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The classical McKay correspondence

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Abstract: The McKay correspondence is an interesting connection between many different areas of mathematics. The connecting element of the McKay correspondence is a special family of graphs called the Dynkin diagrams. In this thesis, we will study the classical McKay correspondence, which is an interesting connection between finite subgroups of SL(2,C) and Dynkin diagrams without oriented edges. Moreover, there are two ways how to get the Dynkin diagrams from the groups. In the first chapter of the thesis, we will provide a classification for both the finite subgroups and Dynkin diagrams. The second chapter uses the tools of the representation theory to construct the corresponding graph from the irreducible representations of the group. In the third part, we let the group act on the two-dimensional complex vector space. We then factor out this action to construct an algebraic variety with one singular point and find the Dynkin diagram in this singularity.

Keywords: McKay correspondence, McKay graphs, Dynkin diagrams

Contents

Introduction **2**

1 Finite subgroups of $SL(2,\mathbb{C})$ and Dynkin diagrams **3**
1.1 Classification of finite subgroups of $SL(2,\mathbb{C})$ 3
1.2 Dynkin diagrams 11

2 The algebraic correspondence **15**
2.1 Introduction to the representation theory of finite groups 15
2.2 McKay graphs and the algebraic correspondence 16

3 The geometric correspondence **21**
3.1 Introduction to algebraic geometry 21
3.2 Geometric invariant theory 22
3.3 Singularities 23

Conclusion **27**

Bibliography **28**

Introduction

In this thesis, we will study the classical McKay correspondence, which is a bijection between nontrivial finite subgroups of $SL(2, C)$ and the simply laced Dynkin diagrams. This correspondence has two parts, the algebraic and the geometric. Both describe a way how to construct a Dynkin diagram from a given subgroup of $SL(2, C)$, but the tools they are using are very different. We will firstly classify the finite subgroups of $SL(2, C)$ and the simply laced Dynkin diagrams and then cover both of the correspondences.
The structure of this thesis is inspired by Hemelsoet [2018], which covers similar topics. In this thesis, I've reached similar results, however, I tried to present my own proofs of the theorems.

1. Finite subgroups of $SL(2, \mathbb{C})$ and Dynkin diagrams

1.1 Classification of finite subgroups of $SL(2, \mathbb{C})$

Our first goal is to classify all the finite subgroups of $SL(2, \mathbb{C})$ up to conjugation. Fix any finite $G < SL(2, \mathbb{C})$ with more than two elements. Recall from linear algebra and group theory:

- Every matrix $g \in G$ satisfies $g^{|G|} = I$, where I is the identity matrix.
- Over algebraically closed field, any matrix is conjugate to a matrix in Jordan canonical form.
- A Jordan block B of dimension $d > 1$ cannot satisfy an equality $B^n = I$ for $n > 0$, therefore all matrices in G must be diagonalizable.
- Two diagonalizable matrices commute if and only if they share a common eigenbasis. A proof of this statement can be found in Horn and Johnson [2012] (Theorem 1.3.12)
- Since $G < SL(2, \mathbb{C})$ and for diagonalizable matrices the determinant is the product of eigenvalues, the eigenvalues of any matrix from G must be mutually inverse.

Now we can start the classification. Denote $\mathcal{A}$ the set of all maximal abelian subgroups of G and a subgroup $Z < G$ depending on the parity of G: $Z = \{I, -I\}$ for $|G|$ even and $Z = \{I\}$ for $|G|$ odd.

Proposition 1.1.1. *The matrix $-I$ is in G if and only if $|G|$ is even.*

Proof. If $|G|$ is odd, then the order of any $g \in G$ is also odd, therefore $-I \notin G$. If $|G|$ is even, then it must contain an element of order 2. As shown above, this matrix must be conjugate to a matrix of the form $diag(\lambda, \lambda^{-1})$. Since order of this matrix is 2, the eigenvalues must satisfy the equation $\lambda^2 = 1$, therefore $\lambda \in \{1, -1\}$. $\lambda = 1$ gives us a matrix I of order 1, $\lambda = -1$ gives us $-I$. These matrices commute with any other matrix, therefore conjugation doesn't change them and we see that $-I$ is the only possible matrix of order 2, which can be in G. □

Proposition 1.1.2. *Let $A_1, A_2 \in \mathcal{A}$, $A_1 \neq A_2$. Then $A_1 \cap A_2 = Z$.*

Proof. Since Z is a subgroup of the center $Z(SL(2, \mathbb{C}))$, it is also in the center $Z(G)$. If we take a group generated by the elements of A_1 and elements of Z, its generators commute, therefore the group is abelian. From maximality of A_1 this means that $Z \leq A_1$. Similarly $Z \leq A_2$, therefore $Z \leq A_1 \cap A_2$. Now suppose $b \in (A_1 \cap A_2) \setminus Z$. The only matrices in G, which have an eigenspace of dimension $d > 1$ are in Z, therefore b has only one eigenbasis B up to permuting and scaling the vectors. From which it follows that for any matrix in G commuting with b, B is an eigenbasis. Denote $A = \{g \in G | B \text{ is an eigenbasis of } g\}$. This set is closed under multiplication, inverses and contains I, therefore a subgroup of G. From the property of sharing an eigenbasis, A is abelian. Therefore $A_1 \leq A \geq A_2$ and from the maximality of A_1, A_2 follows $A_1 = A = A_2$. $\square$

Corollary 1.1.3. $|G| + (|\mathcal{A}| - 1)|Z| = \sum_{A \in \mathcal{A}} |A|$

Proof. Every element of $G \setminus Z$ is in some $A \in \mathcal{A}$, from the Proposition in exactly one, therefore $|G| - |Z| = \sum_{A \in \mathcal{A}}(|A| - |Z|) = \sum_{A \in \mathcal{A}} |A| - |\mathcal{A}| \cdot |Z|$. $\square$

Proposition 1.1.4. *For every $A \in \mathcal{A}$ there exists a matrix $M \in GL(2, \mathbb{C})$ and a natural number n, such that $A = \{M \cdot diag(e^{\frac{2k\pi i}{n}}, e^{-\frac{2k\pi i}{n}}) \cdot M^{-1} | k \in \{0, \dots, n-1\}\}$.*

Proof. The set Z is not a maximal abelian subgroup of G (the group generated by elements of Z and any other element is abelian), therefore for all $A \in \mathcal{A}$ it holds $A \setminus Z \neq \emptyset$. According to the proof of previous proposition, if we choose any $b \in a \setminus Z$ with eigenbasis B, then $A = \{g \in G | B \text{ is an eigenbasis of } g\}$. Therefore if we define M as the matrix, whose columns are the vectors of B, A will be of the form $\{M \cdot diag(\lambda_i, \lambda_i^{-1}) \cdot M^{-1} | i \in \{1, \dots, |A|\}\}$ for some $\lambda_i \in \mathbb{C}$. Since the order of every $a \in A$ is a divisor of $|G|$, only possible λ_i are those of the form $e^{\frac{2k\pi i}{|G|}}$ for $k \in \{0, \dots, |G| - 1\}$. Therefore A is a subgroup of a cyclic group $\{M \cdot diag(e^{\frac{2k\pi i}{|G|}}, e^{-\frac{2k\pi i}{|G|}}) \cdot M^{-1} | k \in \{0, \dots, |G| - 1\}\}$, from which it follows it is of the form we claimed. $\square$

Proposition 1.1.5. *If $A \in \mathcal{A}$ and g is in the normalizer $N(A)$, then either $gag^{-1} = a$ for every $a \in N(A) \setminus A$, or $gag^{-1} = a^{-1}$ for every $a \in N(A) \setminus A$.*

Proof. Since A is cyclic, any homomorphism from A is given by its value on a generator of A. The map $a \mapsto gag^{-1}$ is a homomorphism and conjugation doesn't change the eigenvalues of a matrix, therefore only possible values of gag^{-1} are a and a^{-1}. $\square$

Proposition 1.1.6. *For $A \in \mathcal{A}$, either $N(A) = A$ or $|N(A)| = 2|A|$.*

Proof. Define a homomorphism $N(A) \to Aut(A)$, $g \mapsto \varphi_g, \varphi_g(a) = gag^{-1}$. The image of this homomorphism has one or two elements, the kernel consists of those elements, which commute with all elements from A, from maximality of A this kernel is A. $\square$

Theorem 1.1.7. *For odd $|G|$, G is cyclic. For even $|G|$, G is cyclic or satisfy the equality $|\mathcal{A}| = \frac{|G|}{4} + 1$.*

Proof. When we consider the action of conjugation on $\mathcal{A}$, i.e. $\varphi : G \to S(\mathcal{A})$, $\varphi_g(A) = gAg^{-1}$ and denote the orbit of A as $Conj(A)$, then $|Conj(A)| = \frac{|G|}{|N(A)|}$. Therefore if we denote $l_A = \frac{|N(A)|}{|A|}$, we get an equality $|A| = \frac{|G|}{l_A |Conj(A)|}$. From Corollary 1.1.3 then follows

$$1 + \frac{(|\mathcal{A}| - 1)|Z|}{|G|} = \sum_{A \in \mathcal{A}} \frac{1}{l_A |Conj(A)|}.$$

Conjugate groups have conjugate normalizers, therefore if we define $\mathcal{C}$ a set of orbits $\{Conj(A) | A \in \mathcal{A}, C = Conj(A)\}$, we can rewrite right-hand side of the equation and get

$$1 + \frac{(|\mathcal{A}| - 1)|Z|}{|G|} = \sum_{A \in \mathcal{A}} \frac{1}{l_A |Conj(A)|} = \sum_{C \in \mathcal{C}} \sum_{A \in C} \frac{1}{l_A |Conj(A)|} = \sum_{C \in \mathcal{C}} \frac{1}{l_C},$$

where l_C is defined as l_A of any $A \in C$. But since all l_A are 1 or 2, the right-hand side is now some integral multiple of $\frac{1}{2}$. Therefore left-hand side must be also, so $|G|$ divides $2(|\mathcal{A}|-1)|Z|$. For $|G|$ odd, all $A \in \mathcal{A}$ must also have odd order, therefore at least 3. For $|G|$ even, they all contain $-I$, therefore they must have even order, therefore at least 4. In both cases, $A \setminus Z$ always contains at least 2 elements, which are not in any other $A_i \setminus Z$, therefore $|G| > 2|\mathcal{A}|$. This means that $2(|\mathcal{A}|-1)|Z| < |G| \cdot |Z|$. For $|G|$ odd this means $2(|\mathcal{A}|-1)|Z| = 0|G|$, therefore $|\mathcal{A}| = 1$ and $\mathcal{A} = \{G\}$. Since all $A \in \mathcal{A}$ are cyclic, G is cyclic. For even $|G|$, we can either have $2(|\mathcal{A}|-1)|Z| = 0|G|$ and for the same reason is G cyclic, or $2(|\mathcal{A}|-1)|Z| = 1|G|$, from which follows the statement. □

Remark 1.1.8. *From the proof it also follows that in the case of non-cyclic groups,* $\sum_{C \in \mathcal{C}} \frac{1}{l_C} = \frac{3}{2}$*, so* $\mathcal{C}$ *has either two elements with one having* $l_{C_1} = 1$ *and the other* $l_{C_1} = 2$*, or three elements with* $l_{C_1} = l_{C_2} = l_{C_3} = 2$

From now on, assume G is not cyclic (and therefore $|G|$ is even and $|Z| = 2$).

Proposition 1.1.9. *If* A_1*,* A_2 *or* A_1*,* A_2*,* A_3*, are representatives of different elements of* $\mathcal{C}$*, then* $\frac{1}{|G|} + \frac{1}{4} = \sum_i \frac{1}{l_{A_i}|A_i|}$

Proof. Follows from the equality $\frac{|G|}{4} + 1 = \mathcal{A} = \sum_i |Conj(A_i)|$ when divided by $|G|$ and using the equality $|A| = \frac{|G|}{l_A |Conj(A)|}$. □

Proposition 1.1.10. *If* $|\mathcal{C}| = 3$*, the only values of* $|A_i|$ *and* $|G|$ *satisfying the equality* $\frac{1}{G} + \frac{1}{4} = \sum_i \frac{1}{l_{A_i}|A_i|}$ *are (up to permutation of* A_i*):*

Type:	$\lvert A_1\rvert$	$\lvert A_2\rvert$	$\lvert A_3\rvert$	$\lvert G\rvert$
I	4	4	$2k$	$4k$
II	4	6	6	24
III	4	6	8	48
IV	4	6	10	120

If $|\mathcal{C}| = 2$*, the only possible values satisfying the equation (where* $l_{A_1} = 1$*,* $l_{A_2} = 2$*) are:*

Type:	$\lvert A_1\rvert$	$\lvert A_2\rvert$	$\lvert G\rvert$
V	4	$2k$	$4k$
VI	6	4	24

Proof. Every $A \in \mathcal{A}$ contains the element $-I$ of order 2, therefore $|A|$ is even. $\{I, -I\}$ is not maximal, therefore $|A| \geq 4$. If $|\mathcal{C}| = 3$, we may WLOG assume $|A_1| \leq |A_2| \leq |A_3|$. Then $|A_1| = 4$, since $6 \leq |A_1| \leq |A_2| \leq |A_3|$ implies $\sum_i \frac{1}{2|A_i|} \leq \frac{1}{4} < \frac{1}{4} + \frac{1}{|G|}$. If $|A_2| = 4$, any value $|A_3| = 2k$ satisfies the equation with $|G| = 4k$. If $|A_2| = 6$, the only possible values for $|A_3|$ are 6, 8 and 10, otherwise again $\sum_i \frac{1}{2|A_i|} \leq \frac{1}{4} < \frac{1}{4} + \frac{1}{|G|}$. For the same reason $8 \leq |A_2|$ does not lead to a solution. If $|\mathcal{C}| = 2$ and $l_{A_1} = 1$, $l_{A_2} = 2$, then $|A_1| = x$, $|A_2| = y$ is a solution of the equation if and only if $|A_1| = |A_2| = x$, $|A_3| = y$ is a solution of the case $|\mathcal{C}| = 3$. □

Proposition 1.1.11. *The group* G *has* $2 + \sum_i \frac{|A_i|-2}{l_{A_i}}$ *conjugacy classes.*

Proof. Besides $\pm I$, there are $|A_i| - 2$ elements in each A_i, therefore there exist $(|A_i|-2)|Conj(A_i)| = (|A_i|-2)\frac{|G|}{l_{A_i}|A_i|}$ such elements in $\bigcup Conj(A_i)$. Each of these elements commutes only with its maximal abelian subgroup, therefore has conjugacy class of size $\frac{|G|}{|A_i|}$. Therefore $\bigcup Conj(A_i) \setminus \{\pm I\}$ splits into $\frac{|A_i|-2}{l_{A_i}}$ conjugacy classes. If we sum over i and add the conjugacy classes $\{I\}$ and $\{-I\}$, we get the expression $2 + \sum_i \frac{|A_i|-2}{l_{A_i}}$. □

Proposition 1.1.12. *For every $n \in \mathbb{N}$, $SL(2, \mathbb{C})$ has exactly one cyclic subgroup of order n up to conjugation.*

Proof. The group $\{diag(e^{\frac{2k\pi i}{n}}, e^{-\frac{2k\pi i}{n}}) | k \in \{0, \dots, n-1\}\}$ satisfies the properties, therefore existence is proved. From Proposition 1.1.4 it follows that any two such groups are conjugate. □

Proposition 1.1.13. *The subgroup of type I exists if and only if k is even. The subgroup of type V exists if and only if k is odd and greater than 1. In both cases, the group is unique up to conjugation.*

Proof. Let G be group of the type I or V. According to Proposition 1.1.4, we can find a matrix S with eigenvalues $e^{\frac{i\pi}{k}}$, $e^{-\frac{i\pi}{k}}$ in the $2k$-element cyclic subgroup and a matrix T with eigenvalues $e^{\frac{i\pi}{2}}$, $e^{-\frac{i\pi}{2}}$ in one of the 4-element maximal cyclic subgroups. Since these matrices do not commute and $|Conj(\langle S \rangle)| = \frac{|G|}{l_{\langle S \rangle} |\langle S \rangle|} = 1$ (in other words, $\langle S \rangle$ is normal), it must hold $TST^{-1} = S^{-1}$. Let $B = (b_1, b_2)$ be a basis, such that

$$[S]_B = \begin{pmatrix} e^{\frac{i\pi}{k}} & 0 \\ 0 & e^{-\frac{i\pi}{k}} \end{pmatrix}.$$

If we denote the elements of $[T]_B$ as

$$[T]_B = \begin{pmatrix} q & r \\ s & t \end{pmatrix},$$

we can compute its inverse

$$[T]_B^{-1} = \frac{1}{qt - rs} \begin{pmatrix} t & -r \\ -s & q \end{pmatrix},$$

where from $T \in SL(2, \mathbb{C})$ it follows that $qt - rs = det(T) = 1$, and get a matrix equality

$$\begin{pmatrix} qte^{\frac{i\pi}{k}} - rse^{-\frac{i\pi}{k}} & qr(e^{-\frac{i\pi}{k}} - e^{\frac{i\pi}{k}}) \\ st(e^{\frac{i\pi}{k}} - e^{-\frac{i\pi}{k}}) & qte^{-\frac{i\pi}{k}} - rse^{\frac{i\pi}{k}} \end{pmatrix} = [TST^{-1}]_B = [S^{-1}]_B = \begin{pmatrix} e^{-\frac{i\pi}{k}} & 0 \\ 0 & e^{\frac{i\pi}{k}} \end{pmatrix}.$$

From the equality of top right corners we learn that either $q = 0$ or $r = 0$, from the equality of bottom left corners we learn that either $s = 0$ or $t = 0$. Since $det(T) = qt - rs = 1$, this gives us only two possibilities, $[T]_B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ or $[T]_B = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$ for some $\lambda \in \mathbb{C}^*$, out of which only the second one gives us the equality of the diagonal elements. If we now define a basis $B' = (\lambda b_1, b_2)$, we get

$$[S]_{B'} = \begin{pmatrix} e^{\frac{i\pi}{k}} & 0 \\ 0 & e^{-\frac{i\pi}{k}} \end{pmatrix}, \ [T]_{B'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since G must be generated by S and T (the subgroup $\langle S, T\rangle$ contains an element of order $2k$ and is not abelian, there is no such subgroup of $SL(2, \mathbb{C})$ with order less than $4k$), this proves the uniqueness. To show the existence, let

$$S = \begin{pmatrix} e^{\frac{i\pi}{k}} & 0 \\ 0 & e^{-\frac{i\pi}{k}} \end{pmatrix}, \; T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $G = \langle S, T\rangle$. Since $T\langle S\rangle T^{-1} = \langle S\rangle$, $\langle S\rangle$ is a normal subgroup with $2k$ elements. Since $T \notin \langle S\rangle$, but $T^2 = -I \in \langle S\rangle$, $\left|\frac{\langle S,T\rangle}{\langle S\rangle}\right| = 2$. Therefore $\langle S, T\rangle$ has $4k$ elements. It is not abelian and contains an element of order $2k$, therefore it is either type I or type V. If k is odd, it can't be of the type I, because the number of sets conjugate to A_1 would have to be $\frac{|G|}{|A_1| \cdot l_{A_1}} = \frac{k}{2}$, which is not an integer. If k is even, then $S^{k/2}TS^{-k/2} = S^{k/2}S^{k/2}T = -IT = T^3 = T^{-1}$, from which it follows that $l_{\langle T\rangle} = 2$, therefore G is a group of the type I. □

Proposition 1.1.14. *A group of type II does not exist. A group of type VI exists and is unique up to conjugation.*

Proof. Let G be a group of the type *II* or *VI*. In both cases, there are four maximal abelian subgroup of order 6, therefore eight matrices with eigenvalues $e^{\frac{\pi i}{3}}$ and $e^{-\frac{\pi i}{3}}$. Denote them M_1, M_2, M_3, M_4, M_1^{-1}, M_2^{-1}, M_3^{-1} and M_4^{-1}. The group $\langle M_1\rangle$ acts on $\{M_2, M_3, M_4, M_2^{-1}, M_3^{-1}, M_4^{-1}\}$ by conjugation. Since an element of the set commute with an element of the group if and only if the element of the group is $\pm I$, the stabilizer of every element is $\{\pm I\}$ and there are two orbits of size 3. Suppose $M_i^{-1} = M_1^n M_i M_1^{-n}$. Then for any $M_j = M_1^m M_i M_1^{-m}$ it holds $M_j^{-1} = M_1^m M_i^{-1} M_1^{-m} = M_1^{n+m} M_i M_1^{-n-m} = M_1^n M_j M_1^{-n}$, therefore if one element in the orbit has its inverse in the same orbit, than this holds for all elements of the orbit. Therefore this can't happen, since the orbits have odd number of elements. Therefore $|Conj(\langle M_2\rangle)| \geq 3$ and G is of type *VI*. WLOG, we can therefore assume M_1, M_2, M_3 and M_4 lie in the same conjugacy class, and M_1 act on the rest by conjugation as permutation $(M_2\ M_3\ M_4)$.

Since G is of the type *VI*, none of the M_i is conjugate to its inverse, therefore orbit of M_1 under the action of conjugating by $\langle M_2\rangle$ is $\{M_1, M_3, M_4\}$. Let M_2 act on this set as a permutation $(M_1\ M_i\ M_j)$ for $\{i, j\} = \{3, 4\}$. In the case $i = 3$, $j = 4$ we reach a contradiction, since conjugating by $M_3 = M_1M_2M_1^{-1}$ would give us a permutation $(M_2\ M_3\ M_4) \circ (M_1\ M_3\ M_4) \circ (M_2\ M_3\ M_4)^{-1} = (M_1\ M_4\ M_2)$, but $M_3 = M_2M_1M_2^{-1}$ would give us $(M_1\ M_3\ M_4) \circ (M_2\ M_3\ M_4) \circ (M_1\ M_3\ M_4)^{-1} = (M_1\ M_2\ M_4)$. Therefore $i = 4$, $j = 3$. From this follows that the matrix $M_1M_2^{-1}$ acts as a permutation $(M_2\ M_3\ M_4) \circ (M_1\ M_4\ M_3)^{-1} = (M_1\ M_4)(M_2\ M_3)$. Therefore $M_1M_2^{-1}$ is a group element of order 4. Let $B = (b_1, b_2)$ be a basis, such that

$$[M_1]_B = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}.$$

Suppose

$$[M_2^{-1}]_B = \begin{pmatrix} q & r \\ s & t \end{pmatrix},$$

therefore

$$[M_1 M_2^{-1}]_B = \begin{pmatrix} qe^{\frac{\pi i}{3}} & re^{-\frac{\pi i}{3}} \\ se^{\frac{\pi i}{3}} & te^{-\frac{\pi i}{3}} \end{pmatrix}.$$

The matrix M_2^{-1} is in $SL(2, \mathbb{C})$, therefore $qt - rs = 1$. It has eigenvalues $e^{\frac{\pi i}{3}}$ and $e^{-\frac{\pi i}{3}}$, therefore $q + t = tr([M_2^{-1}]_B) = 1$. The matrix $M_1 M_2^{-1}$ has eigenvalues $\pm i$, therefore $qe^{\frac{\pi i}{3}} + te^{-\frac{\pi i}{3}} = tr([M_1 M_2^{-1}]_B) = 0$. From the last two equalities follows $0 = qe^{\frac{\pi i}{3}} + (1-q)e^{-\frac{\pi i}{3}} = \frac{1-\sqrt{3}i}{2} + \sqrt{3}iq$, $q = \frac{3+\sqrt{3}i}{6}$, $t = \frac{3-\sqrt{3}i}{6}$. Therefore

$$[M_2^{-1}]_B = \begin{pmatrix} \frac{3+\sqrt{3}i}{6} & r \\ -\frac{2}{3r} & \frac{3-\sqrt{3}i}{6} \end{pmatrix}$$

for some $r \in \mathbb{C}^*$. Again, we can change the basis to $B' = (rb_1, b_2)$ and then

$$[M_1]_{B'} = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}$$

$$[M_2^{-1}]_{B'} = \begin{pmatrix} \frac{3+\sqrt{3}i}{6} & 1 \\ -\frac{2}{3} & \frac{3-\sqrt{3}i}{6} \end{pmatrix}.$$

The group $\langle M_1, M_2 \rangle$ must be the whole G, since there is no subgroup of $SL(2, \mathbb{C})$ with order less then 24 and two noncommuting elements of order 6. This proves the uniqueness. To prove the existence, let

$$M_1 = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{3-\sqrt{3}i}{6} & -1 \\ \frac{2}{3} & \frac{3+\sqrt{3}i}{6} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} \frac{3-\sqrt{3}i}{6} & \frac{1-\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{3} & \frac{3+\sqrt{3}i}{6} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} \frac{3-\sqrt{3}i}{6} & \frac{1+\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{3} & \frac{3+\sqrt{3}i}{6} \end{pmatrix}$$

By direct computation, one can show that this set of matrices is closed under under conjugation by M_1 and M_2, which act on it as the already mentioned permutations $(M_2\ M_3\ M_4)$ and $(M_1\ M_2\ M_4)$ respectively. Therefore the action of conjugation of M_1, M_2, M_3 and M_4 by $\langle M_1, M_2 \rangle$ can be viewed as a group homomorphism $\langle M_1, M_2 \rangle \to A_4$. This homomorphism has kernel $\{\pm I\}$, since there is no other element commuting with all of these matrices. Therefore $\langle M_1, M_2 \rangle$ is a group of order at most 24, since it contains two noncommuting elements of order 6, it is of the type *VI*. □

Proposition 1.1.15. *A group of type III does exist and is unique up to conjugation.*

Proof. Let G be a group of the type *III*. There it contains 6 matrices with eigenvalues $e^{\frac{i\pi}{4}}$ and $e^{\frac{i\pi}{4}}$, denote them M_1, M_2, M_3, M_1^{-1}, M_2^{-1} and M_3^{-1}. The set

$\{M_2, M_3, M_2^{-1}, M_3^{-1}\}$ is closed under conjugation by $\langle M_1\rangle$. Since the only elements from $\langle M_1\rangle$, which commute with some element from $\{M_2, M_3, M_2^{-1}, M_3^{-1}\}$, are $\pm I$, the kernel of this action is $\{\pm I\}$ and the action is transitive. The conjugation of M_2 by M_1 cannot be M_2^{-1}, otherwise the conjugation of M_2^{-1} by M_1 would be M_2 and the orbit would have only these two elements. Therefore we can WLOG assume M_1 acts on $\{M_2, M_3, M_2^{-1}, M_3^{-1}\}$ as the permutation $(M_2\ M_3\ M_2^{-1}\ M_3^{-1})$. For the same reason, the matrix M_2 acts on $\{M_1, M_3, M_1^{-1}, M_3^{-1}\}$ as a permutation $(M_1^{\pm 1}\ M_3\ M_1^{\mp 1}\ M_3^{-1})$. The matrix $M_1M_2^2$ therefore acts on the set of all eight matrices as $(M_2\ M_3\ M_2^{-1}\ M_3^{-1}) \circ (M_1^{\pm 1}\ M_3\ M_1^{\mp 1}\ M_3^{-1})^2$, therefore as a permutation $(M_1\ M_1^{-1})(M_2\ M_3)(M_2^{-1}\ M_3^{-1})$. From this follows $M_1M_2^2$ is an element of order 4. Let $B = (b_1, b_2)$ be a basis, such that

$$[M_2]_B = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}.$$

Suppose

$$[M_1]_B = \begin{pmatrix} q & r \\ s & t \end{pmatrix},$$

therefore

$$[M_1M_2^2]_B = \begin{pmatrix} iq & -ir \\ is & -it \end{pmatrix}.$$

Since M_1 has eigenvalues $e^{\frac{i\pi}{4}}$ and $e^{\frac{i\pi}{4}}$, its trace is $\sqrt{2}$. Since $M_1M_2^2$ has order 4, it has eigenvalues $\pm i$ and its trace is zero. Therefore $q + t = \sqrt{2}$, $iq - it = 0$, which has solution $q = t = \frac{\sqrt{2}}{2}$. Since $M_1 \in SL(2, \mathbb{C})$, it must be of the form

$$[M_1]_B = \begin{pmatrix} \frac{\sqrt{2}}{2} & r \\ -\frac{1}{2r} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Again, we change the basis to $B' = (\sqrt{2}rb_1, b_2)$ to get

$$[M_2]_{B'} = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}$$

$$[M_1]_{B'} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Since M_1 and M_2 does not commute and both have order 8, the group $\langle M_1, M_2\rangle$ must be the whole G, which proves the uniqueness. To prove the existence, let

$$M_2 = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}$$

$$M_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Since the set of 32 vectors

$$\left\{ \omega v \;\middle|\; v \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} \right\}, \omega \in \left\{ \pm 1, \pm i, \pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i \right\} \right\}$$

is closed under multiplying by M_1 and M_2, it is closed under multiplying by all matrices from $\langle M_1, M_2\rangle$. These vectors generate the space $\mathbb{C}^2$, therefore a matrix is uniquely determined by its action on these vectors and the $\langle M_1, M_2\rangle$ can be embedded to S_{32}. Therefore it is finite and since it contains two noncommuting elements of order 8, it is of the type *III*. □

Proposition 1.1.16. *A group of type IV exists and is unique up to conjugation.*

Proof. Let G be a group of type IV. It contains a matrix M_1 with eigenvalues $e^{\frac{i\pi}{5}}$ and $e^{-\frac{i\pi}{5}}$. Let $B = (b_1, b_2)$ be a basis, such that

$$[M_1]_B = \begin{pmatrix} e^{\frac{\pi i}{5}} & 0 \\ 0 & e^{-\frac{\pi i}{5}} \end{pmatrix}.$$

Let M_2 be another matrix with the same eigenvalues, which generates a different maximal abelian subgroup. Suppose

$$[M_2]_B = \begin{pmatrix} q & r \\ s & t \end{pmatrix}.$$

Since M_2 has eigenvalues $e^{\frac{i\pi}{5}}$ and $e^{-\frac{i\pi}{5}}$, it has trace $q + t = 2cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{4}$. Since M_1M_2 has order 3, 5, 6, 10 or 4, its trace $e^{\frac{\pi i}{5}}q + e^{-\frac{\pi i}{5}}t$ can be $2cos(2\pi\alpha)$ for $\alpha \in \{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{1}{6}, \frac{2}{6}, \frac{1}{4}\}$. The same holds for the matrix $M_1^{-1}M_2$ with the trace $e^{-\frac{\pi i}{5}}q + e^{\frac{\pi i}{5}}t$. If we add these equations, we get $2cos(2\pi\alpha_1) + 2cos(2\pi\alpha_2) = 2cos(\frac{\pi}{5})q + 2cos(\frac{\pi}{5})t = 4cos^2(\frac{\pi}{5})$. The only solution with possible α_1, α_2 is $\alpha_1 = \frac{1}{10}$, $\alpha_2 = \frac{1}{6}$ or $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{1}{10}$. We can WLOG assume M_1M_2 has eigenvalues $e^{\frac{i\pi}{5}}$ and $e^{-\frac{i\pi}{5}}$, otherwise we can redefine M_2 to be M_2^{-1} and then $M_1M_2^{new} = (M_2^{old}M_1^{-1})^{-1}$, which is a matrix conjugated to $M_1^{-1}M_2^{old}$ and have the same eigenvalues. We can then compute q and t from $q + t = \frac{1+\sqrt{5}}{4}$ and $e^{\frac{\pi i}{5}}q + e^{-\frac{\pi i}{5}}t = \frac{1+\sqrt{5}}{4}$ and get $q = \frac{1+\sqrt{5}}{4}\frac{1-e^{\frac{\pi i}{5}}}{1-e^{\frac{2\pi i}{5}}} = \cdot\frac{1+\sqrt{5}}{4(1+e^{\frac{\pi i}{5}})}$, $t = \cdot\frac{(1+\sqrt{5})e^{\frac{\pi i}{5}}}{4(1+e^{\frac{\pi i}{5}})}$. The same way as in the previous cases, we can change the basis, such that both matrices M_1, M_2 have with respect to that basis some fixed form. Since there is no other subgroup of $SL(2, \mathbb{C})$ with two noncommuting matrices of order 10, this shows uniqueness. Since elements of this matrices are in very complicated form, we will not prove the existence this way. The proof of the can be found in Lamotke [1986] (page 35). □

Definition 1.1.17. *The cyclic group of order n is denoted by C_n. The group of type I or V of order $4n$ is called the binary dihedral group and is denoted by BD_{4n}. The group of type VI is called the binary tetrahedral group and is denoted by BT. The group of type III is called the binary octahedral group and denoted by BO. The group of type IV is called the binary icosahedral group and denoted by BI.*

The names of groups BT, BO and BI are derived from the names of the platonic solids, because the quotient group $G/\{\pm I\}$ is isomorphic to the rotation group of the corresponding solid. More about this also in Lamotke [1986].

1.2 Dynkin diagrams

Definition 1.2.1. *A multigraph is a triple (V, E, f), where V is a nonempty finite set, E is a finite set and f is a map assigning to every element of E a set of the form $\{u, v\}$ for some $u, v \in V$. The elements of the set V are called vertices, the elements of E edges. Note that we allow edges e for which $f(e) = \{v, v\}$, such edges are called loops. A submultigraph of a multigraph (V, E, f) is a multigraph (V', E', f') satisfying $V' \subseteq V$, $E' \subseteq E$, $f' = f|_{E'}$.*

Definition 1.2.2. *Two multigraphs (V, E, f) and (V', E', f') are isomorphic if there exists a bijection of vertices $\alpha : V \to V'$ and a bijection of edges $\beta : E \to E'$, which for all edges e, $f(e) = \{u, v\}$ satisfy $f'(\beta(e)) = \{\alpha(u), \alpha(v)\}$. In this thesis, isomorphic graphs will be considered as same.*

Definition 1.2.3. *A multigraph (V, E, f) is said to be connected if for any two vertices $u, v \in V$ there exists a walk from u to v, which means there exists a natural number n, vertices $w_1, \ldots, w_n$ and edges $e_1, \ldots, e_{n-1}$ such that $w_1 = u$, $w_n = v$ and $f(e_i) = \{w_i, w_{i+1}\}$.*

Definition 1.2.4. *An adjacency matrix of a graph (V, E, f) is a square matrix M of order $|V|$, whose rows and columns are indexed by vertices and whose elements are defined as $M_{u,v} = |\{e \in E | f(e) = \{u, v\}\}|$.*

Remark 1.2.5. *A multigraph is (up to isomorphism) uniquely determined by its adjacency matrix.*

Definition 1.2.6. *Let M be a real matrix of order n. Its spectral norm $||M||$ is defined as max $\left\{ \frac{||Mv||}{||v||} \;\middle|\; 0 \neq v \in \mathbb{R}^n \right\}$, where $||v||$ is the Euclidean norm of a vector v.*

Remark 1.2.7. *Since the value $\frac{||Mv||}{||v||}$ depends only on the direction of the vector and not the magnitude, the supremum of the values $\frac{||Mv||}{||v||}$ for nonzero vectors is the same as supremum for only the unit vectors. The set of all unit vectors is compact, therefore the maximum in the definition is attained.*

Definition 1.2.8. *A simply laced Dynkin diagram is a connected multigraph without loops, whose adjacency matrix has the spectral norm $||M|| < 2$. An extended simply laced Dynkin diagram is a connected multigraph without loops, whose adjacency matrix has the spectral norm $||M|| = 2$. In this thesis, we will call them only Dynkin diagrams, resp. extended Dynkin diagrams.*

Proposition 1.2.9. (Perron-Frobenius) *Let M be a symmetric matrix with nonnegative elements. Then there exists a nonzero eigenvector x_M with nonnegative components corresponding to an eigenvalue $||M||$.*

Proof. Since M is symmetric, the vector space $\mathbb{R}^n$ has an orthogonal basis $B = (b_1, \ldots, b_n)$ consisting of eigenvectors of M, $Mb_i = \lambda_i b_i$. WLOG suppose $|\lambda_1| \geq \cdots \leq |\lambda_n|$. For a vector $v = \sum a_i b_i$, $a_i \in \mathbb{R}$ then it holds $||Mv||^2 = ||\sum a_i \lambda_i b_i||^2 = \sum a_i^2 \lambda_i^2 \geq \sum a_i^2 \lambda_n^2 = \lambda_n^2 ||v||$. For the vector b_n it holds $||Mb_n||^2 = \lambda_n^2 ||b_n||$, therefore $|\lambda_n|$ is the spectral norm of M. Now define a vector x_M componentwise as the absolute value of b_n, i. e. if $b_n = (\beta_1, \ldots, \beta_n)^T$, then $x_M = |b_n| = (|\beta_1|, \ldots, |\beta_n|)^T$.

Then compomnentwise $Mx_M = M|b_n| \geq |Mb_n| = |\lambda_n b_n| = ||M|| \cdot |b_n| = ||M|| x_M$. But this inequality can't be strict in any component, otherwise would $||Mx_M||$ be greater than $||M|| \cdot ||x_M||$, which is a contradiction with the definition of the spectral norm. Therefore $Mx_M = ||M|| x_M$. □

Remark 1.2.10. *For a general matrix with nonnegative elements, the vector x_M does not have to be unique, we will denote by x_M any fixed vector with these properties.*

Corollary 1.2.11. *Let M be an adjacency matrix with nonnegative elements and x its eigenvector corresponding to an eigenvalue λ, which has only positive components. Then $||M|| = \lambda$.*

Proof. Since x_M is nonzero and has only nonnegative components, the value $x^T x_M$ is positive. Therefore $\lambda x^T x_M = (Mx)^T x_M = x^T M x_M = ||M|| x^T x_M$ implies $||M|| = \lambda$. □

Proposition 1.2.12. *Let $G = (V, E, f)$ be a connected multigraph, M its adjacency matrix. Then any nonzero eigenvector x_M with nonnegative components has all components strictly positive.*

Proof. The vector x_M is nonzero and has only nonnegative components, therefore it has some positive component. Suppose there is some vertex, such that the corresponding component of x_M is zero. Since G is connected, we can find an edge e, $f(e) = \{u, v\}$, such that the components $x_{M,u}$, $x_{M,v}$ of vector x_M satisfy $x_{M,u} > 0$ and $x_{M,v} = 0$. Since M and x_M are nonnegative, $(Mx_M)_v = \sum_{x \in V} M_{v,x} x_{M,x} \geq M_{v,u} x_{M,u} \geq x_{M,u} > 0$. But x_M is an eigenvector of M, which leads to a contradiction. □

Proposition 1.2.13. *Let $G = (V, E, f)$ be a connected multigraph and M its adjacency matrix. Then x_M is defined uniquely up to scaling.*

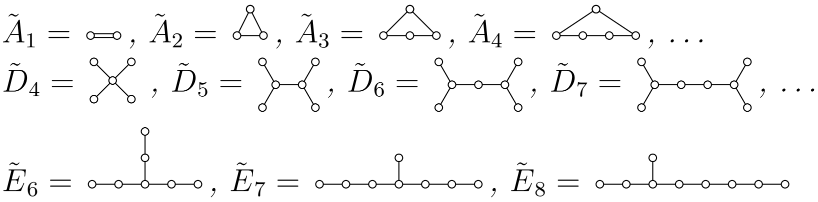
Proof. Suppose x_M and x'_M are two eigenvectors with strictly positive components. Since they correspond to the same eigenvalue, any linear combination of them is also an eigenvector. Let ε be the smallest real number, such that $x_M - \epsilon x'_M$ has a zero component. This vector is an eigenvector with nonnegative components and one of the components is zero, therefore from the previous Proposition it follows it is the zero vector and $x_M = \epsilon x'_M$. □

Proposition 1.2.14. *Let $G = (V, E, f)$ be a connected multigraph with adjacency matrix M and $G' = (V', E', f')$ its proper connected submultigraph with adjacency matrix M'. Then $||M|| > ||M'||$.*

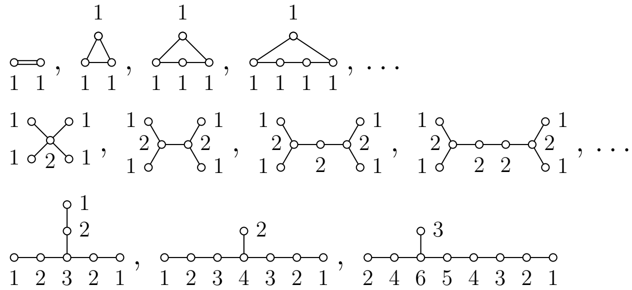
Proof. Define a vector $w \in \mathbb{R}^V$ as $w_u = x_{M',u}$ if $u \in V'$ and $w_u = 0$ otherwise. Define a matrix N of order $|V|$ as $N_{u,v} = M'_{u,v}$ if $u, v \in V'$ and 0 otherwise. Since G is connected and G' is its proper submultigraph, there must be an edge e, $f(e) = \{u, v\}$, such that at least one of the vertices u, v is in V', WLOG u. Then $(Mw)_v \geq (Nw)_v + x_{M',u} > ||M'|| w_v$ and for other vertices $i \in V$ it holds $(Mw)_i \geq (Nw)_i = ||M'|| w_i$, therefore $||Mw|| > ||M'|| \cdot ||w||$ and $||M|| > ||M'||$. □

Corollary 1.2.15. *An extended Dynkin diagram can't be a proper submultigraph of another extended Dynkin diagram.*

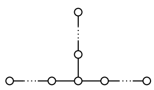
Theorem 1.2.16. *Every extended Dynkin diagram either is in one of the infinite families* $\{\tilde{A}_n | n \geq 1\}$, $\{\tilde{D}_n | n \geq 4\}$ *or is one of three graphs* $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$*:*



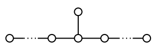
Proof. For each of the graphs on the list define a vector $x \in \mathbb{R}^V$ by assigning these values to vertices:



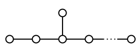
We can check that for each multigraph, the vector x is an eigenvector of the adjacency matrix M corresponding to the eigenvalue 2 (it can be seen from the picture since for every vertex, the sum of values of its neighbours is twice the value of the vertex, if one counts multiplicities in the multigraph $\tilde{A}_1$). From Corollary 1.2.11 then follows these multigraphs are extended Dynkin diagrams. Now suppose there exists an extended Dynkin diagram, which is not on the list. Since $\tilde{A}_1$ is not its submultigraph, it can't have multiple edges. Since none other $\tilde{A}_n$ is its submultigraph, it must be a tree. Since $\tilde{D}_4$ is not its submultigraph, its vertices must have a degree at most 3. Since none other $\tilde{D}_n$ is its submultigraph, it must have at most one vertex of degree 3. On the other hand, it must have a vertex of degree 3, otherwise, it would be a submultigraph of some $\tilde{A}_n$. Therefore it consists of one vertex of degree 3 and three "branches" coming from it:



Since $\tilde{E}_6$ is not its submultigraph, the length of one of these "branches" must be 1.

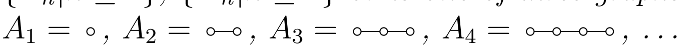


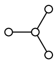
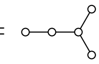
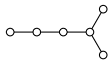
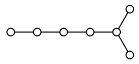
The other two must be longer than 1, otherwise, it would be a submultigraph of some $\tilde{D}_n$. On the other hand, one of them must be shorter than 3, otherwise, $\tilde{E}_7$ would be its multigraph.

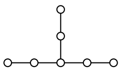
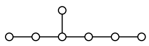
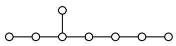


But then either it is a submultigraph of $\tilde{E}_8$ or $\tilde{E}_8$ is its submultigraph. $\square$

Corollary 1.2.17. *Every Dynkin diagram either is in one of the infinite families* $\{A_n | n \geq 1\}$, $\{D_n | n \geq 4\}$ *or is one of three graphs* E_6, E_7, E_8*:*



$D_4 =$  , $D_5 =$ , $D_6 =$ , $D_7 =$ , $\ldots$

$E_6 =$ , $E_7 =$ , $E_8 =$ 

Proof. In the proof of the theorem, we actually showed that every connected multigraph either is a submultigraph of an extended Dynkin diagram or contains an extended Dynkin diagram as a submultigraph. Therefore the Dynkin diagrams are exactly connected proper submultigraphs of the extended Dynkin diagrams. □

Remark 1.2.18. *The extended Dynkin diagrams are called extended because we get them when we "extend" Dynkin diagrams by one new vertex (in a suitable way). The letters A, D and E come from the classification of more general Dynkin diagrams, which uses the letters B, C, F and G for the not-simply-laced ones.*

2. The algebraic correspondence

2.1 Introduction to the representation theory of finite groups

Definition 2.1.1. *Let G be a finite group. A representation of G over a field $\mathbb{K}$ is a group homomorphism $\rho : G \to GL(n, \mathbb{K})$ for some $n \in \mathbb{N}$. The number n is called the dimension of the representation and denoted $dim(\rho)$. An invariant subspace of the representation is a vector subspace $V \subseteq \mathbb{K}^n$, such that for all $g \in G$ and $v \in V$ holds $\rho(g)v \in V$. A representation is said to be irreducible (or simple) if it does not have any invariant subspaces other than 0 and $\mathbb{K}^n$. It is said to be faithful if ρ is injective.*

Definition 2.1.2. *Let ρ_1, ρ_2 be two representations of G over a field $\mathbb{K}$. A homomorphism (resp. isomorphism) of vector spaces $f : \mathbb{K}^{dim(\rho_1)} \to \mathbb{K}^{dim(\rho_2)}$ is said to be a homomorphism (resp. isomorphism) of representations $\rho_1 \to \rho_2$ if for every $g \in G$ and $v \in \mathbb{K}^{dim(\rho_1)}$ it holds $f(\rho_1(g)v) = \rho_2(g)f(v)$.*

Definition 2.1.3. *Let ρ_1, ρ_2 be two representations of G over a field $\mathbb{K}$. The direct sum $\rho_1 \oplus \rho_2$ is defined as a representation of dimension $dim(\rho_1) + dim(\rho_2)$ assigning to $g \in G$ a block diagonal matrix $\begin{pmatrix} \rho_1(G) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$. The tensor product $\rho_1 \otimes \rho_2$ is defined using the tensor product (sometimes called Kronecker product) of matrices as a representation of dimension $dim(\rho_1) \cdot dim(\rho_2)$, $g \mapsto \rho_1(g) \otimes \rho_2(g)$. For iteration of these operations we will write $n\rho = \rho \oplus \cdots \oplus \rho$, $\rho^n = \rho \otimes \cdots \otimes \rho$.*

Proposition 2.1.4. *Up to isomorphism, the direct sum and the tensor product are both commutative and associative and the tensor product is distributive over the direct sums.*

Proof. The proof is done just by rewriting definitions and (in case of commutativities) permuting indices of rows and columns of the resulting matrices. □

Theorem 2.1.5. (Mashke) *Let G be a finite group and $\mathbb{K}$ be a field, such that $char(\mathbb{K})$ does not divide the order of G. Then every representation of G over $\mathbb{K}$ is isomorphic to a direct sum of irreducible representations. This decomposition is unique up to permutation and isomorphism of the irreducible representations.*

The proof can be found in Singh [2010] (Chapter 3).

Theorem 2.1.6. *Let G be a finite group, $\mathbb{K}$ algebraically closed with characteristic not dividing $|G|$. Then the number of irreducible representations of G over $\mathbb{K}$ (up to isomorphism) is the same as the number of conjugacy classes of G.*

The proof can be found in Singh [2010] (Chapter 11).

Theorem 2.1.7. *Let G be a finite group, $\mathbb{K}$ algebraically closed with characteristic not dividing $|G|$ and $\rho_1, \ldots, \rho_n$ be representatives of its irreducible representations. Then $\sum_{i=1}^{n} dim(\rho_i)^2 = |G|$.*

The proof can be found in Singh [2010] (Chapter 11).

Definition 2.1.8. *The character* χ_ρ *of a representation* ρ *is defined as a map* $G \to \mathbb{K}, g \mapsto tr(\rho(g))$*. The dual representation* ρ^* *is a representation of the same dimension defined as* $g \mapsto (\rho(g)^{-1})^T$

Proposition 2.1.9. *Let* ρ_1*,* ρ_2 *be two representations of a finite group* G *over the field* $\mathbb{K}$*. Then the characters have following properties:*

1. $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$
2. $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$
3. $\chi_{\rho_1^*}(g) = \chi_{\rho_1}(g^{-1})$
4. *If* $\mathbb{K} = \mathbb{C}$*,* $\chi_{\rho_1}(g^{-1}) = \overline{\chi_{\rho_1}(g)}$*.*
5. *If* $\mathbb{K} = \mathbb{C}$*,* $|\chi_{\rho_1}(g)| \leq dim(\rho_1)$*.*

Proof. The properties *1.* and *3.* follows straight from the definitions. For property *2.* suppose $\rho_1(g)$ has elements $d_1, \ldots, d_n$ on the diagonal and $\rho_2(g)$ has elements $e_1, \ldots, e_m$ on the diagonal. Then the diagonal of $(\rho_1 \otimes \rho_2)(g)$ consists of the elements of the form $d_i e_j$, from which it follows that $\chi_{\rho_1 \otimes \rho_2}(g) = \sum_{i,j} d_i e_j = \left(\sum_i d_i\right)\left(\sum_j e_j\right) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$. For the proof of property *4.* find a Jordan canonical form of $\rho_1(g)$. The elements on the diagonal must be complex numbers c_i of the absolute value 1, otherwise the matrix couldn't satisfy $\rho_1(g)^{|G|} = I$. Therefore $\rho_1(g^{-1}) = \rho_1(g)^{-1}$ has eigenvalues $c_i^{-1} = \overline{c_i}$ and $\chi_{\rho_1^*}(g) = \sum_i \overline{c_i} = \overline{\chi_{\rho_1}(g)}$. *5.* follows from triangle inequality and the fact that the eigenvalues of $\rho_1(g)$ have absolute value 1.

□

Theorem 2.1.10. *Let* G *be a finite group,* $\mathbb{K}$ *algebraically closed with characteristic not dividing* $|G|$ *and* $\rho_1, \ldots, \rho_n$ *the list of all simple representations up to isomorphism. Then* $\sum_{g \in G} \chi_{\rho_i}(g)\chi_{\rho_j}(g^{-1}) = \delta_{ij}|G|$*.*

The proof can be found in Singh [2010] (Chapter 10).

Corollary 2.1.11. *Let* $\mathbb{K}$ *be an algebraically closed field with characteristic zero,* $\rho_1, \ldots, \rho_n$ *be the list of irreducible representations up to isomorphism and* ρ *be any representation with decomposition* $\rho = \bigoplus a_i \rho_i$*. Then* $a_i = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g)\chi_\rho(g^{-1})$*.*

Proof. $\sum_{g \in G} \chi_{\rho_i}(g)\chi_\rho(g^{-1}) = \sum_j \sum_{g \in G} \chi_{\rho_i}(g) a_j \chi_{\rho_j}(g^{-1}) = \sum_j \delta_{ij} a_j |G| = a_i|G|$ □

Corollary 2.1.12. *Let* $\mathbb{K}$ *algebraically closed with characteristic zero. Then the character uniquely determines the representation (up to isomorphism).*

2.2 McKay graphs and the algebraic correspondence

In this section, all representations will be considered over $\mathbb{C}$.

Proposition 2.2.1. *Let G be a finite group, $\rho_1, \ldots, \rho_n$ be the list of its irreducible representations up to isomorphism and ρ be a representation satisfying $\rho^* \cong \rho$. For $i, j \in \{1, \ldots, n\}$ define numbers a_{ij} as $\rho_i \otimes \rho \cong \bigoplus_j a_{ij}\rho_j$. Then $a_{ij} = a_{ji}$.*

Proof. From Corollary 2.1.11 it follows that

$$a_{ij} = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_j}(g)\chi_{\rho\otimes\rho_i}(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_j}(g)\chi_{\rho}(g^{-1})\chi_{\rho_i}(g^{-1}) =$$

$$= \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_j}(g)\chi_{\rho}(g)\chi_{\rho_i}(g^{-1}) = \frac{1}{|G|}\sum_{h\in G}\chi_{\rho\otimes\rho_j}(h^{-1})\chi_{\rho_i}(h) = a_{ji},$$

where the middle equality on the second line is using a substitution $h := g^{-1}$. □

Definition 2.2.2. *Let G be a finite group. A self-dual representation is a representation satisfying $\rho \cong \rho^*$. The McKay graph of G and self-dual ρ is a multigraph, whose vertices are the irreducible representation of G and its adjacency matrix consists of the coefficients a_{ij}.*

Proposition 2.2.3. *Let (V, E, f) be a McKay graph of a group G and a self-dual representation ρ, let M be its adjacency matrix. Then $||M|| = dim(\rho)$.*

Proof. Define a vector $x \in \mathbb{R}^V$ componentwise as $x_i = dim(\rho_i)$. For any $i \in \{1, \ldots, n\}$ from $\rho_i \otimes \rho \cong \bigoplus_j a_{ij}\rho_j$ it follows that $dim(\rho_i)\cdot dim(\rho) = \sum_j a_{ij} dim(\rho_j)$, therefore $x_i dim(\rho) = \sum_j a_{ij} dim(\rho_j)$ and therefore $dim(\rho)x = Mx$. The statement then follows from Corollary 1.2.11. □

Definition 2.2.4. *Let G be a finite group. Its trivial representation ρ_{triv} is the constant homomorphism $G \to GL(1, \mathbb{C}), g \mapsto 1$. Let G be a finite subgroup of $SL(2, \mathbb{C})$. Its canonical representation is the embedding $G \to GL(2, \mathbb{C}), g \mapsto g$.*

Theorem 2.2.5. (Burnside) *Let G be a finite group, ρ a faithful representation of G and ρ_i an irreducible representation. Then there exists $n_i \in \mathbb{N}$ such that ρ_i is contained in the decomposition of ρ^{n_i} into irreducible representations.*

Proof. In other words, we want to show that $\sum_{g\in G}\chi_{\rho_i}(g)\left(\overline{\chi_\rho(g)}\right)^{n_i} > 0$ for some $n_i \in \mathbb{N}$. Lets assume the converse, $\forall n \in \mathbb{N} : \sum_{g\in G}\chi_{\rho_i}(g)\left(\overline{\chi_\rho(g)}\right)^{n} = 0$. Since all group elements are of finite order, the matrices $\rho(g)$ are diagonalizable and have eigenvalues of the form $e^{\frac{2k\pi i}{|G|}}$, therefore $|\chi_\rho(g)| \leq dim(\rho)$. Let $\gamma = \{g \in G | dim(\rho) = |\chi_\rho(g)|\}$. Then for $g \in \gamma$, $\rho(g)$ must have only one eigenvalue, otherwise the triangle inequality $|\lambda_1 + \cdots + \lambda_{dim(\rho)}| \leq 1 + \cdots + 1 = dim(\rho)$ would be strict. Therefore $\rho(g) = \lambda_g I$ for some $\lambda_g = e^{\frac{2k\pi i}{|G|}}$. From ρ being faithful it follows that for distinct elements of γ are the numbers λ_g distinct. Let us now choose $k \in \mathbb{Z}$ and define a sequence of functions $f_{k,n} : G \to \mathbb{C}, g \mapsto \frac{\chi_\rho(g)^{|G|n+k}}{dim(\rho)^{|G|n+k}}$. As $n \to \infty$, this sequence is constant at $g \in \gamma$ and decreases exponentially (in the absolute value) at $g \notin \gamma$, therefore it has a pointwise limit f_k, which has the value λ_g^k at $g \in \gamma$ and 0 otherwise. When we view maps $G \to \mathbb{C}$ as vectors in $\mathbb{C}^G$, the assumption $\sum_{g\in G}\chi_{\rho_i}(g)\left(\overline{\chi_\rho(g)}\right)^{n} = 0$ tells us that $f_{k,n}$ and χ_{ρ_i} are orthogonal. From continuity of the dot product it follows that f_k and χ_{ρ_i} are orthogonal. For a vector $v \in \mathbb{C}^G$ define its restriction $\tilde{v} \in \mathbb{C}^\gamma$ by omitting the components outside γ. Since f_k are zero outside γ, it means that $\tilde{f}_k$ and $\tilde{\chi}_{\rho_i}$ are orthogonal. If we name the elements of γ as $g_1, \dots, g_{|\gamma|}$, we can combine the equations for $k \in \{0, \dots, |\gamma| - 1\}$ into one matrix equation.

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_{g_1} & \lambda_{g_2} & \lambda_{g_3} & \dots & \lambda_{g_{|\gamma|}} \\ \lambda_{g_1}^2 & \lambda_{g_2}^2 & \lambda_{g_3}^2 & \dots & \lambda_{g_{|\gamma|}}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{g_1}^{|\gamma|-1} & \lambda_{g_2}^{|\gamma|-1} & \lambda_{g_3}^{|\gamma|-1} & \dots & \lambda_{g_{|\gamma|}}^{|\gamma|-1} \end{pmatrix} \cdot \overline{\begin{pmatrix} \chi_{\rho_i}(g_1) \\ \chi_{\rho_i}(g_2) \\ \chi_{\rho_i}(g_3) \\ \vdots \\ \chi_{\rho_i}(g_{|\gamma|}) \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In the matrix on the left we recognise the Vandermonde matrix, which is regular, since λ_{g_i} are pairwise. This means that χ_{ρ_i} is zero on γ, but since the neutral element e is in γ and $\chi_{\rho_i}(e) = dim(\rho_i)$, this is a contradiction.

□

Corollary 2.2.6. *Let G be a finite group and ρ a self-dual faithful representation. Then the McKay graph of G and ρ is connected.*

Proof. By induction, we will prove that if the decomposition of ρ^n contains irreducible $\tilde{\rho}$, then there exist a walk from ρ_{triv} to $\tilde{\rho}$. For $n = 1$ the statement holds, since if $\tilde{\rho}$ is contained in $\rho = \rho \otimes \rho_{triv}$, it is a neighbor of ρ_{triv} by definition of the McKay graph. Suppose the statement holds for n and $\tilde{\rho}$ is contained in ρ^{n+1}. When we decompose ρ^n as $\bigoplus a_i \rho_i$, we learn that $\tilde{\rho}$ is contained in $\bigoplus a_i \rho_i \otimes \rho$. From uniqueness of the decomposition it must be contained in some $\rho_i \otimes \rho$ such that $a_i > 0$. When we take a walk from ρ_{triv} to ρ_i and extend it by a vertex $\tilde{\rho}$ (and some suitable edge between ρ_i and $\tilde{\rho}$), we get a walk to $\tilde{\rho}$. Now, to get a walk between general vertices u, v, one can take a walk from u to ρ_{triv} and follow it by a walk from ρ_{triv} to v. □

Proposition 2.2.7. *Let G be a finite group, ρ be an irreducible representation of G and g be an element of the center of G. Then $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. We know that $\rho(g)$ is diagonalizable, let λ be an eigenvalue and V the corresponding eigenspace. We want to show that V is an invariant subspace of ρ. Let $v \in V$, $h \in G$. Then $\rho(g)\rho(h)v = \rho(h)\rho(g)v = \lambda\rho(h)v$, therefore $\rho(h)v \in V$. Since ρ is irreducible, this means $V = \mathbb{C}^{dim(\rho)}$. □

Corollary 2.2.8. *Let G be a finite abelian group. Then all of its irreducible representations are one-dimensional.*

Proof. Since all matrices $\rho(g)$ are of the form λI, any subspace of $\mathbb{C}^{dim(\rho)}$ is invariant. □

Proposition 2.2.9. *Let G be a finite group with a nontrivial center, ρ_i an irreducible representation and ρ_j a faithful irreducible representation. Then ρ_i does not appear in the decomposition of $\rho_j \otimes \rho_i$.*

Proof. Suppose it does. Let $g \neq e$ be an element of the center. From Proposition 2.2.7 it follows that for any irreducible ρ_k, $\chi_{\rho_k}(g) = \lambda_k dim(\rho_k)$ for some complex unit λ_k. Let $\bigoplus a_k \rho_k$ be the decomposition of $\rho_j \otimes \rho_i$. When we apply the absolute value to the equality

$$\lambda_i \lambda_j \sum a_k dim(\rho_k) = \lambda_i \lambda_j \sum a_k \chi_{\rho_k}(e) = \lambda_i \lambda_j \chi_{\rho_j \otimes \rho_i}(e) =$$

$$= \lambda_i \lambda_j dim(\rho_j) \cdot dim(\rho_i) = \chi_{\rho_j \otimes \rho_i}(g) = \sum a_k \chi_{\rho_k}(g) = \sum a_k \lambda_k dim(\rho_k),$$

we get $\sum a_k \dim(\rho_k) = |\sum a_k \lambda_k \dim(\rho_k)|$. On the other hand, the statement $\sum a_k |\lambda_k| \dim(\rho_k) \leq |\sum a_k \lambda_k \dim(\rho_k)|$ follows from the triangle equality, which is an equality if and only if all of the nonzero summands have the same argument $arg(z) = \frac{z}{|z|}$. This means λ_k is the same for all k satisfying $a_k > 0$. Since there exists a loop from ρ_i to ρ_i, we know that $a_i > 0$, therefore

$$\lambda_i \lambda_j \sum a_k dim(\rho_k) = \sum a_k \lambda_k dim(\rho_k).$$

We can cancel out $\lambda_i \sum a_k dim(\rho_k)$ in and get $\lambda_j = 1$, which is in contradiction with ρ_j being faithful. □

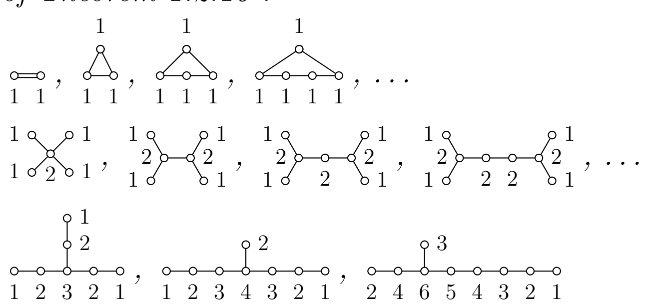
Proposition 2.2.10. *Let G be a nontrivial finite subgroup of $SL(2, \mathbb{C})$ and ρ be its canonical representation. Then ρ is self-dual and the McKay graph of G and ρ has no loops.*

Proof. Since for every $g \in G$ the eigenvalues of $\rho(g)$ are complex conjugates of each other, the character χ_ρ is real-valued. Therefore $\chi_{\rho^*} = \chi_\rho$ and from Corollary 2.1.12 $\rho \cong \rho^*$. If ρ is irreducible, then the proof follows from Proposition 2.2.9 (all nontrivial finite subgroups of $SL(2, \mathbb{C})$ have a nontrivial center). If not, it decomposes into two one-dimensional representations ρ_1, ρ_2. Since eigenvalues of matrices in $SL(2, \mathbb{C})$ are inverse to each other, ρ_1 and ρ_2 must be dual. From this follows that both ρ_1 and ρ_2 are faithful, since if $g \in Ker(\rho_1)$, then $g \in Ker(\rho_2)$ and $g \in Ker(\rho_1 \oplus \rho_2)$, which contradicts the fact that canonical representation is by definition faithful. If ρ_i is now any irreducible representation, it does not appear in the decomposition of $(\rho_1 \otimes \rho_i) \oplus (\rho_2 \otimes \rho_i) \cong \rho \otimes \rho_i$, which is what we wanted to prove. □

Corollary 2.2.11. *Let G be a nontrivial finite subgroup of $SL(2, \mathbb{C})$ and ρ its canonical representation. Then the McKay graph of G and ρ is an extended Dynkin diagram.*

Proof. Follows from Proposition 2.2.3, Corollary 2.2.6 and Proposition 2.2.10 □

Proposition 2.2.12. *Let G be a nontrivial finite subgroup of $SL(2, \mathbb{C})$. Then the dimension vector x used in the proof of Proposition 2.2.3 is the same as the eigenvector x_M for the corresponding extended Dynkin diagram used in the proof of Theorem 1.2.16 :*



Proof. Both of these vectors are eigenvectors of the adjacency matrix with positive components, from Proposition 1.2.13 follows $x = \alpha x_M$ for some $\alpha \in \mathbb{R}$. Since components of x are positive integers, $\alpha \in \mathbb{N}$. Since the trivial representation is an irreducible representation with dimension 1, $\alpha \leq 1$. $\square$

Theorem 2.2.13. *The extended Dynkin diagrams corresponding to nontrivial finite subgroups of* $SL(2, \mathbb{C})$ *are:*

Group:	*Diagram:*
C_n	$\tilde{A}_{n-1}$
BD_{4n}	$\tilde{D}_{n+2}$
BT	$\tilde{E}_6$
BO	$\tilde{E}_7$
BI	$\tilde{E}_8$

Proof. From *Theorem* 2.1.6 the diagram determines the number of conjugacy classes, from Proposition 2.2.12 and Theorem 2.1.7 it determines the order of the group:

Diagram:	$\lvert G\rvert$	Conjugacy classes
$\tilde{A}_{n-1}$	n	n
$\tilde{D}_{n+2}$	$4n$	$n+3$
$\tilde{E}_6$	24	7
$\tilde{E}_7$	48	8
$\tilde{E}_8$	120	9

Therefore for a given nontrivial finite subgroup of $SL(2, \mathbb{C})$, the diagram in the statement of the Theorem is the only possible one. $\square$

3. The geometric correspondence

3.1 Introduction to algebraic geometry

In this chapter, the reader is expected to know the basics of commutative algebra and algebraic geometry. All algebras in this chapter are considered associative, commutative and unital and over the field $\mathbb{C}$.

Definition 3.1.1. *By the n-dimensional affine space $\mathbb{A}^n$ we mean $\mathbb{C}^n$ as a set (later equipped with a topology). Let $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$. The common zero set of S is the set $V(S) := \{a \in \mathbb{A}^n | \forall f \in S : f(a) = 0\}$. Subsets of $\mathbb{A}^n$ of the form $V(S)$ are called affine algebraic sets. Let $X \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Then the ideal of X is the ideal $I(X) := \{f \in \mathbb{A}[x_1, \ldots, x_n] | \forall x \in X : f(x) = 0\}$.*

Proposition 3.1.2. *The intersections and finite unions of algebraic sets are algebraic sets. The empty set and the set $\mathbb{A}^n$ are algebraic sets. The complements of algebraic sets therefore form a topology on $\mathbb{A}^n$.*

Proof. Straight from the definition follows $V(\bigcup S_i) = \bigcap V(S_i)$ and $V(S_1 S_2) = V(S_1) \cup V(S_2)$ (where $S_1 S_2 = \{st | s \in S_1, t \in S_2\}$). The empty set is equal to $V(1)$ and the set $\mathbb{C}^n$ is equal to $V(0)$. □

Definition 3.1.3. *The topology from the previous Proposition and the induced topologies on subsets of $\mathbb{C}^n$ are called the Zariski topology. Unless stated otherwise, open and closed sets will be from now on considered with respect to the Zariski topology.*

Definition 3.1.4. *Let (X, τ) be a topological space. If for any two closed subsets X_1, X_2 satisfying $X_1 \cup X_2 = X$ holds $X_1 = X$ or $X_2 = X$, X is said to be irreducible (alternatively, if every nonempty open of X subset is dense in X). A subset of X is said to be irreducible if it is irreducible as a topological space with the induced topology. Nonempty irreducible affine algebraic sets are called affine varieties.*

Definition 3.1.5. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. The coordinate algebra of X is the algebra $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(X)$ (which can be considered as an algebra of polynomial functions $X \to \mathbb{C}$).*

Definition 3.1.6. *Let $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ be affine algebraic sets. A map $X \to Y$, $(x_1, \ldots, x_m) \mapsto (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m))$ is a morphism of varieties if the functions $f_1, \ldots, f_n$ are polynomial.*

Example. *As a set, the coordinate algebra of X is therefore the set of all morphisms $X \to \mathbb{A}^n$.*

Theorem 3.1.7. *The coordinate algebra of an affine algebraic set is finitely generated and does not contain any nontrivial nilpotent elements. This assignment between the category of affine algebraic sets and the category of finitely algebras without nontrivial nilpotents is a contravariant equivalence of categories, which maps the morphism of varieties $f : X \to Y$ to a morphism of the coordinate algebras $\tilde{f} : \mathbb{C}[Y] \to \mathbb{C}[X]$, $g \mapsto g \circ f$.*

Proof. The coordinate algebra is finitely generated, because it is a quotient algebra of finitely generated $\mathbb{C}[x_1, \ldots, x_n]$. It does not contain nontrivial nilpotents, since it consists of valued functions with operations defined pointwise. The proof of equivalence of categories can be found in Wallach [2017] (section 1.2.1) or Smith et al. [2000] (section 2.5). Both of these books differ in the terminology, their definition of an affine variety correspond to our definition of an affine algebraic set. In the book Wallach [2017], the symbol $\mathcal{O}(Z)$ stands for the local ring of a set $Z \subseteq \mathbb{A}^n$, which for closed sets is the same as the coordinate algebra. □

3.2 Geometric invariant theory

Definition 3.2.1. *Let G be a finite group and X be an affine variety. By a group action on X we mean a group homomorphism $\varphi : G \to Aut(X)$. The equivalence of the categories lets us view this as a group homomorphism $\tilde{\varphi} : G \to Aut(\mathbb{C}[X])$, $\tilde{\varphi}_g(f) = f \circ \varphi_{g^{-1}}$. The algebra of invariant polynomials is the algebra $\mathbb{C}[X]^G = \{f \in \mathbb{C}[X] | \forall g \in G : \ \tilde{\varphi}_g(f) = f\}$. If this algebra is finitely generated, we denote the corresponding affine variety $X//G$ and call it the GIT-quotient of X. The equivalence of categories then lets us view the inclusion $i : \mathbb{C}[X]^G \to \mathbb{C}[X]$ as a morphism $\pi : X \to X//G$.*

Theorem 3.2.2. *Let $G \subseteq GL(n, \mathbb{C})$ be a finite group acting on $\mathbb{A}^n$ by multiplying the coordinate vector:*

$$\varphi_g \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = g \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then $\mathbb{C}[X]^G$ is finitely generated. Moreover, the ideal $I \subseteq \mathbb{C}[X]$ generated by all homogeneous invariant polynomials is generated by finitely many homogeneous invariant polynomials and these polynomials also generate $\mathbb{C}[X]^G$.

Proof. The proof can be found in $Wallach$ [2017] (section 3.1.4). □

Proposition 3.2.3. *Let G be the group $C_n < SL(2, \mathbb{C})$. Then the affine variety $\mathbb{A}^2//G$ is isomorphic to $V(ab - c^n) \in \mathbb{A}^3$.*

Proof. WLOG assume $C_n = \langle M \rangle$, where $M = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix}$. A polynomial is invariant, if and only if it is invariant under the action of the generators of the group. Since M is diagonal, its action on the monomials changes only the coeficient of the monomial. A polynomial is therefore invariant if and only if all of its monomials are invariant. The matrix M sends $x^k y^l$ to $(e^{-\frac{2\pi i}{n}} x)^k (e^{\frac{2\pi i}{n}})^l = x^k y^l e^{\frac{2\pi i(l-k)}{n}}$, therefore the monomial is invariant if and only if $k \equiv l \pmod{n}$. The ideal I from the previous Theorem is therefore (x^n, y^n, xy) (since these polynomials are invariant and any nonconstant invariant polynomial is divisible by at least one of them). Let $\psi : \mathbb{C}[a, b, c] \to \mathbb{C}[X]^G$ be a homomorphism defined by $\psi(a) = x^n$, $\psi(b) = y^n$, $\psi(c) = xy$. Since the generators of $\mathbb{C}[X]^G$ are in the image, the homomorphism is surjective, therefore $\mathbb{C}[X]^G \cong \frac{\mathbb{C}[a,b,c]}{Ker(\psi)}$. Clearly $ab - c^n \in Ker(\psi)$, therefore $\mathbb{A}//G$ is some subvariety of $V(ab - c^n)$. The projection morphism is defined as $(x, y) \mapsto (x^n, y^n, xy)$. If for $(a, b, c) \in V(ab - c^n)$ holds $a \neq 0$, then

$\pi(\sqrt[n]{a}, \frac{c}{\sqrt[n]{a}}) = (a, b, c)$. If $a = 0$, then $\pi(0, \sqrt[n]{b}) = (a, b, c)$. Therefore π is surjective, therefore $Ker(\psi) = V(ab - c^n)$ and $\mathbb{A}//G$ is the whole affine variety $V(ab - c^n)$. □

3.3 Singularities

Definition 3.3.1. *Let*

Some difficulties arise when one is working with singular varieties. To avoid these problems, we may want to find some different variety, which is nonsingular, but still carries some information about the original variety. It can also help us better understand the singularities. The simplest method of finding these resolutions is blowing up points. However, the result of it usually isn't an affine variety (even if the original variety is), but a more general object. This object can be embedded into the projective space, however, we will follow a more abstract approach and think of it as a topological space (with some additional structure), which is covered by affine varieties, analogously to the way how differential varieties are covered by Euclidianly open subsets of $\mathbb{R}^n$. We will call these objects abstract varieties or just varieties. The precise definition can be found in Wallach [2017] (section 1.4.3)

Definition 3.3.2. *Let X be a variety. A resolution of singularities is a nonsingular variety Y together with a morphism $f : Y \to X$, such that:*

- *The morphism f is proper, i.e. if we equip X and Y with Euclidean topology, the preimage of a compact set is compact.*
- *When restricted, f is an isomorphism of varieties $f^{-1}(X^{reg}) \to X^{reg}$, where X^{reg} is the open subvariety of X consisting of all regular points.*

An exceptional locus of the resolution is the set $f^{-1}(X^{sing})$, where X^{sing} is the set of all singular points $X \setminus X^{reg}$.

Remark 3.3.3. *If $X \subseteq \mathbb{A}^n$ is an affine variety, it can be equipped with the Euclidean topology induced from the by the inclusion $X \to \mathbb{C}^n$. For abstract varieties, the topology can be constructed locally the same way. More about this in Neeman [2007] (chapter 4).*

Definition 3.3.4. *Let p be a point of an affine variety $X \subseteq \mathbb{A}^n$. The blowup of X at the point p is the closure of*

$$\Gamma = \{(x, y) \in (X \setminus \{p\}) \times \mathbb{P}^{n-1} | [x_1 - p_1 : \ \dots \ : x_n - p_n] = [y_1 : \ \dots \ : y_n]\}$$

in $X \times \mathbb{P}^{n-1}$. This variety is denoted $Bl_p(X)$ and there is a canonical morphism $\pi : Bl_p(X) \to X$, $(x, y) \mapsto x$. Again, the set $\pi^{-1}(\{p\})$ is called the exceptional locus.

Remark 3.3.5. *A blowup can be defined even for abstract varieties as in Hauser [2014]. For our purposes suffices to know that the blowup of an abstract variety, looks on the affine cover as the blowup of affine variety if the covering affine variety contains the point p and is an isomorphism otherwise.*

Example. *Let* $X = V(x_1^3 + x_1^2 - x_2^2) \subseteq \mathbb{A}^2$ *and* $p = (0,0)$. *We can cover the projective space by two open subsets, on which* $y_1 \neq 0$, *resp.* $y_2 \neq 0$. *Since the closure commutes with finite unions, we can use the closures of the sets in the open cover to find the closure of* Γ. *The first can be parameterized by* x_1, x_2 *and* $\tilde{x}_2$, *where* $[y_1 : y_2] = [1 : \tilde{x}_2]$. *The corresponding subset of* Γ *is therefore parameterized by*

$$\{(x_1, x_2, \tilde{x}_2) \in \mathbb{A}^3 | x_1 \neq 0,\ x_1^3 + x_1^2 - x_2^2 = 0,\ x_2 = x_1\tilde{x}_2\} =$$
$$= \{(x_1, x_2, \tilde{x}_2) \in \mathbb{A}^3 | x_1 \neq 0,\ x_1 + 1 - \tilde{x}_2^2 = 0,\ x_2 = x_1\tilde{x}_2\}.$$

By taking the closure of this set in $\{(x, y) \in X \times \mathbb{P}^{n-1} | y_1 \neq 0\}$, *we remove the condition* $x_1 \neq 0$ *and therefore get two new points points* $(x_1, x_2, \tilde{x}_2) = (0, 0, \pm 1)$. *Since* x_2 *can be expressed from* x_1 *and* $\tilde{x}_2$, *it is not needed for the parameterization and the covering affine variety is isomorphic to* $V(x_1 + 1 - \tilde{x}_2^2) \subseteq \mathbb{A}^2$.
We could similarly parameterize the open set $y_2 \neq 0$, *however, in this particular example it is not necessary, since the first parameterization covers the whole* Γ.
The morphism $\pi : Bl_p(X) \to X$ *is given by* $(x_1, \tilde{x}_2) \mapsto (x_1, x_2) = (x_1, x_1\tilde{x}_2)$

This process can be generalized to an algorithm for finding affine cover of any blowup of hyperplane, i.e. any blowup $Bl_p(X)$ of a variety of the form $X = V(f) \subseteq \mathbb{A}^n$.

- cover $\mathbb{P}^{n-1}$ by n open subsets U_i defined by $y_i \neq 0$.
- parametrize every $X \times U_i$ by $x_1, \dots, x_n, x_{1,i}, \dots, x_{i-1,i}, x_{i+1,i}, \dots, x_{n,i}$, where $[y_1 : \dots : y_n] = [x_1 - p_1 : \dots : x_{i-1} - p_{i-1} : 1 : x_{i+1} - p_{i+1} : \dots, x_n - p_n]$.
- express the coordinates $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ in f using the coordinates $x_{1,i}, \dots, x_{i-1,i}, x_i, x_{i+1,i} \dots, \dots, x_{n,i}$.
- divide the transformed $f \in \mathbb{C}[x_{1,i}, \dots, x_{i-1,i}, x_i, x_{i+1,i} \dots, \dots, x_{n,i}]$ by the highest possible power of $x_i - p_i$
- find the closure in $X \times U_i$, i.e. add the points with $x_i = p_i$
- on the intersections of multiple maps, find the transition map

Proposition 3.3.6. *The restriction of the blowup morphism is an isomorphism of the varieties* $\pi^{-1}(X \setminus \{p\}) \to X \setminus \{p\}$

Proof. The set $\Gamma \cup (p \times \mathbb{P}^{n-1})$ is closed, therefore this restriction of π is injective. The map

$$X \setminus \{p\} \to (X \setminus \{p\}) \times \mathbb{P}^{n-1}, (x_1, \dots, x_n) \mapsto (x_1 \dots, x_n, [x_1 - p_1 : \ \dots \ : x_n - p_n])$$

is the inverse morphism. □

Proposition 3.3.7. *The blowup morphism is proper.*

Proof. Let $C \subseteq X$ be a compact subset of X (with respect to Euclidean topology) Since X is covered by affine varieties, we can find a compact neighborhood U of $p \in X$. Let $V = \overline{X \setminus U}$. Since on $(X \setminus \{p\}) \times \mathbb{P}^{n-1}$ is π an isomorphism, $V \cap C$ is homeomorphic to $\pi^{-1}(V \cap C)$, therefore $\pi^{-1}(C \cap V)$ is compact. Since $\mathbb{P}^{n-1}$ is compact (Wallach [2017], Lemma 1.29), the set $U \times \mathbb{P}^{n-1}$, therefore the closed subset $\pi^{-1}(U \cap C)$ is also compact. Therefore $\pi^{-1}(C) = \pi^{-1}(U \cap C) \cup \pi^{-1}(V \cap C)$ is compact. □

Corollary 3.3.8. *We can construct a sequence $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$, in which X_{k+1} is a blowup of X_k at some singular point. If X_n is a nonsingular variety, then the composition of the blowup morphisms $X_n \to X$ is a resolution of singularities.*

Theorem 3.3.9. *Let G be a nontrivial finite subgroup of $SL(2, \mathbb{C})$ and $X = \mathbb{A}^2//G$. Then we can get a resolution of X by consequent blowups at singular points. If we then construct a multigraph, whose vertices are the irreducible components of the exceptional locus and the number of edges between components U and V is $|U \cap V|$, we get a non-extended version of the extended Dynkin diagram from Theorem 2.2.13.*

Proof. We will show this for each group separately.

$\mathbf{C_2}$**:** The variety $X = \mathbb{A}^2//G$ is isomorphic to $V(xy - z^2)$. The only singular point is the point $p = (0, 0, 0)$. The blowup is covered by affine varieties $\overline{U_i}$. Using the algorithm at the beginning of the section, we get the affine cover of $Bl_p(X)$:

- $Bl_p(X) \cap (X \times U_x)$ is covered by $V(\tilde{y} - \tilde{z}^2)$ with the morphism $\pi_x : (x, \tilde{y}, \tilde{z}) \mapsto (x, \tilde{y}x, \tilde{z}x, [1 : \tilde{y} : \tilde{z}])$.
- $Bl_p(X) \cap (X \times U_y)$ is covered by $V(\tilde{x} - \tilde{z}^2)$ with the morphism $\pi_y : (\tilde{x}, y, \tilde{z}) \mapsto (\tilde{x}y, y, \tilde{z}y, [\tilde{x} : 1 : \tilde{z}])$.
- $Bl_p(X) \cap (X \times U_y)$ is covered by $V(\tilde{x}\tilde{y} - 1)$ with the morphism $\pi_z : (\tilde{x}, \tilde{y}, z) \mapsto (\tilde{x}z, \tilde{y}z, z, [\tilde{x} : \tilde{y} : 1])$.

All three varieties $V(\tilde{y} - \tilde{z}^2)$, $V(\tilde{x} - \tilde{z}^2)$ and $V(\tilde{x}\tilde{y} - 1)$ are nonsingular, therefore $Bl_p(X)$ is nonsingular, therefore it is a resolution of singularities. We will now study the exceptional locus, i.e. $\pi^{-1}(p)$. In the first set of the affine cover, the preimage of $p = (0, 0, 0)$ consist of the points parameterized by $(x, \tilde{y}, \tilde{z}) = (0, \tilde{z}^2, \tilde{z})$, which is an irreducible set. Similarly for U_y, $\pi^{-1}(0, 0, 0)$ is parameterized by $y = 0$ and $\tilde{x} = \tilde{z}^2$ and for U_z it is parameterized by $z = 0$ and $\tilde{x}\tilde{y} = 1$. Again, these are irreducible sets. On the intersection of the covering sets, we can use the transition maps to find out which of these irreducible subsets are in the same irreducible component of $\pi^{-1}(0)$. For example, for $t \neq 0$, the point parameterized by π_x as $(x, \tilde{y}, \tilde{z}) = (0, t^2, t)$ is the point $(p, [1 : t^2 : t])$ in $X \times \mathbb{P}^2$, which is the same point as $(p, [\frac{1}{t^2} : 1 : \frac{1}{t}])$ parameterized by π_y as $(\tilde{x}, y, \tilde{z}) = (\frac{1}{t^2}, 0, \frac{1}{t})$, which is the same point as $(p, [\frac{1}{t} : t : 1])$ parameterized by π_z as $(\tilde{x}, \tilde{y}, z) = (\frac{1}{t}, t, 0)$. Since these sets are all dense in their respective $\pi_i^{-1}(p)$, we can conclude that they form a single irreducible component of the exceptional locus.

$\mathbf{C_3}$**:** The variety $\mathbb{A}^2//G$ is isomorphic to $V(xy - z^3)$. The blowup is covered by nonsingular affine varieties $\tilde{y} - \tilde{z}^3x$ (for U_x), $\tilde{x} - \tilde{z}^3y$ (for U_y) and $\tilde{x}\tilde{y} - z$ (for U_z). The respective parametrizations of the subsets of exceptional locus are $V(x, \tilde{y})$ (for U_x), $V(y, \tilde{x})$ (for U_y) and $V(z, \tilde{x}\tilde{y})$. Notice that the first two affine algebraic subsets are irreducible, but the third one decomposes as $V(z, \tilde{x}) \cup V(z, \tilde{y})$. After applying the maps π_i, we can see that the exceptional locus is the set of points $\{(p, [x' : y' : z'])|x'y' = 0\}$, which decomposes into two irreducible components $\{(p, [x' : y' : z'])|x' = 0\} \cup \{(p, [x' : y' : z'])|y' = 0\}$. We can see that these two components intersect in one point $(p, [0 : 0 : 1])$.

$\mathbf{C_n}$: We will firstly prove by induction on n that the singularities of $X = \mathbb{A}^2//G$ can be resolved via blowups. For $n = 2$ and $n = 3$ the statement holds, assume $n \geq 4$. The variety $\mathbb{A}^2//G$ is isomorphic to $V(xy - z^n)$. The blowup is covered by nonsingular affine varieties $\tilde{y} - \tilde{z}^3 x^{n-2}$ (for U_x), $\tilde{x} - \tilde{z}^3 y^{n-2}$ (for U_y) and a variety $\tilde{x}\tilde{y} - z^{n-2}$ (for U_z), which is singular at $(0, 0, 0)$. Analogically to C_3, the exceptional locus decomposes as

$$\{(p, [x' : y' : z'])|x'y' = 0\} = \{(p, [x' : y' : z'])|x' = 0\} \cup \{(p, [x' : y' : z'])|y' = 0\}$$

with the intersection point $q = (p, [0 : 0 : 1])$. However, the point q is singular, therefore $Bl_p(X)$ is not a resolution of singularities. Therefore we need to construct another blowup $Bl_q(Bl_p(X))$. Luckily for us, in the affine cover of $Bl_p(X)$, the singularity is covered by the affine variety $\mathbb{A}^2//C_{n-2}$, therefore we already know we can resolve the singularity at q by blowups. Let $f : Y \to Bl_p(X)$ be the resolution. Then Y with the composition of morphisms $\pi \circ f$ (where π is the blowup morphism $Bl_p(X) \to X$) is a resolution of singularities of X. Firstly, the exceptional locus of this resolution is the set $f^{-1}(\{(p, [x' : y' : z'])|x'y' = 0\})$. If we remove the point q from the set $\{(p, [x' : y' : z'])|x'y' = 0\}$, the resulting set lie in $Bl_p(X)^{reg}$, therefore is isomorphic to its preimage $f^{-1}(\{(p, [x' : y' : z'])|x'y' = 0\} \setminus \{q\})$. Denote $l_x = \{(p, [x' : y' : z'])|x' = 0\}$, $l_x = \{(p, [x' : y' : z'])|y' = 0\}$. Two irreducible components of the exceptional locus are therefore the closures of $f^{-1}(l_x \setminus \{q\})$ and $f^{-1}(l_y \setminus \{q\})$ in Y.

We will now prove by induction on n that the Theorem 3.3.9 holds for $G = C_n$ and that the components $\overline{f^{-1}(l_x \setminus \{q\})}$, $\overline{f^{-1}(l_y \setminus \{q\})}$ correspond to the left-most and right-most vertex of the Dynkin diagram. If we want to prove this, we already know from the induction hypothesis how does the exceptional locus of $Y \to Q$ look like, therefore we only need to show that $\overline{f^{-1}(l_x \setminus \{q\})}$ and $\overline{f^{-1}(l_y \setminus \{q\})}$ intersect this locus the way they should. Since the resolution $Y \to Bl_p(X)$ is an isomorphism outside $f^{-1}(q)$, we only need to resolve the affine neighborhood $V(xy - z^{n-2})$ of q (we have renamed the variables from $\tilde{x}$, $\tilde{y}$, z back to x, y, z, so we can use the tilded letters for the variables of the second blowup). Therefore if we denote $\tilde{\pi}$ the morphism of the blowup $Bl_q(V(xy - z^{n-2}))$, we are interested in $\overline{\tilde{\pi}^{-1}(l_x \setminus \{q\})} = \overline{\{(0, y, 0, [1, 0, 0])\}}$, $\overline{\tilde{\pi}^{-1}(l_y \setminus \{q\})} = \overline{\{(x, 0, 0, [1, 0, 0])\}}$. The exceptional locus of this blowup contains one irreducible component $\{(q, [x', y', z']) \in V(xy - z^{n-2}) \times \mathbb{P}^2 | x'y' = z'^2\}$ if $n = 4$, two irreducible components $\{(q, [0, y', z'])\}$ and $\{(q, [x', 0, z'])\}$ if $n > 4$. In either case, we see that $\overline{\tilde{\pi}^{-1}(l_x \setminus \{q\})}$ and $\overline{\tilde{\pi}^{-1}(l_y \setminus \{q\})}$ intersect the components of the locus as they should. Since $\overline{\tilde{\pi}^{-1}(l_x \setminus \{q\})}$ and $\overline{\tilde{\pi}^{-1}(l_y \setminus \{q\})}$ doesn't contain any singular points, this fact is not changed by consequent blowups, therefore the statement is proven.

The resolution for the rest of the groups can be found in Hemelsoet [2018]

□

Conclusion

We have classified finite subgroups of SL(2,C) using their maximal abelian subgroups. This gave us some restrictions on the subgroup, we have then one by one checked the possible types of the subgroup and shown whether the subgroup exists and its uniqueness up to conjugation. After the classification of the Dynkin diagrams and extended Dynkin diagrams, we have continued by showing both parts of the classical McKay correspondence. We have shown that the McKay graph of a subgroup of SL(2,C) is an extended Dynkin diagram and it contains enough information to uniquely determine the group. We also gave an outline of the geometric correspondence by computing GIT-quotients of the two-dimensional complex space and then blowing up the singular points.

Bibliography

Herwig Hauser. Blowups and resolution, 2014.

Nicolas Hemelsoet. Simple lie algebras and singularities. Master's thesis, École polytechnique fédérale de Lausanne, 2018.

R. A. Horn and C. R. Johnson. *Matrix Analysis*. Second edition. Cambridge university press, 2012. ISBN 978-0-521-83940-2.

K. Lamotke. *Regular Solids and Isolated Singularities*. Springer, 1986. ISBN 978-3-322-91767-6.

Amnon Neeman. *Algebraic and Analytic Geometry*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007. doi: 10.1017/CBO9780511800443.

Anupam Singh. Representation theory of finite groups, 2010.

K. E. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves. *An Invitation to Algebraic Geometry*. Springer, 2000. ISBN 978-1-4757-4497-2.

N. R. Wallach. *Geometric Invariant Theory over the Real and Complex Numbers*. Springer, 2017. ISBN 978-3-319-65905-3.