the action of T_N. Because $B \cap A$ has finite dimension by definition of B, **Proposition 1.3.9** implies that $B \cap A$ is spanned by eigenvectors common to all linear maps $t \cdot - : \mathbb{C}[M] \to \mathbb{C}[M]$. However, eigenvectors in $\mathbb{C}[M]$ are characters, so $B \cap A$ is spanned by characters of T_N. Because $f \in B \cap A$ is a $\mathbb{C}$-linear combination of characters χ^m for $m \in \mathcal{B}$, we infer $\chi^m \in A$ for all $m \in \mathcal{B}$. Hence $f \in \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$. □

A direct consequence of this lemma is the proof of the fact that coordinate rings of affine toric varieties are semigroup algebras.

Proposition 1.3.20. *Let V be an affine toric variety. Then there exists an affine semigroup $\mathsf{S} \subseteq M$ such that $V = \operatorname{Specm}(\mathbb{C}[\mathsf{S}])$.*

Proof. Let V contain the algebraic torus T_N with character lattice M. We have observed that $\mathbb{C}[M]$ is the coordinate ring of T_N. The inclusion map $\iota : T_N \hookrightarrow V$, induces the map $\iota^* : \mathbb{C}[V] \to \mathbb{C}[M]$ on coordinate rings. This map is injective because T_N, being open in V, is Zariski dense in V. This means that an extension of a polynomial (thus continuous) map $f : T_N \to \mathbb{C}$ to a polynomial map $\overline{f} : V \to \mathbb{C}$ is (if it exists) uniquely given by $\iota^*(\overline{f}) = \overline{f} \circ \iota = f$, so ι^* is indeed injective. In light of that, we regard $\mathbb{C}[V]$ as a subalgebra of $\mathbb{C}[M]$.

The action of T_N on V is a morphism $T_N \times V \to V$. If we define, as we did before, the action of T_N on $\mathbb{C}[V]$ by $(t \cdot f)(P) = f(t^{-1} \cdot P)$ for $t \in T_N, P \in V$ and $f \in \mathbb{C}[V]$, then $t \cdot f \in \mathbb{C}[V]$ because $t^{-1} \cdot P \in V$, so $\mathbb{C}[V]$ is invariant under the action of T_N. By **Lemma 1.3.19**, we can decompose

$$\mathbb{C}[V] = \bigoplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C} \cdot \chi^m.$$

Therefore, $\mathbb{C}[V] = \mathbb{C}[\mathsf{S}]$ for the semigroup $\mathsf{S} \coloneqq \{m \in M \mid \chi^m \in \mathbb{C}[V]\}$.

Because $\mathbb{C}[V]$ is finitely generated, we can find $f_1, \ldots, f_s \in \mathbb{C}[V]$ such that $\mathbb{C}[V] = \mathbb{C}[f_1, \ldots, f_s]$. Expressing each function f_i in terms of characters following the decomposition of $\mathbb{C}[V]$ above, we obtain a finite set of generators for S. So, S is indeed an affine semigroup. □

We have reached the crux of the section where we collect our findings into one big theorem. It exactly says that the three different views of affine toric varieties we have presented throughout the section are in reality one and the same.

Theorem 1.3.21 (On Building of Affine Toric Varieties)**.** *Let V be an affine variety. The following statements are equivalent.*

1. *V is an affine toric variety.*
2. *$V = V_{\mathcal{A}}$ for a finite subset $\mathcal{A}$ of a lattice.*
3. *$V = \mathbf{V}(I)$ for a toric ideal I.*
4. *$V = \mathrm{Specm}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S.*

Proof. The equivalence $(b) \iff (c)$ is the content of **Proposition 1.3.14** and both $(b) \iff (d)$ and $(d) \implies (a)$ follow from **Proposition 1.3.17**. Finally, the implication $(a) \implies (d)$ is covered by the last **Proposition 1.3.20**. □

The proof of **Proposition 1.3.20** also instructs us how to construct a cone from an affine toric variety. This is the last result we state before falling headlong onto the next section where we have fun making collages out of cones and toric varieties.

Corollary 1.3.22 (Cone From An Affine Toric Variety)**.** *Let V be an affine toric variety with torus T_N whose character lattice is M. Then there exists a rational convex polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ such that $V = \mathrm{Specm}(\mathbb{C}[\sigma^* \cap M])$.*

Proof. From **Proposition 1.3.20**, we know that there exists an affine semigroup S such that $V = \mathrm{Specm}(\mathbb{C}[\mathsf{S}])$. Let $\mathcal{A} := \{m_1, \ldots, m_s\} \subseteq M$ be its set of generators. Taking $\sigma := (\mathrm{Cone}(m_1, \ldots, m_s))^*$ gives $\mathsf{S} = \sigma^* \cap M$. □

2. Abstract Toric Varieties, Fans and Resolution of Singularities

Building upon concepts discussed in the first chapter, we define a 'manifold of toric varieties', which is commonly called an *abstract* toric variety, and is essentially a designation for a complex analytical space that locally looks like an affine toric variety. In a similar manner, we glue together convex polyhedral cones to objects which are called 'fans' and show a neat connection to abstract toric varieties. And, number theorists rejoice, we finish off explaining how singular points on toric surfaces (abstract toric varieties of dimension two) can be removed by inserting new cones whose bases are given by specific continued fractions.

2.1 Abstract Varieties

Before we can start gluing together toric varieties, we would like to try it with your normal boring mundane stereotypical affine varieties first. One should not tackle muffins before knowing how to make dough.

Unsurprisingly, as things get more abstract, category theory drives in, wearing sunglasses and a childish smirk. We hope that readers are at least marginally acquainted with categorical concepts like *morphisms*, *sheaves* or, well, a *category*. We do not wish to make an introduction to category theory inside the thesis itself, as we did in the case of algebraic geometry in **Section 1.1**. Hopefully, we will not need to since not as much machinery is needed for the ensuing constructions to make sense.

This particular section mostly follows Cox et al. [2011], Section 3.0. Oda [1985] gives many useful insights but does the construction of abstract varieties in a much more analytical (meaning less algebraic) manner and Fulton [1993] in a way bypasses abstract varieties altogether. As was the case with all previous sections, proofs are either original or augmented versions of those existent in the original text. The latter case is always mentioned.

The way we construct an abstract variety is similar to the way one constructs a topological manifold, albeit less concrete.

First, given a finite collection $\{V_\alpha\}_{\alpha\in A}$ of affine varieties, we need to find a way to naturally glue them together. The first condition is the ability to move back and forth between them. The following may remind readers of the definition of transition maps within some differential structure. For each V_α, we require Zariski open sets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \simeq V_{\alpha\beta}$ satisfying the following 'compatibility' conditions:

(C1) $g_{\alpha\beta} = g_{\beta\alpha}^{-1} \quad \forall \alpha, \beta \in A$, (C2) $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha} \quad \forall \alpha, \beta, \gamma \in A$.

We daringly believe dear readers are getting lost in the indices. Let us shed some light on the situation. Note that $V_{\beta\alpha} \subseteq V_\alpha$ and $V_{\alpha\beta} \subseteq V_\beta$. A funny analogue to the condition (C1) is that people build two-way roads. The road you take from one city to another is often the same one you take when coming back. Of course, this is not the 'actual' reason this property of transition maps is important. It is more of a comparison to make things settle in better. In this case, we say that the road from $V_{\alpha\beta}$ to $V_{\beta\alpha}$ is the same as the inverse one, from $V_{\beta\alpha}$ to $V_{\alpha\beta}$.

The condition (C2) is possibly harder to decode. It says two related things. First, transition maps preserve intersections. What we mean by this is that starting in the intersection $V_{\beta\alpha} \cap V_{\gamma\alpha} \subseteq V_\alpha$ of open sets corresponding to β and γ inside V_α and then moving to V_β, we end up not only inside $V_{\alpha\beta}$ (which is true by definition of $g_{\beta\alpha}$) but also inside $V_{\gamma\beta}$, the open set in V_β corresponding to γ. Finally, the second condition is that the road

$$V_{\beta\alpha} \cap V_{\gamma\alpha} \xrightarrow{g_{\beta\alpha}} V_{\alpha\beta} \cap V_{\gamma\beta} \xrightarrow{g_{\gamma\beta}} V_{\alpha\gamma} \cap V_{\beta\gamma}$$

is the same as

$$V_{\beta\alpha} \cap V_{\gamma\alpha} \xrightarrow{g_{\gamma\alpha}} V_{\beta\gamma} \cap V_{\alpha\gamma} \; .$$

We finish our construction by gluing the V_α together. Basically, what we do is that we identify points on different varieties which are mutually reachable via the transition maps. Let $Y := \coprod_{\alpha \in A} V_\alpha$. We define the relation $\sim$ on Y in such a way that $a \sim b$ if there exist $\alpha, \beta \in A$ such that $a \in V_\alpha$, $b \in V_\beta$ and $g_{\beta\alpha}(a) = b$. Condition (C1) assures that $\sim$ is symmetric and (C2) that it is transitive. Hence, we define the abstract variety X as the quotient space $Y/\sim$ with the quotient topology. For each $\alpha \in A$, the set

$$U_\alpha := \{[a] \in X \mid a \in V_\alpha\} \tag{2.1.1}$$

is open in X and maps $h_\alpha : V_\alpha \to U_\alpha, a \mapsto [a]$ are homeomorphisms between V_α and U_α. So, X is locally an affine variety.

We now proceed to augment the constructions made in **Section 1.1** for affine varieties and move them to the language of abstract varieties. This basically means defining the local properties of abstract varieties.

We begin with what is typically called a *structure sheaf* or *sheaf of regular functions* of an affine variety. Local rings at points crumble under the inexorable flow of time for being, well, a tad *too* local. Since the elements of X are officially not points but equivalence classes, we must extend our definition of local rings at points to open sets. Before we do that, we let you munch on some sheaf theory.

Definition 2.1.1 (Presheaf)**.** Given X a topological space, a datum $\mathcal{F}$ is called a *presheaf (of sets)* on X if

(PSH1) for each open set $U \subseteq X$ there is a set $\mathcal{F}(U)$,

(PSH2) for each pair of open sets V, U with $V \subseteq U$ there is a restriction map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_U^U = 1_U$,

(PSH3) for each triple of open sets U, V, W such that $W \subseteq V \subseteq U$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.

Elements of $\mathcal{F}(U)$ are normally called *sections* of $\mathcal{F}$ over U and we denote $s_{|V} := \rho_V^U(s)$ for $s \in \mathcal{F}(U)$ if $V \subseteq U$.

A presheaf $\mathcal{F}$ on X is in addition called a *sheaf*, if for every $U \subseteq X$ open, elements of $\mathcal{F}(U)$ are defined locally and can be glued together. Formally, we require:

(SH1) If $\{U_i \mid i \in I\}$ is an open cover of U and $s, t \in \mathcal{F}(U)$ are such that $s_{|U_i} = t_{|U_i}$ for all $i \in I$, then $s = t$.

(SH2) If $\{U_i \mid i \in I\}$ is an open cover of U and $\{s_i \in \mathcal{F}(U) \mid i \in I\}$ is a family of sections satisfying $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all $i, j \in I$, then there exists some $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$ for all $i \in I$.

Now, this was a tedious definition but hopefully the notions of locality and gluing seeped through. It will take a while and many small steps to explain why this notion is useful to our cause. Since abstract varieties are meshes of affine varieties, we shall start with the latter.

Definition 2.1.2 (Regular Function)**.** If V is an affine variety and $U \subseteq V$ an open set, we say that a function $\varphi : U \to \mathbb{C}$ is *regular* if there exists a rational function $r \in \mathbb{C}(V)$ which is defined at each point of U and $\varphi(P) = r(P)$ for each $P \in U$.

The ring of regular functions on U is denoted $\mathcal{O}_V(U)$.

Remark. Observe that a rational function $r \in \mathbb{C}(V)$ is an element of $\mathcal{O}_{V,P}$, the local ring of V at P if r is defined at P. In the preceding definition, we considered $\mathcal{O}_V(U)$ to be those rational functions defined for every P in U. It follows that

$$\mathcal{O}_V(U) = \bigcap_{P \in U} \mathcal{O}_{V,P}.$$

We could have used this as our definition of a regular function.

This description also makes it easy to see that regular functions are continuous in the Zariski topology. On the neighbourhood of each point, they are defined as the quotient of two polynomial, thus continuous maps, with non-zero denominator on said neighbourhood.

Observe that $\mathcal{O}_V$ is a presheaf on V. Not of sets, no, of $\mathbb{C}$-algebras but that does not pose any difficulties. Indeed, the restriction maps ρ_W^U are defined naturally by restricting regular functions on U to regular functions on W. It is, in fact, also a sheaf. We will dedicate a number of lines to the verification of this claim.

Firstly, fix some open $U \subseteq V$ with an open cover $\{U_i \mid i \in I\}$. If $f \neq g \in \mathcal{O}_V(U)$, then by continuity we find an open $W \subseteq U$ such that $f_{|W} \neq g_{|W}$. Since U_i cover U, there exists also some i such that $f_{|U_i} \neq g_{|U_i}$. This sates the needs of (SH1).

Condition (SH2) is not much harder. The construction we are about to do here is, in a more general setting, formalized in category theory as a *product*. We shall not define it properly as we wish to avoid technical preliminaries such as diagrams, cones, limits and whatnot.

Given $f_i \in \mathcal{O}_V(U_i)$ for each $i \in I$, we simply define $f \in \mathcal{O}_V(U)$ by $f_{|U_i} \coloneqq f_i$. U_i cover U so this defines a function on the entirety of U. The condition $f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$ ensures that this definition makes sense, that is, $f(P)$ is indeed defined uniquely for each point $P \in U$. Should P lie in more than one U_i, its images under the corresponding f_i coincide. Finally, f is regular on the entirety of U. If $r_i \in \mathbb{C}(V)$ are the rational functions witnessing the regularity of f_i for each $i \in I$, then gluing them together exactly the same way as we did the f_i gives a rational function $r \in \mathbb{C}(V)$ defined at each point of U and such that $f(P) = r(P)$ for each $P \in U$. Hence, (SH2) is satisfied.

Having defined regular functions on open subsets of V, a natural question arises. What are the 'global' regular functions, $\mathcal{O}_V(V)$? A natural answer comes forth. The only functions that are regular everywhere are polynomial functions.

Proposition 2.1.3. *Let V be an affine variety. Then, $\mathbb{C}[V] = \mathcal{O}_V(V)$.*

Proof. The inclusion $\mathbb{C}[V] \subseteq \mathcal{O}_V(V)$ is clear. A function $f \in \mathbb{C}[V]$ defines a regular function on the entirety of V which is given by $f/1$ around each point $P \in V$.

We use the equality $\mathcal{O}_V(V) = \bigcap_{P \in V} \mathcal{O}_{V,P}$. Fix some $f \in \mathbb{C}(V)$ and define

$$J \coloneqq \{g \in \mathbb{C}[V] \mid fg \in \mathbb{C}[V]\}.$$

J is an ideal of $\mathbb{C}[V]$ because if $fg \in \mathbb{C}[V]$ and $h \in \mathbb{C}[V]$, then $fgh \in \mathbb{C}[V]$, which implies that $gh \in J$. Similarly, if $fg \in \mathbb{C}[V]$ and $fh \in \mathbb{C}[V]$, then $fg + fh = f(g + h) \in \mathbb{C}[V]$. Thus, $g + h \in J$.

Note that $\mathbf{V}(J) = \{P \in V \mid g(P) = 0 \ \forall g \text{ such that } fg \in \mathbb{C}[V]\}$. In the case that f is defined at every point $P \in V$, $\mathbf{V}(J)$ must be empty. Indeed, suppose this is not the case and pick $P \in \mathbf{V}(J)$. For any representation $f = g/h$ where $g, h \in \mathbb{C}[V]$ we would have $h(P) = 0$ because $fh = g \in \mathbb{C}[V]$. Thus, f is not defined at P. This is a contradiction, hence $\mathbf{V}(J) = \emptyset$. By the Weak Nullstellensatz, $J = \mathbb{C}[V]$. In particular, $1 \in J$, so $1f = f \in \mathbb{C}[V]$ by definition of J. □

The last concept we are yet to define are morphisms between open subsets of affine varieties. We, again, take the categorical approach. For V_1, V_2 affine varieties and $U_1 \subseteq V_1, U_2 \subseteq V_2$ open sets, we dub a map

$$\varphi : U_1 \to U_2,$$

a *morphism (of affine varieties)* if the induced map takes regular functions to regular functions, that is, $\varphi^*(\mathcal{O}_{V_2}(U_2)) \subseteq \mathcal{O}_{V_1}(U_1)$ where, recall, $\varphi^*(f) := f \circ \varphi$ for $f \in \mathcal{O}_{V_2}(U_2)$.

A morphism φ is called an *isomorphism* if φ is bijective and its inverse φ^{-1} is a morphism.

Morphisms have a few nice properties we care about and shall endeavour to prove before moving on.

Lemma 2.1.4 (Properties of Morphisms, Exercise 3.0.3 in Cox et al. [2011])**.**

(1) Regular functions on U are precisely morphisms $U \to \mathbb{C}$.

(2) A composition of morphisms is a morphism.

(3) The inclusion map $W \hookrightarrow U$ where $W \subseteq U$ is an open set of V is a morphism.

(4) Morphisms are continuous in the Zariski topology.

Proof.

(1) Let $\varphi \in \mathcal{O}_V(U)$ be a regular function. Since $\mathcal{O}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}[x]$ by **Proposition 2.1.3**, we require that for every $f \in \mathbb{C}[x]$, the composition $f \circ \varphi$ be regular on U. This is true because φ is regular on U and f is a polynomial. Hence, if φ is represented by a rational function r on U, then $f \circ \varphi$ is represented by $f \circ r$, which is rational and defined on U.

    Conversely, let φ be a morphism $U \to \mathbb{C}$. Then the induced map is $\varphi^* : \mathbb{C}[x] \to \mathcal{O}_V(U)$. Meaning, $f \circ \varphi$ is regular on U for every $f \in \mathbb{C}[x]$. Specifically, $1_{\mathbb{C}} \circ \varphi = \varphi$ is regular on U.

(2) Let V_1, V_2, V_3 be affine varieties, $U_i \subseteq V_i, i = 1, 2, 3$ be open sets and $\varphi : V_1 \to V_2, \psi : V_2 \to V_3$ be morphisms. We have $\varphi^*(\mathcal{O}_{V_2}(U_2)) \subseteq \mathcal{O}_{V_1}(U_1)$ and $\psi^*(\mathcal{O}_{V_3}(U_3)) \subseteq \mathcal{O}_{V_2}(U_2)$. Composing gives

$$(\psi \circ \varphi)^*(\mathcal{O}_{V_3}(U_3)) = \varphi^*(\psi^*(\mathcal{O}_{V_3}(U_3))) \subseteq \varphi^*(\mathcal{O}_{V_2}(U_2)) \subseteq \mathcal{O}_{V_1}(U_1),$$

    so $\psi \circ \varphi$ is a morphism.

(3) The map induced by an inclusion map is a restriction map, so it is a morphism. Indeed, if $\iota : W \hookrightarrow U$ is the inclusion map, then for $f \in \mathcal{O}_V(U)$, we have $f \circ \iota = f_{|W}$. If f is regular on U, then it is, in particular, regular on W.

(4) We use the notation from (2). We argue in a way similar to the second paragraph in (1). Since $\varphi : U_1 \to U_2$ is a morphism, the induced map is a $\mathbb{C}$-algebra homomorphism $\mathcal{O}_{V_2}(U_2) \to \mathcal{O}_{V_1}(U_1)$. This means that all compositions $f \circ \varphi$ for $f \in \mathcal{O}_{V_2}(U_2)$ are regular, thus continuous. Choosing f to be successively the projection to each coordinate gives the continuity of φ. □

We took great care to define the notions of regular functions and morphisms so that they are easily scalable from affine varieties to abstract varieties. We leave that as an easy exercise. Alright, alright, quit screaming. We do not.

In the following we consider abstract varieties $X \coloneqq \bigcup_{\alpha\in A} U_\alpha$ and $Y \coloneqq \bigcup_{\beta\in B} U'_\beta$ where U_α, U'_β are as in (2.1.1) and there are transition maps on X and Y satisfying (C1) and (C2). We also denote by h_α the homeomorphism $V_\alpha \simeq U_\alpha, a \mapsto [a]$.

Definition 2.1.5 (Morphism of Abstract Varieties)**.** A Zariski continuous map $\Phi : X \to Y$ is styled a *morphism (of abstract varieties)* if

$$\Phi_{|U_\alpha\cap\Phi^{-1}(U'_\beta)} : U_\alpha \cap \Phi^{-1}(U'_\beta) \to U'_\beta$$

is a morphism of affine varieties for all pairs $(\alpha, \beta) \in A \times B$.

Definition 2.1.6 (Sheaf of Regular Functions of an Abstract Variety)**.** Let $U \subseteq X$ be open and denote $W_\alpha \coloneqq h_\alpha^{-1}(U\cap U_\alpha) \subseteq V_\alpha$. A function $\varphi : U \to \mathbb{C}$ is regular if

$$\varphi \circ h_{\alpha|W_\alpha} : W_\alpha \to \mathbb{C}$$

is regular for every $\alpha \in A$. Thanks to conditions (C1) and (C2), the map $\varphi\circ h_{\alpha|W_\alpha}$ is well-defined and we can thus further define

$$\mathcal{O}_X(U) \coloneqq \{\varphi : U \to C \mid \varphi \text{ is regular}\}.$$

One checks that $\mathcal{O}_X$ is a sheaf on X pretty much exactly the same way as in the case of an affine variety V.

We finish with basic concepts of abstract varieties by defining local rings at points. In the language of sheaf theory, they are exactly stalks of the sheaf $\mathcal{O}_X$.

The local ring $\mathcal{O}_{V,P}$ for an affine variety V and a point $P \in V$ is essentially the set of quotients f/g with $f, g \in \mathbb{C}[V]$ and $g(P) \neq 0$. The set V_g is then an open neighbourhood of P and f/g is regular on V_g. In light of this, we can view elements of $\mathcal{O}_{V,P}$ as regular functions defined on an open neighbourhood of P.

We scale this idea to the level of abstract varieties. There is not much to add, we only need to make sure that regular functions that assume equal values around P are identified. Formally, if U_1, U_2 are open neighbourhoods of a point $P \in X$ and f_1, f_2 regular functions on U_1 and U_2, respectively, we say that they are *equivalent at* P if we can find an open neighbourhood $U \subseteq U_1\cap U_2$ of P such that $f_{1|U} = f_{2|U}$. We write this as $f_1 \sim_P f_2$. Then, *the local ring of* X *at* P is defined as the set of classes of regular functions that are equivalent at P. In symbols,

$$\mathcal{O}_{X,P} \coloneqq \{f \in \mathcal{O}_X(U) \mid U \text{ is a neighbourhood of } P\}/ \sim_P .$$

Remark. Thanks to the relation $\sim_P$, every $\varphi \in \mathcal{O}_{X,P}$ has a well-defined value at P. One can thus consider the map $\mathcal{O}_{X,P} \to \mathbb{C}, \varphi \mapsto \varphi(P)$ and, repeating the steps of the proof of **Proposition 1.1.12**, show that $\mathcal{O}_{X,P}$ is indeed local with the maximal ideal

$$\mathfrak{m}_{X,P} = \{\varphi \in \mathcal{O}_{X,P} \mid \varphi(P) = 0\}.$$

In the following sections, we are going to use the terms *normal*, *separated* and *smooth* abstract variety. We dedicate a few pages to them. Also, to save space and finger muscles, we shall henceforth call abstract varieties simply ‘varieties’ if there is no risk of confusion.

Definition 2.1.7 (Normal Ring)**.** An integral domain R with field of fractions K is called *normal*, seldom *integrally closed*, if every element of K which is integral over R (is a root of a monic polynomial in $R[x]$) lies in R.

Just as in the affine case, we will, most of the time, want our varieties *irreducible*. The definition here is the same as for the affine case. We call Zariski closed subsets of X, *subvarieties* of X. The variety X is then called irreducible if it is the union of no two proper subvarieties.

Definition 2.1.8 (Normal Variety)**.** A variety X is called *normal* if it is irreducible and $\mathcal{O}_{X,P}$ is normal for every point $P \in X$.

From the definition, we see that normality is a local property. This definition is however far from convenient when one is pressed to check normality explicitly. Fortunately, not all is lost as normality of an abstract variety X can be determined by looking at its elementary constituents, the affine varieties V_α. Affine varieties admit a definition of normality through coordinate rings, which are arguably easier to work with, than local rings at points.

Definition 2.1.9 (Normal Affine Variety)**.** Let V be an affine variety. Then V is called *normal* if its coordinate ring $\mathbb{C}[V]$ is a normal ring.

Since abstract varieties are identical to affine varieties in case they are built from only one affine variety, it behooves us to reconcile the two notions in this edge case. We require the following technical lemma from commutative algebra.

Lemma 2.1.10 (Exercise 1.0.7 from Cox et al. [2011])**.**

(1) Let $R_\alpha, \alpha \in A$ be normal integral domains with common field of fractions K. Then, $\bigcap_{\alpha \in A} R_\alpha$ is normal.

(2) Let R be a normal integral domain with field of fractions K and $S \subseteq R$ be a multiplicative set. Then, R_S is normal.

Proof.

(1) Denote $S := \bigcap_{\alpha \in A} R_\alpha$. Clearly, the field of fractions for S is K. Let $s \in K$ be integral over S, that is, there exists $f \in S[x]$ such that $f(s) = 0$. Because S is the intersection of all R_α, we also have $f \in R_\alpha[x]$ for all $\alpha \in A$. In particular, $s \in R_\alpha$ for all $\alpha \in A$ because the R_α are normal. So, $s \in \bigcap_{\alpha \in A} R_\alpha = S$.

(2) Suppose $k \in K$ is integral over R_S. Thus k satisfies some equation

$$k^n + \frac{r_{n-1}}{s_{n-1}} k^{n-1} + \ldots + \frac{r_0}{s_0} = 0$$

where $r_i \in R$ and $s_i \in S$. Let $s := s_0 \cdots s_{n-1}$. Then $s \in S$ because S is multiplicative. By definition of s, $s/s_i \in S$ and so $sr_i/s_i \in R$ for every $i \leq n$. We have

$$s^n k^n + \frac{r_{n-1}}{s_{n-1}} s^n k^{n-1} + \ldots + \frac{r_0}{s_0} s^n = (sk)^n + \frac{sr_{n-1}}{s_{n-1}} (sk)^{n-1} + \ldots + \frac{s^n r_0}{s_0} = 0.$$

Hence, sk is integral over R. R is normal, so $sk \in R$. It follows that $k \in R_S$ because $s \in S$ as was required to show. □

Proposition 2.1.11. *Let V be an affine variety. Then, $\mathbb{C}[V]$ is normal if and only if $\mathcal{O}_{V,P}$ is normal for all $P \in V$.*

Proof. The following proof is largely due to Cox et al. [2011], Proposition 3.0.11.

We know that $\mathbb{C}[V] = \bigcap_{P \in V} \mathcal{O}_{V,P}$. If $\mathcal{O}_{V,P}$ is normal for all $P \in V$, then $\mathbb{C}[V]$ is normal by **Lemma 2.1.10**.

In the other direction, suppose $\mathbb{C}[V]$ is normal and $\alpha \in \mathbb{C}(V)$ is integral over $\mathcal{O}_{V,P}$. Then, α satisfies

$$a_0 + a_1 \alpha + \ldots + \alpha^k = 0$$

for some $a_0, \ldots, a_{k-1} \in \mathcal{O}_{V,P}$. Write $a_i = g_i/f_i$ for $f_i, g_i \in \mathbb{C}[V]$ and $f_i(P) \neq 0$. Take $f := f_1 \cdots f_k$. Then, $a_i \in \mathbb{C}[V]_f$ and $f(P) \neq 0$. $\mathbb{C}[V]_f$ is normal, also by **Lemma 2.1.10**, and $\mathbb{C}[V]_f \subseteq \mathcal{O}_{V,P}$ because $f(P) \neq 0$. It follows that $\alpha \in \mathbb{C}[V]_f \subseteq \mathcal{O}_{V,P}$. This concludes the proof. □

The immediate corollary is one we shall employ heavily in **Section 2.2**.

Corollary 2.1.12. *Let X be an irreducible abstract variety. Then, X is normal if and only if each V_α is normal.*

Proof. Just a few paragraphs later, when we discuss smooth abstract varieties, we will show that $\mathcal{O}_{V_\alpha,P} \simeq \mathcal{O}_{X,P}$ whenever $P \in V_\alpha$. Hence, if X is normal, then all $\mathcal{O}_{X,P}$ are normal. Consequently, all the $\mathcal{O}_{V_\alpha,P}$ for each $\alpha \in A$ and $P \in V_\alpha$ are normal, so all the V_α are normal. The proof of the inverse implication is symmetrical. □

We now define a *smooth* variety. To that end, we need to make sure that both notions – $\dim_P X$, the dimension of X at a point $P \in X$, and $T_P X$, the tangent space to X at P – are scalable from affine varieties to abstract ones. Thankfully, with a little bit of work, they are.

As for $\dim_P X$, we have at our disposal the local rings at points $\mathcal{O}_{X,P}$. The definition of $\dim_P X$ as the Krull dimension of $\mathcal{O}_{X,P}$ thus gets a pass without change.

To show the same for $T_P X$, we need to check that its definition does not depend on the particular choice $\alpha \in A$ such that $P \in V_\alpha$. Concretely, if $P \in X$ is a point that lies in the intersection $V_\alpha \cap V_\beta$ for some $\alpha, \beta \in A$, then we show that $T_{V_\alpha}P$ and $T_{V_\beta}P$ are isomorphic as complex vector spaces. This is most easily seen from the fact that $\mathcal{O}_{V_\alpha,P} \simeq \mathcal{O}_{V_\beta,P}$ because tangent spaces are defined through maximal ideals of these local rings.

A few paragraphs ago, we observed that $\mathcal{O}_{V,P}$ for an affine variety V are really regular functions defined on a neighbourhood of $P \in V$. This begets the idea that perhaps also $\mathcal{O}_{V_\alpha,P} \simeq \mathcal{O}_{X,P}$. The natural map to consider is $[f] \mapsto [f \circ h_\alpha^{-1}]$ where we abuse notation a little and denote by $[\cdot]$ both the equivalence class in $\mathcal{O}_{V_\alpha,P}$ and in $\mathcal{O}_{X,P}$. The map $f \circ h_\alpha^{-1} : U_\alpha \to \mathbb{C}$ is indeed regular at P because U_α is an open neighbourhood of P, h_α^{-1} is a homeomorphism and f is regular. Also, if $f \circ h_\alpha^{-1} \sim_P g \circ h_\alpha^{-1}$, then there exists some neighbourhood $U \subseteq U_\alpha$ of P on which those two functions agree. Since h_α is a homeomorphism, f and g agree on $h_\alpha^{-1}(U)$, which is an open neighbourhood of P in V_α and thus they belong to the same class in $\mathcal{O}_{V_\alpha,P}$. Finally, every $[h] \in \mathcal{O}_{X,P}$ has its pre-image in $[h \circ h_\alpha]$. The fact that $[f] \mapsto [f \circ h_\alpha^{-1}]$ is a ring homomorphism is clear and we have just shown that it is well-defined and bijective. Hence, if $P \in V_\alpha \cap V_\beta$, we claim that $\mathcal{O}_{V_\alpha,P} \simeq \mathcal{O}_{X,P} \simeq \mathcal{O}_{V_\beta,P}$ and $T_P X$ is thus well-defined.

Definition 2.1.13 (Smooth Point of a Variety)**.** A point P of a variety X is said to be *smooth* if $\dim T_P X = \dim_P X$. A variety X is smooth if its every point is smooth.

The remainder of the section is dedicated to separated varieties. We have observed that the Zariski topology does not shake hands with Hausdorff-ness. However, varieties can still be Hausdorff in the classical topology. An important result states that a variety is classically Hausdorff if and only if it is separated in the Zariski topology, which is a notion for varieties as close to Hausdorff-ness as we can get.

We need to digress for a while and talk about products of varieties. We start with their affine little sisters. If V_1 and V_2 are affine varieties, their product $V_1 \times V_2$ can also be given the structure of an affine variety. We do *not* imbue it with the product topology as one can show that this does not generally make $V_1 \times V_2$ into an affine variety. Instead, suppose $V_1 \subseteq \mathbb{C}^n = \operatorname{Specm}(\mathbb{C}[x_1, \ldots, x_n])$ and $V_2 \subseteq \mathbb{C}^m = \operatorname{Specm}(\mathbb{C}[y_1, \ldots, y_m])$. If $\mathbf{I}(V_1) = (f_1, \ldots, f_k)$ and $\mathbf{I}(V_2) = (g_1, \ldots, g_l)$, then we define

$$V_1 \times V_2 \coloneqq \mathbf{V}(f_1, \ldots, f_k, g_1, \ldots, g_l) \subseteq \mathbb{C}^{n+m}.$$

Since f_i and g_j depend on distinct sets of variables, $V_1 \times V_2$ is an affine variety.

This notion can be immediately transported to abstract varieties. If $X = \bigcup_{\alpha \in A} U_\alpha$ and $Y = \bigcup_{\beta \in B} U'_\beta$, then their *product* $X \times Y$ is obtained by gluing $V_\alpha \times V'_\beta$ via the transitions maps $g_{\alpha_1\alpha_2} \times g_{\beta_1\beta_2}$ for $\alpha_1, \alpha_2 \in A$ and $\beta_1, \beta_2 \in B$. Since everything is defined component-wise, the conditions (C1) and (C2) are satisfied almost trivially.

Definition 2.1.14 (Separated Variety)**.** Let X be a variety. Consider the diagonal map $\Delta : X \to X \times X, \Delta(P) \coloneqq (P, P)$. We call X *separated* if $\Delta(X)$ is Zariski closed in $X \times X$.

Theorem 2.1.15. *A variety X is separated if and only if it is Hausdorff in the classical topology.*

Proof. See Cox et al. [2011], Theorem 3.0.17. □

The last thing we do is prove two rather useful properties of separated varieties.

Lemma 2.1.16 (Exercise 3.0.6 in Cox et al. [2011])**.** *Let X be a separated variety. Then,*

(1) If $\varphi, \psi : Y \to X$, where Y is a variety, are morphisms, then $\{y \in Y \mid \varphi(y) = \psi(y)\}$ is closed in Y.

(2) If U, V are affine open subsets of X, then $U \cap V$ is also affine.

Here an affine *subset means a subset of X isomorphic to an affine variety.*

Proof.

(1) Define $\Phi : Y \to X \times X$ by $\Phi(y) = (\varphi(y), \psi(y))$. Denote $Z \coloneqq \{y \in Y \mid \varphi(y) = \psi(y)\}$. Since φ and ψ are continuous (as morphisms), so is Φ. Observe that $Z = \Phi^{-1}(\Delta(X))$ because $\varphi(y) = \psi(y)$ if and only if $y \in Z$. It follows that Z is closed due to continuity of Φ and X being separated.

(2) We first show that $U \cap V$ can be identified with $\Delta(X) \cap (U \times V) \subseteq X \times X$. If $P \in U \cap V$, then $P \in U$ and $P \in V$, in other words, $(P, P) \in U \times V$. Of course, $(P, P) \in \Delta(X)$. Conversely, if $(P, Q) \in \Delta(X) \cap (U \times V)$, then $P = Q$ and thus $(P, P) \in U \times V$. It follows that $P \in U \cap V$. The result now follows because $\Delta(X)$ is closed by assumption on X and $U \times V$ is affine thanks to the way products of affine varieties are constructed. □

2.2 Fans and Toric Varieties

In this section, we define *fans*, basically collections of cones glued together in a natural way. Also, we show a correspondence between abstract toric varieties and fans, a scaled-up version of events occurring in **Section 1.3**.

On the way, we will need to prove a few more essential properties of affine semigroups and cones in general. Since these properties are tied to fans and normal varieties, we decided to include them here rather than incorporate them into **Section 1.2**. That is where we start. In this section, we follow Cox et al. [2011], Section 3.1 and Fulton [1993], Section 1.4.

Suppose σ is a rational convex polyhedral cone gleefully dwelling in the real vector space $N_{\mathbb{R}}$ coming from a lattice $N \simeq \mathbb{Z}^r$ with dual lattice M. We have proven in **Section 1.2**, specifically in Gordan's Lemma, that $\mathsf{S}_\sigma \coloneqq \sigma^* \cap M$ is an affine semigroup. **Theorem 1.3.21** implies that $V_\sigma := \operatorname{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$ is an affine toric variety and every affine toric variety assumes this shape.

We specify our area of interest a little further and require that V_σ always be normal. As one might guess, this puts some necessary conditions on the cone σ. One particular condition also turns out to be sufficient. It seems that whenever σ is *strongly* convex, meaning $\sigma \cap (-\sigma) = \{0\}$, V_σ is automatically normal and vice versa. We shall now proceed to prove both assertions and for that we need to talk about *saturated* affine semigroups.

Definition 2.2.1 (Saturated Affine Semigroup)**.** We call a semigroup $\mathsf{S} \subseteq M$ *saturated*, if for all non-zero $k \in \mathbb{N}$ and $m \in M$, $km \in \mathsf{S}$ implies $m \in \mathsf{S}$.

The following theorem for the proof whereof we thank Cox et al. [2011] and their Theorem 1.3.5 connects normal affine varieties, saturated semigroups and strongly convex cones.

Theorem 2.2.2. *Let V be an affine variety. The following statements are equivalent.*

(1) V is normal.

(2) $V = \operatorname{Specm}(\mathbb{C}[\mathsf{S}])$ where $\mathsf{S} \subseteq M$ is a saturated affine semigroup.

(3) $V = V_\sigma$ for σ a rational strongly convex polyhedral cone.

Proof. By **Theorem 1.3.21**, $V = \operatorname{Specm}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S. By **Proposition 1.3.17**, the torus of V has character lattice $M = \mathbb{Z}\mathsf{S}$. Let $r := \operatorname{rank} M$.

(1) $\Rightarrow$ (2). If V is normal, then $\mathbb{C}[\mathsf{S}] = \mathbb{C}[V]$ is normal. Choose a non-zero $k \in \mathbb{N}$ and $m \in M$ such that $km \in \mathsf{S}$. The character $\chi^{km} : T_N \to \mathbb{C}^*$ is a polynomial function on T_N and thus can be seen as a rational function $V \to \mathbb{C}$ because T_N is Zariski open in V. We have $\chi^{km} \in \mathbb{C}[\mathsf{S}]$ because $km \in \mathsf{S}$. It follows that χ^m is a root of the monic polynomial $X^k - \chi^{km} \in \mathbb{C}[\mathsf{S}][X]$. The normality of $\mathbb{C}[\mathsf{S}]$ implies $\chi^m \in \mathbb{C}[\mathsf{S}]$ and thus $m \in \mathsf{S}$. This means that S is saturated.

(2) $\Rightarrow$ (3). Let $\mathcal{A} \subseteq \mathsf{S}$ be a generating set of S. Then, $\mathsf{S} \subseteq \operatorname{Cone}(\mathcal{A}) \subseteq M_\mathbb{R}$. If $\operatorname{rank} \mathbb{Z}\mathcal{A} = r$, then $\dim \operatorname{Cone}(\mathcal{A}) = r$. Indeed, this is true because we have $\operatorname{rank} \mathbb{Z}\mathcal{A} = \dim \operatorname{span}(\mathcal{A})$. By definition of the dimension of a cone, $\dim \operatorname{Cone}(\mathcal{A}) = \dim \operatorname{span}(\mathcal{A})$. Hence, $\dim \operatorname{Cone}(\mathcal{A}) = \operatorname{rank} \mathbb{Z}\mathcal{A}$.

We show that in this case, $\sigma := \operatorname{Cone}(\mathcal{A})^*$ is strongly convex. For contradiction, suppose that there is a non-zero $u \in \sigma \cap (-\sigma)$. Then, necessarily $\langle u, m \rangle = 0$ for all $m \in \operatorname{Cone}(\mathcal{A})$ because $u \in \sigma$ implies $\langle u, m \rangle \geq 0$ and $u \in -\sigma$ implies $\langle -u, m \rangle = -\langle u, m \rangle \geq 0$, so $\langle u, m \rangle \leq 0$. However, $\operatorname{Cone}(\mathcal{A})$ spans $M_\mathbb{R}$ by the previous paragraph, hence $\langle u, m \rangle = 0$ for all $m \in M_\mathbb{R}$. This means that there is a non-zero vector $u \in M_\mathbb{R}^\perp$. Basic results from linear algebra imply that $\dim M_\mathbb{R}^\perp = 0$, though, so this is not possible. Thus, $\sigma \cap (-\sigma) = \{0\}$ and σ is strongly convex.

We see that $\mathsf{S} \subseteq \sigma^* \cap M$. The final step is to prove that $\sigma^* \cap M \subseteq \mathsf{S}$ in case S is saturated. We present a little trick. Let $\{m_1, \dots, m_r\}$ be the basis of M and denote $M_{\mathbb{Q}} \coloneqq M \otimes_{\mathbb{Z}} \mathbb{Q}$ the vector space over $\mathbb{Q}$ with basis $\{m_1, \dots, m_r\}$. Observe that

$$\sigma^* \cap M_{\mathbb{Q}} = \left\{ \sum_{a \in \mathcal{A}} \lambda_a a \mid \lambda_a \in \mathbb{Q} \wedge \lambda_a \geq 0 \right\}.$$

Choose some concrete element $m \in \sigma^* \cap M$. Then, $m \in \sigma^* \cap M_{\mathbb{Q}}$. This means that we can write

$$m = \sum_{a \in \mathcal{A}} \lambda_a a$$

for some $\lambda_a \in \mathbb{Q}, \lambda_a \geq 0$. Let l be the product of the denominators of all the λ_a. Then, $l \in \mathbb{N}$ and $lm \in \mathbb{N}\mathcal{A} = \mathsf{S}$. Since S is saturated, $m \in \mathsf{S}$.

(3) $\Rightarrow$ (1). This final implication requires the usage of the following lemma.

Lemma 2.2.3. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational strongly convex polyhedral cone and ρ be its edge (face of dimension one). Then, since σ is strongly convex, ρ is a ray and since σ is rational, the semigroup $\rho \cap N$ has precisely one unique generator. We denote this generator by u_ρ and call it the* ray generator *of σ. Moreover, σ is generated by the ray generators of its edges.*

Proof. See Lemma 1.2.15 in Cox et al. [2011]. □

Continuation of the proof of Theorem 2.2.2. The goal is to show that $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^* \cap M]$ is normal when $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex and rational. Denote by $\rho_1, \dots, \rho_r$ the rays of σ. By **Lemma 2.2.3**, σ is generated by $u_{\rho_1}, \dots, u_{\rho_r}$ and $\rho_1, \dots, \rho_r$ are cones, so

$$\sigma^* = \bigcap_{i=1}^{r} \rho_i^*.$$

Indeed, $\langle m, u \rangle \geq 0$ for $m \in M_{\mathbb{R}}$ and all $u \in \sigma$ if and only if $\langle m, u_i \rangle \geq 0$ for all $u_i \in \rho_i$ and all $i \leq r$. Intersecting both sides with M yields

$$\mathsf{S}_\sigma = \bigcap_{i=1}^{r} \mathsf{S}_{\rho_i}.$$

From there, somewhat trivially

$$\mathbb{C}[\mathsf{S}_\sigma] = \bigcap_{i=1}^{r} \mathbb{C}[\mathsf{S}_{\rho_i}].$$

By **Lemma 2.1.10**, it is enough to show that each of $\mathbb{C}[\mathsf{S}_{\rho_i}]$ is normal in order for $\mathbb{C}[\mathsf{S}_\sigma]$ to be normal. Hence, we show that $\mathbb{C}[\mathsf{S}_\rho]$ is normal when ρ is a rational ray in $N_{\mathbb{R}}$. Given the unique generator u_ρ of $\rho \cap N$, we see that u_ρ is *primitive*, that is, $\frac{1}{k} u_\rho \notin N$ for any $k > 1$. In such a case, we can find a basis $e_1, \dots, e_r$ of N such that $e_1 = u_\rho$.

Why is this so? Suppose $n_1, \dots, n_r$ is any basis of N and $u_\rho = \sum_{i=1}^{r} z_i n_i$ for $z_i \in \mathbb{Z}$. By primitivity of u_ρ, we have $\gcd(z_1, \dots, z_r) = 1$. By Bézout's identity from basic algebra, there are $u_1, \dots, u_r \in \mathbb{Z}$ satisfying $u_1 z_1 + \ldots + u_r z_r = 1$. If we write elements of N as vectors with respect to $\{n_1, \ldots, n_r\}$, then the linear transformation sending n_1 to u_ρ, that is, $(1, 0, \ldots, 0)$ to $(z_1, \ldots, z_r)$ is invertible, with inverse

$$(z_1, \ldots, z_r) \mapsto (u_1 z_1 + \ldots + u_r z_r, 0, \ldots, 0).$$

In summary, such linear transformation is a change of basis sending n_1 to u_ρ.

This fact allows us to assume $\sigma = \text{Cone}(e_1)$ and so

$$\mathbb{C}[\mathsf{S}_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

The proof of this is not difficult but would demand we lengthen an already long enough exposition, so we delegate kind and understanding readers to Example 1.2.21 in Cox et al. [2011]. Finally, $\mathbb{C}[x_1, \ldots, x_n]$ is normal and thus the localization

$$\mathbb{C}[x_1, \ldots, x_n]_{x_2 \cdots x_n} = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

is normal by **Lemma 2.1.10**. With this we exhale in satisfaction and close the lid on the proof of **Theorem 2.2.2**. □

We now know that strongly convex cones are in one-to-one correspondence with normal affine varieties. The same is true for smooth cones and smooth affine varieties. The proof of this fact however, requires developing the theory of convex cones a little further than we intend to. We take the following statements from Cox et al. [2011], mostly for consistency, but readers are encouraged to confront Fulton [1993], Section 1.4 and Oda [1985], Proposition 1.3 and Theorem 1.4 for equivalent results in a slightly different setting.

Theorem 2.2.4 (Smoothie)**.** *Let σ be a strongly convex rational polyhedral cone. Then, V_σ is smooth if and only if σ is smooth. Moreover, all smooth affine toric varieties are of this form.*

Proof. See for instance Cox et al. [2011], Theorem 1.3.12. □

Our expedition into areas of cones yet unexplored is to end rather disappointingly with two, seemingly random and mostly technical results that we also only cite for reasons of space and time. These however, turn out to be the relics of the past which bring all the ancient civilizations together. Or so we would like to say but reality is scarcely so grand.

Proposition 2.2.5. *Let $\tau := H_m \cap \sigma$ be a face of a strongly convex rational polyhedral cone σ for $m \in \sigma^*$. Then, $\mathsf{S}_\sigma + \mathbb{Z}(-m) = \mathsf{S}_\tau$ and, moreover, the semigroup algebra $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\tau^* \cap M]$ is the localization of $\mathbb{C}[\mathsf{S}_\sigma]$ at the character $\chi^m \in \mathbb{C}[\mathsf{S}_\sigma]$. That is, $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}$.*

Proof. See Cox et al. [2011], Proposition 1.3.16. □

Lemma 2.2.6. *Let $\sigma_1, \sigma_2 \subseteq N_\mathbb{R}$ be polyhedral cones that meet along a common face $\tau = \sigma_1 \cap \sigma_2$ (this means that τ is the face of both σ_1 and σ_2 and constitutes their intersection). Then there exists $m \in \sigma_1^* \cap (-\sigma_2)^* \cap M$ such that*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2.$$

Proof. Right from our holy book on toric varieties, Cox et al. [2011], Lemma 1.2.13. □

An immediate consequence of **Proposition 2.2.5** is that

$$(V_\sigma)_{\chi^m} = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}) = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\tau]) = V_\tau$$

for $\tau = H_m \cap \sigma$ a face of σ. Coupling it with **Lemma 2.2.6** gives not only $V_\tau = (V_{\sigma_1})_{\chi^m} \subseteq V_{\sigma_1}$ but also $V_\tau = (V_{\sigma_2})_{\chi^{-m}} \subseteq V_{\sigma_2}$. Put more plainly, if σ_1 and σ_2 meet along a face τ, then the affine toric variety V_τ lies in $V_{\sigma_1} \cap V_{\sigma_2}$. We will need this fact to transition between V_{σ_1} and V_{σ_2} once we glue them together.

There is one last step needed before we define fans and abstract toric varieties and also show how to construct the latter from the former. The inverse construction, a way to somehow suck out a fan from an abstract toric variety, must regrettably stay shrouded in mystery as the crucial theory of orbit-cone correspondence is in the end too spacious to cover.

First, a technical lemma. Enjoy.

Lemma 2.2.7. *Let σ_1, σ_2 be cones. Then,*

$$\sigma_1^* + \sigma_2^* = (\sigma_1 \cap \sigma_2)^*.$$

Proof. It is easier to show that

$$\sigma_1^* \cap \sigma_2^* = (\sigma_1 + \sigma_2)^*.$$

The inclusion from left to right is obvious because

$$\langle m, u_1 \rangle \geq 0 \wedge \langle m, u_2 \rangle \geq 0 \implies \langle m, u_1 + u_2 \rangle = \langle m, u_1 \rangle + \langle m, u_2 \rangle \geq 0$$

for all $m \in \sigma_1^* \cap \sigma_2^*$ and $u_1 \in \sigma_1, u_2 \in \sigma_2$.

To prove the inverse inclusion, choose $m \in (\sigma_1 + \sigma_2)^*$. Since $0 \in \sigma_2$, this gives $\langle m, u_1 \rangle \geq 0$ for all $u_1 \in \sigma_1$ and symmetrically $0 \in \sigma_1$ implies $\langle m, u_2 \rangle \geq 0$ for all $u_2 \in \sigma_2$. It follows that $m \in \sigma_1^* \cap \sigma_2^*$.

Finally, the desired equality is achieved by replacing σ_i with σ_i^* for $i = 1, 2$ and taking the dual of both sides. Concretely, the previous implies

$$\sigma_1 \cap \sigma_2 = (\sigma_1^* + \sigma_2^*)^*$$

and thus

$$(\sigma_1 \cap \sigma_2)^* = \sigma_1^* + \sigma_2^*.$$

□

And here comes the last proposition we need to build toric varieties from fans.

Proposition 2.2.8. *Let σ_1 and σ_2 meet along a common face τ. Then,*

$$\mathsf{S}_\tau = \mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2}.$$

Proof. Save for a few added details due to Cox et al. [2011], Proposition 3.1.3.

The inclusion $\mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2} \subseteq \mathsf{S}_\tau$ follows from the previous **Lemma 2.2.7** because $\sigma_1^* + \sigma_2^* = (\sigma_1 \cap \sigma_2)^* = \tau^*$.

For the reverse, take some $t \in \mathsf{S}_\tau$ and find $m \in \sigma_1^* \cap (-\sigma_2)^* \cap M$, whose existence ensures **Lemma 2.2.6**, satisfying

$$\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_m.$$

Applying **Proposition 2.2.5** to σ_1 yields $t = s + k(-m)$ for adequate $s \in \mathsf{S}_{\sigma_1}$ and $k \in \mathbb{N}$. Since $-m \in \sigma_2^* \cap M = \mathsf{S}_{\sigma_2}$, we have $t \in \mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2}$, as desired. □

Definition 2.2.9 (Fan)**.** A *fan* Σ in $N_\mathbb{R}$ is a collection of finitely many strongly convex rational polyhedral cones such that

(F1) Each face of $\sigma \in \Sigma$ is also a cone in Σ for all $\sigma \in \Sigma$.

(F2) If $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 (and thus lies in Σ).

We also define the *support* of Σ as

$$|\Sigma| \coloneqq \bigcup_{\sigma \in \Sigma} \sigma$$

and $\Sigma(r)$ denotes the set of r-dimensional cones in Σ.

Definition 2.2.10 (Toric Variety)**.** An (abstract) variety X is called *toric* if X contains a torus $T_N \simeq (\mathbb{C}^*)^r$ as a Zariski open subset and the action of T_N on itself extends to an action $T_N \times X \to X$ of T_N on X.

With all this ammunition in stock, we can build a toric variety from a fan quite easily. Let Σ be a fan in $N_\mathbb{R}$ and consider $V_\sigma = \operatorname{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$ for every $\sigma \in \Sigma$. We glue these affine toric varieties together.

Let $\sigma_1, \sigma_2 \in \Sigma$ and $\tau \coloneqq \sigma_1 \cap \sigma_2$. By **Proposition 2.2.5** and **Lemma 2.2.6**, we obtain the isomorphism

$$g_{\sigma_2,\sigma_1} : (V_{\sigma_1})_{\chi^m} \xrightarrow{\sim} (V_{\sigma_2})_{\chi^{-m}}$$

which restricts to the identity map on τ.

We check the compatibility conditions (C1) and (C2) for g_{σ_2,σ_1}.

As for (C1), we have

$$(V_{\sigma_1})_{\chi^m} \xrightarrow{\quad g_{\sigma_2,\sigma_1} \quad} (V_{\sigma_2})_{\chi^{-m}} \xrightarrow{\quad g_{\sigma_1,\sigma_2} \quad} (V_{\sigma_1})_{\chi^m}$$

and so $g_{\sigma_1,\sigma_2} \circ g_{\sigma_2,\sigma_1}$ is the identity map on $(V_{\sigma_1})_{\chi^m}$ as both g_{σ_2,σ_1} and g_{σ_1,σ_2} are isomorphisms.

Let $\sigma_3 \in \Sigma$ be another cone in Σ and $m' \in M$ such that $\sigma_1 \cap \sigma_3 = H_{m'} \cap \sigma_1 = H_{m'} \cap \sigma_3$. Since $m \in M$ satisfies $\sigma_1 \cap \sigma_2 = H_m \cap \sigma_1 = H_m \cap \sigma_2$, observe that we have

$$H_{m+m'} \cap \sigma_1 = H_m \cap H_{m'} \cap \sigma_1 = (H_m \cap \sigma_1) \cap (H_{m'} \cap \sigma_1) = \sigma_1 \cap \sigma_2 \cap \sigma_3.$$

Moreover, $(V_{\sigma_1})_{\chi^m} \cap (V_{\sigma_1})_{\chi^{m'}} = (V_{\sigma_1})_{\chi^{m+m'}}$ because $(V_{\sigma_1})_{\chi^m} \simeq V_{\sigma_1} \setminus \mathbf{V}(\chi^m)$ and so

$$(V_{\sigma_1} \setminus \mathbf{V}(\chi^m)) \cap (V_{\sigma_1} \setminus \mathbf{V}(\chi^{m'})) = V_{\sigma_1} \setminus (\mathbf{V}(\chi^m) \cup \mathbf{V}(\chi^{m'})) = V_{\sigma_1} \setminus \mathbf{V}(\chi^m \chi^{m'}).$$

It follows that

$$\begin{aligned} g_{\sigma_2,\sigma_1}((V_{\sigma_1})_{\chi^m} \cap (V_{\sigma_1})_{\chi^{m'}}) &= g_{\sigma_2,\sigma_1}((V_{\sigma_1})_{\chi^{m+m'}}) = (V_{\sigma_2})_{\chi^{-m-m'}} \\ &= (V_{\sigma_2})_{\chi^{-m}} \cap (V_{\sigma_2})_{\chi^{-m'}} \end{aligned}$$

as demanded by (C2). The equality $g_{\sigma_3,\sigma_1} = g_{\sigma_3,\sigma_2} \circ g_{\sigma_2,\sigma_1}$ is proven by a very similar set of equalities as above but is notationally cumbersome and at this stage does not provide any new insight to the matter at hand. We have decided to skip it.

By gluing together all the V_σ for $\sigma \in \Sigma$, we obtain a variety X, which we label X_Σ. The following result is the essence of this section.

Theorem 2.2.11 (Toric Variety From A Fan)**.** *Let Σ be a fan in $N_\mathbb{R}$. Then, X_Σ is a separated normal toric variety.*

Proof. We, again, present an augmented version of the proof of Theorem 3.1.5 in the book by Cox et al. [2011]. We will also use Proposition 1.2.12 in the same book stating that a cone $\sigma \in \Sigma$ is strongly convex if and only if $\{0\}$ is a face of σ.

Since all cones of Σ are strongly convex by definition of a fan, $\{0\}$ is a face of every $\sigma \in \Sigma$. Hence, we have $\mathbb{C}[\mathsf{S}_\sigma] \subseteq \mathbb{C}[\mathsf{S}_{\{0\}}] = \mathbb{C}[M]$ for $\{0\}^* = M_\mathbb{R}$. This implies $T_N = \mathrm{Specm}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^r \subseteq V_\sigma$ for all $\sigma \in \Sigma$. For the same reason, also $T_N \subseteq \mathrm{Specm}(\mathbb{C}[V_\sigma]_{\chi^m}) = (V_\sigma)_{\chi^m}$ which means that gluing the V_σ together identifies all the underlying tori. So, we have $T_N \subseteq X_\Sigma$.

The transition maps g_{σ_2,σ_1} viewed as morphisms $V_{\sigma_1} \to V_{\sigma_2}$, restrict to identity maps on $V_{\sigma_1 \cap \sigma_2}$. The torus T_N acts on each V_σ by a morphism $T_N \times V_\sigma \to V_\sigma$. We see that the action $T_N \times V_{\sigma_1} \to V_{\sigma_1}$ and the action $T_N \times V_{\sigma_2} \to V_{\sigma_2}$ are compatible on the intersections $V_{\sigma_1} \cap V_{\sigma_2}$ thanks to the transition maps. This means there is an action $T_N \times X \to X$ of T_N on X given on the neighbourhood of each $P \in X$ by the action of T_N on V_σ if $\sigma \in \Sigma$ is such that $P \in V_\sigma$.

On our quest to prove that X is normal, we first observe that X is irreducible. This is because each V_σ is irreducible and, being an affine toric variety, contains T_N. As we have perceived in the previous paragraph, the tori of V_σ are glued together by the transition maps so, if we had $X = X_1 \cup X_2$ for some subvarieties $X_1, X_2 \subseteq X$, then for instance X_1 would have to contain all (the quotients of) the tori. However, these are Zariski open subsets of the irreducible affine varieties

V_σ, hence dense in V_σ. X_1, being closed, thus contains all the $U_\sigma = h_\sigma(V_\sigma)$ which forces X_2 to be $\emptyset$.

Each V_σ is normal by **Theorem 2.2.2**. Hence, X is normal by **Corollary 2.1.12**.

It remains to check that X_Σ is separated. For that, we need a little lemma which says that separated-ness is a local property.

Lemma 2.2.12. *If X be the variety obtained by gluing together $V_\alpha, \alpha \in A$ along the transition maps $g_{\beta\alpha} : V_{\beta\alpha} \simeq V_{\alpha\beta}$. Then, X is separated if the image of $\Delta : V_{\alpha\beta} \to V_\alpha \times V_\beta, P \mapsto (P, g_{\alpha\beta}(P))$ is Zariski closed for all $\alpha, \beta \in A$.*

Proof. Let Δ_X denote the typical diagonal map $\Delta_X(P) = (P, P) \in X \times X$. Observe that under the assumptions of the lemma, $\Delta_X(X)$ naturally has the structure of a variety. Indeed, since $\Delta(V_{\alpha\beta})$ is closed in $V_\alpha \times V_\beta$, it is an affine variety and we have the canonical isomorphisms $\Gamma_{\beta\alpha} : \Delta(V_{\alpha\beta}) \simeq \Delta(V_{\beta\alpha})$ given by $(P, g_{\alpha\beta}(P)) \mapsto (g_{\beta\alpha}(P), g_{\beta\alpha}(g_{\alpha\beta}(P)) = (g_{\beta\alpha}(P), P)$ because the $g_{\alpha\beta}$ are also isomorphisms. It follows that, being defined via $g_{\alpha\beta}$, the maps $\Gamma_{\alpha\beta}$ satisfy compatibility conditions (C1) and (C2) and so we can glue together $\Delta(V_{\alpha\beta})$ to get $\Delta_X(X)$. This proves that $\Delta_X(X)$ is a subvariety of $X \times X$ and is thus closed. □

Continuation of the proof of Theorem 2.2.11. By **Lemma 2.2.12** it is enough to show that for all $\sigma_1, \sigma_2 \in \Sigma$, the image of the diagonal map

$$\Delta : V_{\sigma_1 \cap \sigma_2} \to V_{\sigma_1} \times V_{\sigma_2}$$

is Zariski closed. However, the map induced by Δ is the $\mathbb{C}$-algebra homomorphism

$$\Delta^* : \mathbb{C}[\mathsf{S}_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}_{\sigma_2}] \to \mathbb{C}[\mathsf{S}_{\sigma_1 \cap \sigma_2}].$$

given by $\chi^m \otimes \chi^n \mapsto \chi^m \chi^n = \chi^{m+n}$. By **Proposition 2.2.8**, Δ^* is surjective, thus

$$\mathbb{C}[\mathsf{S}_{\sigma_1 \cap \sigma_2}] \simeq (\mathbb{C}[\mathsf{S}_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}_{\sigma_2}]) / \ker(\Delta^*).$$

Looking at maximal spectra of these rings implies that $\Delta(V_{\sigma_1 \cap \sigma_2})$ is a subvariety of $V_{\sigma_1} \cap V_{\sigma_2}$ and is, therefore, closed. □

Remark. In the proof of the theorem above, we quietly and inconspicuously used the fact that $\mathbb{C}[V_1 \times V_2] \simeq \mathbb{C}[V_1] \otimes_{\mathbb{C}} \mathbb{C}[V_2]$ for affine varieties V_1 and V_2. This, however, is not hard to prove.

Suppose a regular function $f : V_1 \times V_2 \to \mathbb{C}$ is given by a pair of regular functions $f_1 : V_1 \to \mathbb{C}, f_2 : V_2 \to \mathbb{C}$. The map $f = (f_1, f_2) \mapsto f_1 \otimes f_2$ is a $\mathbb{C}$-algebra homomorphism thanks to the properties of the tensor product. Indeed, we have

$$(f_1 + f_1') \otimes (f_2 + f_2') = (f_1 \otimes f_2) + (f_1' \otimes f_2')$$

and

$$(f_1 f_1') \otimes (f_2 f_2') = (f_1 \otimes f_2)(f_1' \otimes f_2).$$

It is clearly bijective because $f_1 \otimes f_2 = f_1' \otimes f_2'$ if and only of $f_1 = f_1$ and $f_2' = f_2'$ and each pair of regular functions f_1, f_2 defines a regular function $f = (f_1, f_2)$ on $V_1 \times V_2$. This remark was perhaps unnecessarily long for readers acquainted with tensor products of rings but we have shown that $\mathbb{C}[V_1 \times V_2] \simeq \mathbb{C}[V_1] \otimes_{\mathbb{C}} \mathbb{C}[V_2]$.

The fact that every separated normal toric variety comes from an underlying fan is also true, as was already mentioned, but, unfortunately, we are not going to make the necessary effort to see this conclusion. It requires diving into the correspondence between orbits of the action of T_N on X and individual cones of the fan Σ, and will remain, just barely, out of the scope of this work.

We end on a similarly unsatisfactory note in the case of smooth varieties. We call a fan *smooth* if its every cone is smooth. Since smoothness is a local property dependant on the dimensions of tangent spaces and local rings at points, which always lie inside one of the glued-together affine toric varieties, an immediate consequence of the aptly dubbed Smoothie theorem is that X_Σ is smooth if and only if Σ is smooth. The idea that smooth toric varieties always come from smooth fans will, likewise, stay unexplored.

2.3 Resolution of Singularities on Toric Surfaces

The preceding text provided a very general view of toric varieties and their correspondence to fans in real vector spaces. In this last section of the thesis we take a look at a specific types of toric varieties, specifically toric varieties of dimension two, which we call *toric surfaces*.

We know that a toric surface coming from a fan is always normal and separated but not necessarily smooth. We study the singular points of such surfaces and discover that there is in fact an algorithmic way to smooth them out, 'resolve' them, so to speak, by refining the defining fan.

There are, however, a few pieces of theory we did not address properly, or at all, that are necessary for the following results to make sense. We shall either cite or prove them as we go.

The content of this section is a somewhat complex conglomerate of Cox et al. [2011], Sections 10.1 and 10.2, Fulton [1993] Sections 2.2 and 2.6 and last but not least Oda [1985], Section 1.6.

We would like to understand the structure of V_σ for σ is strongly convex rational polyhedral cone of dimension two. Let $\{e_1, e_2\}$ be the basis of N. If σ is smooth, then by definition its generators form a $\mathbb{Z}$-basis of N so we can suppose $\sigma = \text{Cone}(e_1, e_2)$. Indeed, if this is not the case, consider the isomorphism $N \simeq N'$ where $N' \simeq \mathbb{Z}^2$ is the lattice with basis the generators of σ and look at σ as a cone in $N'_\mathbb{R} \simeq N_\mathbb{R}$.

Then, $\mathsf{S} = \sigma^* \cap M$ are simply the linear combinations of the elements of the dual basis, which we label $\{\varepsilon_1, \varepsilon_2\}$, with non-negative integral coefficients. In other words, $\mathsf{S} = \mathbb{N}\{\varepsilon_1, \varepsilon_2\}$, hence

$$\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\chi^{\varepsilon_1}, \chi^{\varepsilon_2}] \simeq \mathbb{C}[x, y].$$

It follows that $V_\sigma = \text{Specm}(\mathbb{C}[\mathsf{S}_\sigma]) \simeq \text{Specm}(\mathbb{C}[x, y]) = \mathbb{C}^2$ so the case when σ is smooth is not too interesting.

Before we tackle the general case, we consider the cone $\sigma := \text{Cone}(e_2, ke_1 - e_2)$ for some $k \in \mathbb{N}, k \geq 2$ and on we march to wage war against V_σ.

If $k \geq 2$, then σ is of course not smooth. What does V_σ look like in this case? Consider the lattice N' with basis $\{e_2, ke_1 - e_2\}$. Denote by σ' the cone with the same generators as σ but viewed as a cone in $N'_\mathbb{R}$. Then, σ' is smooth and so $V_{\sigma'} \simeq \mathbb{C}^2$ by the observations already made. To describe V_σ, we need the following result.

Proposition 2.3.1. *Let N' is a sublattice of N of finite index (meaning N/N' is finite) and $\sigma \subseteq N'_\mathbb{R} \simeq N_\mathbb{R}$ a strongly convex rational polyhedral cone. Denote by σ' the cone with the same generators as σ considered as a cone in $N'_\mathbb{R}$. Then*

(a) there is a natural isomorphism

$$N/N' \simeq \ker(T_{N'} \to T_N),$$

(b) $G := N/N'$ acts on $V_{\sigma'}$ and the morphism $V_{\sigma'} \to V_\sigma$ induced by the inclusion $N' \hookrightarrow N$ is surjective and has kernel G. Therefore,

$$V_{\sigma'}/G \simeq V_\sigma.$$

Proof. We cannot prove this statement unless we dive into orbits of torus action and deepen our understanding of convex cones. Thus, for a full proof, see for instance Oda [1985], Proposition 1.25. □

This proposition enables us to find an explicit form for V_σ if we can do so first for N/N'. We can write N and N' as $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $N' = \mathbb{Z}e_2 \oplus \mathbb{Z}(ke_1 - e_2) = k\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. This makes it easy to see that $N/N' \simeq \mathbb{Z}/k\mathbb{Z}$. We (hopefully) know this group quite well. One important fact to us is that, $\mathbb{Z}/k\mathbb{Z}$ is isomorphic to the multiplicative subgroup of $\mathbb{C}^*$ consisting of k-th roots of unity, which we will denote by μ_k. We study the action of N/N' on $V_{\sigma'} \simeq \mathbb{C}^2$.

Denote by M' the dual lattice to M. The inclusion $N' \subseteq N$ implies $M \subseteq M'$. Recall that we denote $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ and $M_\mathbb{Q} := M \otimes_\mathbb{Z} \mathbb{Q}$. The bilinear pairing $M \times N \to \mathbb{Z}$ induces a bilinear pairing $M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q}$. Given that we are interested in $\mu_k \simeq N/N'$, we are, in the light of this discovery, prompted to define the map

$$\begin{aligned} M'/M \times N/N' &\to \mathbb{C}^*, \\ (m' + M, u + N') &\mapsto e^{2\pi i \langle m', u \rangle} \end{aligned}$$

which basically maps $m' + M$ and $u + N'$ to the $\langle m', u \rangle$-th root of unity. This statement makes sense because $\langle m', u \rangle$ is in this scenario often a rational number. Indeed, if the basis of N' is $\{e_2, ke_1 - e_2\}$, then it is easy to calculate that the basis of M' is $\{\frac{1}{k}\varepsilon_1 + \varepsilon_2, \frac{1}{k}\varepsilon_1\}$. Hence $M' \simeq \mathbb{Z}(\frac{1}{k}\varepsilon_1 + \varepsilon_2) \oplus \mathbb{Z}\frac{1}{k}\varepsilon_1 \simeq \frac{1}{k}\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ and so

$$M'/M \simeq (\frac{1}{k}\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2)/(\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2) \simeq \frac{1}{k}\mathbb{Z}/\mathbb{Z}.$$

As before, we will denote equivalence classes by $[\cdot]$, be it in M'/M, N/N' or any other quotient as long as the context is clear. Using the lattice isomorphism above, we can write

$$\langle [a/k], [b] \rangle = \frac{ab}{k}.$$

for $[a/k] \in \frac{1}{k}\mathbb{Z}/\mathbb{Z} \simeq M'/M$ and $[b] \in \mathbb{Z}/k\mathbb{Z} \simeq N/N'$. This shows that we actually have a mapping

$$M'/M \times N'/N \to \mu_k.$$

We ascertain it is well-defined. Suppose $[a/k] = [a'/k]$ and $[b] = [b']$, Then, $a \equiv a' \pmod{k}$ and $b \equiv b' \pmod{k}$, so $ab \equiv a'b' \pmod{k}$. It follows that $e^{2\pi iab/k} = e^{2\pi ia'b'/k}$. This bilinear mapping thus induces an isomorphism $N'/N \simeq \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mu_k)$.

The last line finally gives us the form of the action of $G := N'/N$ on $V_{\sigma'}$. If we now denote the basis of M' as $\{m_1, m_2\}$, the previous paragraph implies that G acts on $\mathbb{C}[V_{\sigma'}] = \mathbb{C}[\chi^{m_1}, \chi^{m_2}] \simeq \mathbb{C}[x, y]$ by

$$[u] \cdot \chi^{m'} = e^{2\pi i\langle m', u\rangle}\chi^{m'}$$

for $m' \in (\sigma')^* \cap M'$ and $u = le_1, l = 0, \ldots, k-1$. We calculate that

$$\langle m_1, e_1 \rangle = \left\langle \frac{1}{k}\varepsilon_1 + \varepsilon_2, e_1 \right\rangle = \frac{1}{k}$$

and similarly

$$\langle m_2, e_1 \rangle = \frac{1}{k}.$$

Hence, if we set up the isomorphism $\mu_k \simeq N/N'$ by mapping $e^{2\pi il/k} \mapsto [le_1]$, then we have

$$e^{\frac{2\pi il}{k}} \cdot (x, y) = (e^{\frac{2\pi il}{k}}x, e^{\frac{2\pi il}{k}}y)$$

for all $l \leq k-1$ and $(x, y) \in V_{\sigma'}$ as $x = \chi^{m_1}$ and $y = \chi^{m_2}$. From this and **Proposition 2.3.1** we finally get the result that $V_\sigma \simeq V_{\sigma'}/\mu_k \simeq \mathbb{C}^2/\mu_k$ if $\sigma = \mathrm{Cone}(e_2, ke_1 - e_2)$ and the action of μ_k on $\mathbb{C}^2$ is as above.

You might wonder why we went through all this stuff just to study this one special case. We are glad you wonder and beg you put your mind at ease. The cone $\mathrm{Cone}(e_2, ke_1 - e_2)$ is actually not all that special. A general two-dimensional strongly convex cone is not too different. We prove the following combinatorial result.

Proposition 2.3.2 (Characterizing Two-Dimensional Strongly Convex Cones)**.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a two-dimensional strongly convex rational polyhedral cone. Then, there exists a basis $\{e_1, e_2\}$ of N such that*

$$\sigma = \mathrm{Cone}(e_2, ke_1 - le_2)$$

for $k > 0, 0 \leq l < k$ and $\gcd(k, l) = 1$.

Proof. Suppose $\sigma = \text{Cone}(u_1, u_2)$ for *primitive* vectors $u_1, u_2 \in N$, which we identify with $\mathbb{Z}^2$. Because of primitiveness, we can suppose one of u_1, u_2 is part of the canonical basis of $\mathbb{Z}^2$, for instance $u_2 = (0, 1)$, and the other is (k, z) for some positive integer k and $z \in \mathbb{Z}$. Since the matrix

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

for $c \in \mathbb{Z}$, is a lattice automorphism of $\mathbb{Z}^2$ and

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ ck + z & 1 \end{pmatrix},$$

we see that we can choose z arbitrarily modulo k. So, we take $z := -l$ for some $0 \leq l < k$. The equality $\gcd(k, l) = 1$ stems from the fact that u_1 is primitive. Hence, $\sigma = \text{Cone}(e_2, ke_1 - le_2)$ for $k > 0$, $0 \leq l < k$ and $\gcd(k, l) = 1$. $\square$

This proposition gives us a great advantage in dealing with V_σ for arbitrary strongly convex rational cone σ. As a matter of fact, the calculations covered in the last few pages are almost directly applicable to the situation when $\sigma = \text{Cone}(e_2, ke_1 - le_2)$.

If N' now has the basis $\{e_2, ke_1 - le_1\}$, then the dual basis of M' is $\{\frac{l}{k}\varepsilon_1 + \varepsilon_2, \frac{1}{k}\varepsilon_1\} =: \{m_1, m_2\}$. Consequently, $\langle m_1, e_1 \rangle = 1/k$ and $\langle m_2, e_1 \rangle = l/k$, so $N/N' \simeq \mu_k$ acts on $\mathbb{C}^2$ by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^l y)$$

where $\zeta := e^{2\pi i/k}$ and , again, $x = \chi^{m_1}, y = \chi^{m_2}$.

With part of the preliminaries explained and proven, we are ready to progress to the resolution of singularities on toric surfaces. On the level of affine toric varieties, the isomorphism $V_{\sigma'}/G \simeq V_\sigma$, with notation as above, implies that V_σ has only finitely many singular points because $V_{\sigma'}$ is smooth and $G = N/N'$ is finite. The idea stays the same for the toric variety X_Σ where Σ is a fan. Since every point of X_Σ lies in some $V_\sigma, \sigma \in \Sigma$ whereof there are finitely many, X_Σ also has only finitely many singular points. Unfortunately, we lack the resources to provide a rigorous argument for this fact and instead must delegate tolerant readers to Cox et al. [2011], Section 3.2 for a general discussion and Theorem 3.2.6 for the main result.

The idea behind smoothing out X_Σ is to subdivide cones of Σ which are not smooth into multiple new cones, which are. To formalize this, we first need to touch upon toric morphisms and refinements of fans.

Definition 2.3.3 (Toric Morphism)**.** Let $X_{\Sigma_1}, X_{\Sigma_2}$ be normal toric varieties coming from fans Σ_1 in $(N_1)_\mathbb{R}$ and Σ_2 in $(N_2)_\mathbb{R}$. We call a morphism

$$\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$$

toric if φ maps the torus $T_{N_1} \subseteq X_{\Sigma_1}$ into $T_{N_2} \subseteq X_{\Sigma_2}$ and $\varphi_{|T_{N_1}}$ is a group homomorphism.

An important results states that lattice maps which are compatible with fans yield toric morphisms between the corresponding toric varieties.

First things first, reusing the notation from **Definition 2.3.3**, we say that a $\mathbb{Z}$-linear map $\Phi : N_1 \to N_2$ is *compatible* with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$, one can find a cone $\sigma_2 \in \Sigma_2$ such that $\Phi(\sigma_1) \subseteq \sigma_2$.

Now, the result.

Theorem 2.3.4. *Let N_1, N_2 be lattices and Σ_i a fan in $(N_i)_{\mathbb{R}}, i = 1, 2$. If $\Phi : N_1 \to N_2$ is a $\mathbb{Z}$-linear map compatible with Σ_1 and Σ_2, then there exists a toric morphism $\overline{\Phi} : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\overline{\Phi}_{|T_{N_1}}$ is the map*

$$\Phi \otimes 1 : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \to N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Proof. See Cox et al. [2011], Theorem 3.3.4. □

Finally, we elucidate what we mean by 'resolution of singularities'. Recall that a map $f : X \to Y$ between topological spaces X and Y is called *proper* if $f^{-1}(K)$ is compact for every compact $K \subseteq Y$. Let X be a surface (two-dimensional variety) and denote by X_{sing} the set of singular points of X. A proper morphism $\varphi : Y \to X$ is called a *resolution of singularities* if Y is a smooth surface and φ induces an isomorphism

$$Y \setminus \varphi^{-1}(X_{\text{sing}}) \simeq X \setminus X_{\text{sing}}$$

of varieties.

In the case of toric varieties, resolutions of singularities are induced by the refinements of fans corresponding to said varieties. If Σ is a fan in $N_{\mathbb{R}}$, we say that a fan Σ' is a *refinement* of Σ if every cone in Σ is a union of cones in Σ'. The identity map $N \to N$ is then compatible with Σ and Σ' and we obtain a toric morphism $X_{\Sigma'} \to X_{\Sigma}$. In case Σ' is smooth, then $X_{\Sigma'}$ is a smooth variety.

In light of this, our strategy of smoothing out toric varieties will be as follows. Consider an affine toric variety V_σ for σ a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Suppose V_σ has only one singular point P_σ. In particular, σ is not smooth. We refine σ by inserting a new ray, effectively subdividing it into two smaller cones, which together form a fan Σ. We will soon show this always yields a refinement such that the resulting fan is smooth. The corresponding toric morphism $\varphi_\sigma : X_\Sigma \to V_\sigma$ restricts to an isomorphism

$$X_\Sigma \setminus \varphi_\sigma^{-1}(P_\sigma) \simeq V_\sigma \setminus P_\sigma$$

and is proper, so it is a resolution of singularities. This process is iterable and scales to generic toric surfaces with finite number of singularities. One just needs to make a refinement successively for every singularity. Since X_Σ is obtained by gluing together V_σ for all $\sigma \in \Sigma$, the final resolution of singularities is a gluing of all the φ_σ. We must omit the full proof of this statement and instead make one more reference to Cox et al. [2011], Example 10.1.8. Thankfully, with what is available to us at the moment, we can at least prove that a smooth refinement always exists. This is the content of the following claim, for the content of which we thank Cox et al. [2011] and their Theorem 10.1.10.

Theorem 2.3.5 (Resolution of Singularities on Toric Surfaces)**.** *Let Σ be a fan in $N_{\mathbb{R}}$ and X_Σ a normal toric surface. There exists a smooth fan Σ' refining Σ such that the induced toric morphism $\varphi : X_{\Sigma'} \to X_\Sigma$ is a resolution of singularities.*

Proof. Thanks to the facts contained in the preceding few paragraphs, we know that if such a refinement exists, then the identity map on the lattice N induces a resolution of singularities.

Let $\sigma_1, \ldots, \sigma_r$ be the two-dimensional cones in a fan Σ. Let N_i be the sublattice of N with basis consisting of the ray generators of σ_i for every i. We define $\operatorname{mult}(\sigma_i) \coloneqq |N/N_i|$ and

$$s(\Sigma) \coloneqq \sum_{i=1}^{r} (\operatorname{mult}(\sigma_i) - 1).$$

We will prove the existence of a smooth refinement of Σ by induction on $s(\Sigma)$.

If $s(\Sigma) = 0$, then either $r = 0$ or $\operatorname{mult}(\sigma_i) = 1$ for all i as $\operatorname{mult}(\sigma_i)$ is a positive integer. It is clear that in this case Σ is already smooth because it is either empty or the generators of σ_i form the basis of $N_i \simeq N$ for each i. We take this as the base case for the induction hypothesis.

Assume that we can find a smooth refinement for every Σ with $s(\Sigma) < s$ for some $s \in \mathbb{N}$ and consider a fan Σ with $s(\Sigma) = s$. If $s \geq 1$ and Σ is not smooth, then there exists a non-smooth cone $\sigma_i \in \Sigma$ for some i. By **Proposition 2.3.2**, $\sigma_i = \operatorname{Cone}(e_2, ke_1 - le_2)$ for parameters $k > 0, 0 \leq l < k$ and $\gcd(k, l) = 1$. Consider the refinement Σ' of Σ constructed by inserting the ray $\rho \coloneqq \operatorname{Cone}(e_1)$ which subdivides σ_i into $\sigma_i' = \operatorname{Cone}(e_2, e_1)$ and $\sigma_i'' = \operatorname{Cone}(e_1, ke_1 - le_2)$. We shall show that $s(\Sigma') < s(\Sigma)$. For all $j \neq i$, the values $\operatorname{mult}(\sigma_j)$ remain unchanged. The cone σ_i' is smooth because $\{e_1, e_2\}$ is the basis of N. Hence, $\operatorname{mult}(\sigma_i') - 1 = 0$ and σ_i' contributes nothing to the sum $s(\Sigma')$. We must deal with the cone σ_i''.

The matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

written with respect to the basis $\{e_1, e_2\}$, defines a $\mathbb{Z}$-linear mapping which is also an automorphism of N. It thus takes the cone σ_i'' to $\operatorname{Cone}(e_2, le_1 + ke_2)$ while maintaining the value $\operatorname{mult}(\sigma_i'')$. We need to make use of the following easy number-theoretical lemma.

Lemma 2.3.6 (Can We Divide?, Exercise 10.1.1 in Cox et al. [2011])**.** *Let $u \in \mathbb{Z}, v > 0$. Then, there exist unique $x, y \in \mathbb{Z}$ such that $0 \leq y < v$ and $u = xv - y$.*

Proof. Let $-y \coloneqq u \bmod v$. Then, $0 \leq y < v$ and there exists unique $x \in \mathbb{Z}$ such that $xv - y = u$. □

Continuation of the proof of Theorem 2.3.5. By **Lemma 2.3.6**, we write

$$k = xl - y$$

for some $x \in \mathbb{Z}$ and $0 \leq y < l$. Since $\gcd(k, l) = 1$, we also have $\gcd(l, y) = 1$. Hence, $\sigma_i'' = \text{Cone}(e_2, le_1 - ye_2)$. Because $l < k$, if N_i'' is the sublattice generated by the generators of σ_i'', we obtain

$$\text{mult}(\sigma_i'') = |N/N_i''| = |\mathbb{Z}/l\mathbb{Z}| < |\mathbb{Z}/k\mathbb{Z}| = |N/N_i| = \text{mult}(\sigma_i).$$

So, $s(\Sigma') < s(\Sigma)$ and the proof follows by induction. □

The last thing we take a look at this thesis before we part ways forever and ever is the promised algorithmic way to construct refinements of fans, hence also resolution of singularities on toric surfaces, through continued fraction expansion.

We know that we can find a basis $\{e_1, e_2\}$ of N such that a strongly convex cone is of the form $\text{Cone}(e_2, ke_1 - le_2)$ for $k > 0$ and $l \in \mathbb{Z}$. We shall call these integers, the *parameters* of the cone. In the proof of **Theorem 2.3.5**, we subdivided the cone $\sigma := \text{Cone}(e_2, ke_1 - le_2)$ into $\sigma' := \text{Cone}(e_2, e_1)$, which is smooth, and $\sigma'' := \text{Cone}(e_1, ke_1 - le_1)$ which may not be. We used **Lemma 2.3.6** to find $b_1 \geq 2$ and $0 \leq l_1 < l$ such that $k = b_1 l - l_1$ and thus write the latter possibly non-smooth cone as $\text{Cone}(e_2, le_1 - l_1 e_2)$, that is, with parameters (l, l_1).

Iterating this operation – taking N'' to be the sublattice of N with basis being the generators of σ'', we again subdivide σ'' into a smooth cone and a possibly non-smooth cone with parameters l_1, l_2, where $0 \leq l_2 < l_1$ is defined by $l = b_2 l_1 - l_2$ for adequate $b_2 \geq 2$. We get the sequence

$$\begin{aligned} k &= b_1 l - l_1, \\ l &= b_2 l_1 - l_2, \\ &\vdots \\ l_{r-3} &= b_{r-1} l_{r-2} - l_{r-1}, \\ l_{r-2} &= b_r l_{r-1} \end{aligned} \tag{2.3.1}$$

which computes the parameters of the cones constructed by each additional subdivision of the original non-smooth cone σ. Since $k > l > \ldots > l_{r-1} > \ldots$ is a strictly decreasing sequence of natural numbers, it terminates for some $r \in \mathbb{N}$ with $l_r = 0$. Dividing the i-th equation by l_{i-1} (where we take $l_0 := l$) in (2.3.1), we obtain

$$\begin{aligned} \frac{k}{l} &= b_1 - \frac{l_1}{l}, \\ \frac{l}{l_1} &= b_2 - \frac{l_2}{l_1}, \\ &\vdots \\ \frac{l_{r-3}}{l_{r-2}} &= b_{r-1} - \frac{l_{r-1}}{l_{r-2}}, \\ \frac{l_{r-2}}{l_{r-1}} &= b_r. \end{aligned} \tag{2.3.2}$$

Since we also have $b_i \geq 2$ for all $i \leq r$, we mingle all these equations together to obtain the continued fraction expansion

$$\frac{k}{l} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\dots - \cfrac{1}{b_r}}} =: [[b_1, \dots, b_r]] \tag{2.3.3}$$

Before we formulate our final theorem of the section *and* of the thesis about the, hopefully not shocking, relation of these continued fraction expansions and constructions of resolution of singularities on toric surfaces through refinements, we need one number-theoretical result.

Proposition 2.3.7. *Let $k > l > 0$ be coprime integers and $k/l = [[b_1, \dots, b_r]]$. Define p_i and q_i recursively by the following rule.*

$$p_0 := 1, \quad q_0 := 0,$$
$$p_1 = b_1, \quad q_1 := 1,$$

and for all $2 \leq i \leq r$

$$p_i := b_i p_{i-1} - p_{i-2},$$
$$q_i := p_i q_{i-1} - q_{i-2}.$$

Then,

(1) the sequences $(p_i)_{i=0}^r, (q_i)_{i=0}^r$ are integral and increasing,

(2) $[[b_1, \dots, b_r]] = p_i/q_i$ for all $i \leq r$,

(3) $p_{i-1}q_i - p_i q_{i-1} = 1$ for all $i \leq r$,

(4) we have the following strictly decreasing sequence

$$\frac{d}{k} = \frac{p_r}{q_r} < \frac{p_{r-1}}{q_{r-1}} < \dots < \frac{p_1}{q_1}.$$

Proof. Can be found in most books covering the basics of number theory and continued fractions but Cox et al. [2011] also generously provide their own proof in Proposition 10.2.2. □

With this final result, whose proof we also only cite since it is somewhat technical and only summarizes in a rigorous way the observations made thus far using the notation and content of **Proposition 2.3.7**, we bid farewell to our kind readers.

Theorem 2.3.8. *Let $\sigma := \mathrm{Cone}(e_2, ke_1 - le_2)$ be a strongly convex rational polyhedral cone with parameters k, l. Let $u_0 = e_2$ and let p_i, q_i denote the integers from **Proposition 2.3.7**. Construct vectors*

$$u_i := p_{i-1}e_1 - q_{i-1}e_2$$

and cones

$$\sigma_i := \mathrm{Cone}(u_{i-1}, u_i)$$

for $1 \leq i \leq r+1$. Then,

(1) each σ_i is a smooth cone and u_{i-1}, u_i are its ray generators,

(2) $\sigma_i \cap \sigma_{i+1} = \text{Cone}(u_i)$ for each i,

(3) $\bigcup_{i=1}^{r+1} \sigma_i = \sigma$, so the fan Σ consisting of σ_i and all their faces is a smooth refinement of σ,

(4) the induced toric morphism $X_\Sigma \to V_\sigma$ is a resolution of singularities.

Proof. Can be found in Cox et al. [2011], Theorem 10.2.3. □

Conclusion

We have defined and studied affine toric varieties, objects dancing on the border between combinatorics and algebraic geometry. We have seen how convex polyhedral cones, closed convex sets in the real euclidean space, determine the structure of affine toric varieties, basically special sets of solutions to a finite number of polynomial equations. We have also briefly discussed algebraic tori and proven many intrinsic qualities of cones and affine toric varieties separately.

We did not stop at the affine plane. We have introduced the concept of abstract variety and then glued together affine toric varieties to construct abstract toric varieties – complex analytical spaces only locally homeomorphic to affine toric varieties but still retaining combinatorial properties. We demonstrated that there are also natural gluing methods on the other side of the spectrum, creating fans of convex polyhedral cones. The wild and overly optimistic idea that gluing together cones might somehow produce abstract toric varieties was actually right on the money as we took great pains to show.

We ended with a glimpse into the theory of singularities on toric surfaces and devised an algorithmic way to remove those through refinement of the corresponding fans of two-dimensional cones.

The correspondence between combinatorics and algebraic geometry was what originally prompted me to write a thesis on the topic of toric varieties and I sincerely hope I have managed to convey some of the fascination I have felt in an accessible manner. I expect I shall dedicate myself to finding more of these correspondences in the near future. The podium where many mathematical disciplines come together to perform a play is where lies my curiousity, reward and fun.

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