FACULTY OF MATHEMATICS AND PHYSICS
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BACHELOR THESIS

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Topological properties of algebraic curves

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Study programme: General mathematics
Study branch: Mathematical structures

Prague 2021

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I want to thank my supervisor for guiding me the whole time and for all of his helpful suggestions. I also want to thank my parents, who relentlessly supported me through ups and downs. I also want to acknowledge Mgr. Vítězslav Kala, Ph.D. who made me believe in myself when I needed it the most.

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Abstract: The thesis aims to present a theory about algebraic curves over complex numbers from the topological perspective. The main result proved in the thesis is the classical degree-genus formula which states that in the projective setting, non-singular algebraic curves are compact surfaces whose genus depends only on the degree of the curve itself.

The presented proof relies heavily on algebraic topology; it is shown that the curve acts as a covering space for the projective line (without a finite set of images of ramified points), then a suitable triangulation of a projective line is lifted to the curve. Later, we discuss how our result relates to the popular definition of genus as the number of handles attached to the sphere. Finally, we briefly go through singular curves showing that the degree-genus formula cannot, in general, be applied to them.

Keywords: algebraic curves, surfaces, genus, degree-genus formula

Contents

Introduction

Algebraic curves played a central role in the history of mathematics, with their rigorous study taking off in the 19th century by Bernhard Riemann, Felix Klein and David Hilbert.

This thesis aims to provide an accessible proof of a degree-genus formula - a classical result relating the topology of a planar algebraic curve with the degree of a polynomial defining it.

Planar algebraic curve C is essentially a zero set of a two-variable polynomial $P(x, y)$ in $\mathbb{C}^2$. It can be seen as $\mathbb{C}^2$, a four-dimensional vector space over $\mathbb{R}$, quotiented by the range of $P(x, y)$, which is two-dimensional over $\mathbb{R}$. So intuitively, C should be two-dimensional over $\mathbb{R}$.

We will show that C is indeed a two-dimensional manifold, i. e. a surface. Moreover, if we are instead working in a projective space, which is a compactification of the corresponding affine space, then C is a compact surface.

Compact connected surfaces are fully classified up to homeomorphism - they include orientable surfaces, which are homeomorphic to a sphere with g handles attached to it, and unorientable surfaces. Thus, the goal of the thesis is to show that a sufficiently general projective curve is topologically a sphere with

$$g = \frac{1}{2}(d-1)(d-2)$$

handles attached to it.

To prove the degree-genus formula, we will find a triangulation of C. The advantage of this method is that we will get for free that C is orientable. However, more work is necessary to find the number of handles of C. We will not be fully rigorous in working out the number of handles. Moreover, we will also not be able to prove that C is connected.

The core of the proof comes from the book Kirwan [1992].

Section 1.1 introduces the reader to projective spaces. It will be shown that they are Hausdorff, compact and that they contain the affine space as a subspace. Furthermore, a notion of a projective transformation of a given space will be discussed. We will show that they are homeomorphisms that are easy to find. This section is entirely based on Kirwan [1992], Chapter 2 with trivial modifications of proofs.

Section 1.2 introduces algebraic curves. Basic notions ubiquitous later on will be defined, and the relation between affine and projective algebraic curves will be discussed. The section is based on Kirwan [1992], Chapters 2 and 3, although the reader is expected to refer to Fulton [1989] for a more general algebraic point of view.

Section 1.3 builds a complex-analytic theory that will be further required to verify that one assumption to a crucial lifting criterion will be satisfied. The section is based on Kirwan [1992], Appendix B. The proof of Theorem 1.17 is a modification of a proof of Theorem B.1 in Kirwan [1992].

Section 1.4 aims to introduce the reader to the necessary pieces of algebraic topology. The crucial notions of covering spaces and lifting will be defined, and a lifting criterion sufficiently general to our purposes will be proved. The lifting criterion (precisely Theorem 1.27, Theorem 1.28 and Remark 1.29) is based on

Kirwan [1992], Appendix C, Section 1, however the buildup to it (the rest of the section) is based on Wilton [2019], Chapters 1 and 2. The first four paragraphs of the proof of Theorem 1.27 are mine, albeit probably well-known in this generality. The proof of Theorem 1.28 is an easy but nontrivial modification of the proof of Lemma C.7 from Kirwan [1992].

Chapter 2 introduces the setting of C being a branched cover of $\mathbb{P}^1$. The notion of a ramification point will be defined. We will then determine the number of ramification points in a sufficiently general case and show that such a case can always be reached. The section entirely follows Kirwan [1992], Chapter 4, Section 2. Contrary to that, Remark 2.6 is originally an exercise from Kirwan [1992].

Chapter 3 is the core of the thesis. First, we will define the Euler number and genus of a nonsingular planar projective curve and prove that they are topological invariants associated with the curve. Then, the main result of the thesis, the degree-genus formula, will be proved, showing that the genus defined from the Euler number depends only on the degree of C and agrees with the degree-genus formula. Next, we will discuss that the former definition of genus agrees with the definition of genus as a number of handles, presenting an informal argument in Remark 3.15 (this argument is my work) to support it while at the same time referring to the literature for a fully rigorous proof. Finally, we will show in a Remark 3.16, which was originally an exercise in Kirwan [1992], that the degree-genus formula cannot be applied to singular curves. The rest of the chapter mostly follows Kirwan [1992], Chapter 4, Section 3 except for Claim 3.7, where the part (i) comes from Kirwan [1992], Appendix C, Section 1 and part (ii) is proved in a more general form in Proposition 5.28 in Kirwan [1992]. The proof of Claim 3.7 is a modification of the proof of Proposition 5.28 in Kirwan [1992]. Additionally, Corollary 3.13 was originally an exercise in Kirwan [1992] and Remark 3.8 was given in Kirwan [1992] without proof.

In the thesis, we will freely use knowledge from the compulsory calculus (especially partial derivatives and Euclidean topology on $\mathbb{C}^n$), linear algebra (linear independence and linear automorphisms) and abstract algebra (polynomials, algebraic closedness of $\mathbb{C}$) courses, and from the specialized bachelor general topology course (continuous maps, homeomorphisms, compactness, Hausdorff spaces, connectedness, neighborhoods). On the other hand, knowledge from the course on the algebraic curves is explained or at least recalled. All knowledge exceeding the bachelor curriculum in complex analysis and algebraic topology is appropriately explained.

1. Preliminaries

1.1 Projective spaces and projective transformations

Definition 1.1. Complex projective space $\mathbb{P}^n$ *of dimension* n *is the set* $\mathbb{C}^{n+1} \setminus \{0\}$ *quotiented by the equivalence relation* $\sim$ *where* $a \sim b$ *if and only if there exists* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that* $a = \lambda b$. $\mathbb{P}^n$ *is then given a usual quotient topology of the Euclidean topology on* $\mathbb{C}^{n+1} \setminus \{0\}$ *(i. e.* $G \subset \mathbb{P}^n$ *is open if and only if* $\pi^{-1}(G) \subset \mathbb{C}^{n+1} \setminus \{0\}$ *is open for the natural projection* $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$*). Moreover, we call* $\mathbb{P}^1$ *the projective line and* $\mathbb{P}^2$ *the projective plane.*

It is easy to see that there is a bijection between equivalence classes of $\sim$ and one-dimensional subspaces of $\mathbb{C}^{n+1}$. We can therefore identify the elements of $\mathbb{P}^n$ with these subspaces.

Definition 1.2. *Let* $x \in \mathbb{P}^n$ *be represented by* $(x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. *The* $(n+1)$*-tuple of complex numbers* $(x_0, x_1, \ldots, x_n)$ *is said to be a set of* homogeneous coordinates *for* x. *We write* $x = [x_0, x_1, \ldots, x_n]$.

Clearly, $[x_0, x_1, \ldots, x_n]$ represents the same element as any scalar multiple of it.

Remark 1.3. Let

$$U_n = \{[x_0, x_1, \ldots, x_n] \in \mathbb{P}^n : x_n \neq 0\}.$$

Consider a mapping $\theta_n : \mathbb{C}^n \to U_n$ defined by

$$(x_0, x_1, \ldots, x_{n-1}) \mapsto [x_0, x_1, \ldots, x_{n-1}, 1].$$

The map θ_n is continuous because it is just a composition of an embedding of $\mathbb{C}^n$ into $\mathbb{C}^{n+1}$ and the natural projection onto $\mathbb{P}^n$.

The mapping $\psi : U_n \to \mathbb{C}^n$ given by

$$[x_0, x_1, \ldots, x_n] \mapsto \left(\frac{x_0}{x_n}, \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right).$$

is an inverse to θ_n. By continuity of each $\frac{x_i}{x_n}$ for every $0 \leq i \leq n-1$ the set $\pi^{-1}(\psi^{-1}(G))$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$ whenever G is an open subset of $\mathbb{C}^n$. Hence $\psi^{-1}(G)$ is open in $\mathbb{P}^n$ and also in U_n.

It follows that θ_n is a homeomorphism onto its image, and the affine space $\mathbb{C}^n$ can be viewed as a subspace of $\mathbb{P}^n$. The remaining part of $\mathbb{P}^n$ has a zero last coordinate, hence it can be identified with $\mathbb{P}^{n-1}$.

Definition 1.4. *A* projective transformation *of* $\mathbb{P}^n$ *is a mapping* $f : \mathbb{P}^n \to \mathbb{P}^n$*, for which there exists a linear automorphism* $\alpha : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *such that* $\pi(\alpha(x)) = f(\pi(x))$ *for every* $x \in \mathbb{C}^{n+1} \setminus \{0\}$. *We say that* f *is a projective transformation defined by* α.

Lemma 1.5. *Any projective transformation is a homeomorphism.*

Proof. First, we prove that any projective transformation f is continuous. Both α and π are continuous, therefore their composition $f \circ \pi$ is also continuous, hence $\pi^{-1}(f^{-1}(G))$ is open for any open $G \subset \mathbb{P}^n$. By definition of the quotient topology, $f^{-1}(G)$ is open if and only if $\pi^{-1}(f^{-1}(G))$ is open. Thus f is also continuous.

Let $g : \mathbb{P}^n \to \mathbb{P}^n$ be a projective transformation defined by α^{-1}. Then

$$f \circ g \circ \pi = f \circ \pi \circ \alpha^{-1} = \pi \circ \alpha \circ \alpha^{-1} = \pi$$

holds when we restrict the domain to $\mathbb{C}^{n+1} \setminus \{0\}$. From that, it follows that $f \circ g$ is an identity on $\mathbb{P}^n$. Similarly, we get that $g \circ f$ is an identity, hence f is a homeomorphism. □

Lemma 1.6. *Let $p_0, \ldots, p_n$ and q be $n+2$ distinct points in $\mathbb{P}^n$ and let $u_0, \ldots, u_n$ and v be any of their respective preimages in $\pi : \mathbb{C}^{n+1} \to \mathbb{P}^n$. Suppose that the matrix $(u_0|\cdots|u_n|v)$ is invertible if we erase a single column from it (no matter which one). Then there exists a projective transformation (in fact unique) taking p_i to $[0, \ldots, 0, 1, 0, \ldots, 0]$ where 1 is in the i-th place, and taking q to $[1, \ldots, 1]$.*

Proof. Let α be a linear automorphism of $\mathbb{C}^{n+1}$ taking the basis $(u_0, \ldots, u_n)$ to the canonical basis. By the rank assumption,

$$\alpha(v) = (\lambda_0, \ldots, \lambda_n)$$

where $\lambda_0, \ldots \lambda_n$ are all nonzero complex numbers. Therefore the composition of α with the linear automorphism defined by the diagonal matrix

$$\begin{pmatrix} \frac{1}{\lambda_0} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_1} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\lambda_n} \end{pmatrix}$$

defines a projective transformation taking p_i to

$$\left[0, \ldots, 0, \frac{1}{\lambda_i}, 0, \ldots, 0\right] = [0, \ldots, 0, 1, 0, \ldots, 0]$$

and q to $[1, \ldots, 1]$. □

Also, note that all of $p_0, \ldots, p_n$ and q need not be explicitly defined. Indeed, we can always produce any suitable missing values of $u_0, \ldots u_n$ and v by completing the basis provided that the given u_i and v are linearly independent.

Claim 1.7. *$\mathbb{P}^n$ is compact and Hausdorff as a topological space.*

Proof. First, let us prove the compactness. Let S_{n+1} be the unit sphere in $\mathbb{C}^{n+1}$ i. e.

$$S_{n+1} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} \left|x_i^2\right| = 1\}.$$

The sphere S_{n+1} is closed and bounded in $\mathbb{C}^{n+1}$, therefore it is compact. The restriction of a continuous map π on S_{n+1} is again continuous, hence the image of $\pi \restriction S_{n+1}$ is compact as well. So it remains to prove that $\pi : S_{n+1} \to \mathbb{P}^n$ is surjective.

Suppose $[x_0, x_1, \ldots, x_n] \in \mathbb{P}^n$. Then $\lambda = \sum_{i=0}^{n} |x_i^2| > 0$. Furthermore, $(\lambda^{\frac{1}{2}} x_0, \lambda^{\frac{1}{2}} x_1, \ldots, \lambda^{\frac{1}{2}} x_n) \in S_{n+1}$ and $\pi(\lambda^{\frac{1}{2}} x_0, \lambda^{\frac{1}{2}} x_1, \ldots, \lambda^{\frac{1}{2}} x_n) = [x_0, x_1, x_2, \ldots, x_n]$, as desired.

Next, suppose that p and q are distinct points of $\mathbb{P}^n$. By Lemma 1.6 and Lemma 1.5 there is a homeomorphic projective transformation $f : \mathbb{P}^n \to \mathbb{P}^n$ taking both p and q inside

$$U_n = \{[x_0, x_1, \ldots, x_n] \in \mathbb{P}^n : x_n \neq 0\}.$$

By Remark 1.3 U_n is mapped homeomorphically onto a Hausdorff space $\mathbb{C}^n$. Hence there are disjoint open neighborhoods G_p and G_q of $f(p)$ and $f(q)$ respectively in U_n.

It remains to prove that G_p and G_q are open in $\mathbb{P}^n$ as then $f^{-1}(G_p)$ and $f^{-1}(G_q)$ would be disjoint open neighborhoods of p and q respectively. The set U_n is open in $\mathbb{P}^n$ since $\pi^{-1}(U_n)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. By the definition of a subspace topology, there is an open set G'_p in $\mathbb{P}^n$ such that $G_p = G'_p \cap U_n$. Consequently, G_p is open in $\mathbb{P}^n$ as an intersection of two open sets. Similarly, G_q is open in $\mathbb{P}^n$ finishing the proof.

□

1.2 Algebraic curves

We will start with a definition of affine and projective curves, and then we will examine how these relate.

Definition 1.8. *Let $P(x, y)$ be a complex non-constant two-variable polynomial without repeated factors. The* affine algebraic curve *C in $\mathbb{C}^2$ defined by $P(x, y)$ is*

$$C = \left\{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\right\}.$$

Definition 1.9. *Let $P(x, y, z)$ be a complex non-constant three-variable homogeneous polynomial without repeated factors. The* projective algebraic curve *C in $\mathbb{P}^2$ defined by $P(x, y, z)$ is*

$$C = \left\{[x, y, z] \in \mathbb{P}^2 : P\,[x, y, z] = 0\right\}.$$

If $P(x, y, z)$ is a non-constant homogeneous polynomial of degree d defining a projective algebraic curve C_P, then the polynomial $P(x, y, 1)$ defines an affine algebraic curve C_A such that $[x, y, 1] \in C_P \Leftrightarrow (x, y) \in C_A$. The only way this process can fail is when $P(x, y, 1)$ is a constant polynomial, so it fails precisely when $P(x, y, z)$ is a scalar multiple of z.

On the other hand, given a $P(x, y)$ of degree d, we can construct a projective curve by multiplying each monomial of degree d' by $z^{d-d'}$. Then again $[x, y, 1] \in C_P \Leftrightarrow (x, y) \in C_A$. It is also not difficult to verify that in both ways, irreducibility is preserved. Hence there is a natural correspondence between the sets of all affine and projective algebraic curves (except for the projective curve defined by the polynomial $P(x, y, z) = z$).

Definition 1.10. *A point* $[a, b, c]$ *on a projective algebraic curve* $C \subset \mathbb{P}^2$ *defined by* $P(x, y, z)$ *is called* singular *if*

$$\frac{\partial P}{\partial x}(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial P}{\partial z}(a, b, c) = 0.$$

We say that C *is* singular *if it contains at least one singular point.*

Definition 1.11. *A projective algebraic curve* C *is called a* projective line *if it is defined by a linear polynomial.*

The tangent line *to a projective curve* $C \subset \mathbb{P}_2$ *defined by a homogeneous polynomial* $P(x, y, z)$ *at a non-singular point* $[a, b, c]$ *is the line*

$$\frac{\partial P}{\partial x}(a, b, c)x + \frac{\partial P}{\partial y}(a, b, c)y + \frac{\partial P}{\partial z}(a, b, c)z = 0.$$

The following definition is the same for both affine and projective curves.

Definition 1.12. *The* degree *of an algebraic curve* C *is the degree of the polynomial* P *defining* C*.* C *is called* irreducible *if* P *is irreducible. An irreducible algebraic curve* D *defined by* Q *is a* component *of* C *if* Q *divides* P*.*

At first glance, it is not clear that the degree of C is well-defined (there might be another polynomial R defining C). However, by a corollary of Hilbert's Nullstellensatz (see Fulton [1989], Chapter 1, Section 7 for the affine case, Fulton [1989], Chapter 4, Section 2 for the projective case) R generates the same ideal as P. Thus the degrees of P and R are the same, so the degree of a curve is well-defined.

Projective algebraic curves exhibit many properties, in particular, the following fabled theorem due to Bézout.

Theorem 1.13. *Let* C *and* D *be two projective curves of degree* n *and* m *respectively, which do not have a common component. Then* C *and* D *intersect precisely at* mn *points (counting multiplicities).*

In particular, C *and* D *intersect exactly at* mn *pairwise distinct points if each* $p \in C \cap D$ *is a nonsingular point in both* C *and* D *and the tangent lines to* C *and* D *at* p *differ.*

Proof. See Fulton [1989], Chapter 5, Section 3. □

In the later chapters, we will need the following technical lemma.

Lemma 1.14. *(Euler's relation) If* $P(x, y, z)$ *is a homogeneous polynomial of a degree* d *then*

$$x\frac{\partial P}{\partial x}(x, y, z) + y\frac{\partial P}{\partial y}(x, y, z) + z\frac{\partial P}{\partial z}(x, y, z) = dP(x, y, z).$$

Proof. Just a straightforward computation of partial derivatives. □

1.3 Complex analysis

In the following chapters, we will use several theorems from algebraic topology. However, we first need to develop some complex-analytic tools which will allow us to use this machinery. First, let us recall an important application of Cauchy's residue theorem.

Theorem 1.15. (The argument principle) *Suppose $\gamma : [0, 1] \to \mathbb{C}$ is a circle in $\mathbb{C}$. If f is meromorphic inside and on γ and contains no zeros or poles on γ, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z - P$$

where Z and P are respectively the number of zeros and poles of f in the interior of γ (counting multiplicities).

Proof. See Krantz [1999], Chapter 5, Section 2. □

Remark 1.16. The previous theorem holds for any simple closed curve γ. The formulation above was chosen to avoid the ambiguous notion of a curve since we will only use the theorem for a circle.

Theorem 1.17. *Suppose that $A(z, w)$ is a polynomial with complex coefficients which satisfies*

$$A(z_0, w_0) = 0.$$

Moreover, suppose that the single-variable polynomial $A(z_0, w)$ has a zero of order m at w_0. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then the polynomial $A(z, w)$ has at least m zeros (counting multiplicities) in the disc

$$\{w \in \mathbb{C} : |w - w_0| < \varepsilon\} .$$

Proof. The polynomial $A(z_0, w)$ has finitely many zeros, therefore we can choose $\varepsilon' \geq \varepsilon$ so that

$$S = \{w \in \mathbb{C} : |w - w_0| = \varepsilon'\}$$

satisfies $A(z_0, s) \neq 0$ for every $s \in S$.

Since A is a continuous function of z and w, for every $s \in S$ there exists a $\delta_s > 0$ such that

$$\max (|v - s|, |z - z_0|) \leq \delta_s \implies A(z, v) \neq 0.$$

The subset S is compact and contained in the union of the open subsets

$$\{v \in \mathbb{C} : |v - s| < \delta_s, s \in S\} ,$$

and thus there exists a finite subset $\{s_1, \dots, s_k\}$ of S such that

$$S \subset \bigcup_{1 \leq i \leq k} \{w \in \mathbb{C} : |w - s_i| < \delta_{s_i}\} .$$

Let

$$\delta = \min(\delta_{s_1}, \dots, \delta_{s_k}) > 0.$$

Putting this together, if $|s - w_0| = \varepsilon'$ and $|z - z_0| < \delta$, then $A(z, s) \neq 0$.

Define a simple closed curve $\gamma : [0,1] \to \mathbb{C}$ by

$$\gamma(t) = w_0 + \varepsilon' e^{2\pi i t}.$$

As the polynomial $A(z,w)$ does not have any poles, Theorem 1.15 tells us for each fixed z satisfying $|z - z_0| < \delta$ that the number of zeros of the function $A(z,w)$ of w (counted with multiplicities) inside γ is

$$n(z) = \frac{1}{2\pi i} \int_\gamma \frac{\frac{\partial A}{\partial w}(z,w)}{A(z,w)} dw.$$

By our assumption, $A(z_0, w)$ has a zero of multiplicity m at w_0, hence we have $n(z_0) \geq m$.

As a contour integral of a meromorphic function, $n(z)$ is clearly a continuous and integer-valued function whenever $|z - z_0| < \delta$. Hence, $n(z)$ is a constant meaning that $n(z) \geq m$ whenever $|z - z_0| < \delta$. If $\varepsilon = \varepsilon'$, we are clearly done, otherwise we can choose the same δ as well. ☐

Corollary 1.18. *Let $A(z,w)$ be a polynomial with complex coefficients such that for every fixed $z \in \mathbb{C}$ the polynomial $A(z,w)$ in w is monic of degree n. Let*

$$C = \left\{(z,w) \in \mathbb{C}^2 : A(z,w) = 0\right\}$$

and define $\varphi : C \to \mathbb{C}$ by

$$\varphi(z,w) = z.$$

Then any $z_0 \in \mathbb{C}$ has an open neighborhood U in $\mathbb{C}$ such that each connected component of $\varphi^{-1}(U)$ contains at most one point of $\varphi^{-1}(z_0)$.

Proof. If

$$\varphi^{-1}(\{z_0\}) = \{(z_0, w_1), \ldots, (z_0, w_k)\}$$

then

$$A(z_0, w) = \prod_{1 \leq i \leq k} (w - w_i)^{m_i}$$

where $m_1, \ldots, m_k$ are positive integers satisfying

$$m_1 + \cdots + m_k = n.$$

Choose $\varepsilon > 0$ so that $|w_i - w_j| > 2\varepsilon$ whenever $i \neq j$. Then by Theorem 1.17 there exists some $\delta_i > 0$ such that if $|z - z_0| < \delta_i$ then the polynomial $A(z,w)$ in w has at least m_i roots in the disc

$$D_i = \{w \in \mathbb{C} : |w - w_i| < \varepsilon\}$$

for each $1 \leq i \leq k$.

Let

$$\delta = \min(\delta_1, \ldots, \delta_k) > 0.$$

The discs D_i are disjoint and the sum of the m_i is n, therefore if $|z - z_0| < \varepsilon$ then all the roots of $A(z,w)$ lie in $D_1 \cup \cdots \cup D_k$, and hence

$$\varphi^{-1}\left(\{z \in \mathbb{C} : |z - z_0| < \delta\}\right) \subset \mathbb{C} \times (D_1 \cup \cdots \cup D_k).$$

Thus every connected component of

$$\varphi^{-1}\left(\{z \in \mathbb{C} : |z - z_0| < \delta\}\right)$$

is a subset of $C \times D_i$ for some $1 \leq i \leq k$, so it contains at most one point of $\varphi^{-1}(\{z_0\})$. ☐

1.4 Algebraic topology

Notation. Throughout the rest of the thesis, I will denote the interval $[0, 1]$. Similarly, we define

$$\begin{aligned} \triangle &= \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\right\}, \\ \triangle^o &= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\right\}, \\ \triangle^{\setminus V} &= \triangle \setminus \{(0, 0), (0, 1), (1, 0)\}. \end{aligned}$$

Throughout this section, any map is required to be continuous.

The exact geometrical shape of the triangle does not concern us, as we will be interested only in the topological properties of any triangle.

We will soon see that a projective algebraic curve behaves like a covering space for the complex projective line $\mathbb{P}^1$ excluding a finite set of images of ramification points. However, we must first start by developing some general theory about covering spaces.

Definition 1.19. *Let $f, g : X \to Y$ be maps between topological spaces X, Y. A* hompotopy *from f to g is a map*

$$H : X \times I \to Y$$

such that or each $x \in X$

$$H(x, 0) = f(x), \qquad\qquad H(x, 1) = g(x).$$

Definition 1.20. *A* path *in a topological space X is a map $\gamma : I \to X$. Suppose that $\gamma, \delta : I \to X$ are paths with common endpoints (i. e. $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$). A* homotopy from γ to δ as paths *is a homotopy*

$$H : I \times I \to X$$

which also satisfies for each $x \in I$

$$H(0, x) = \gamma(0), \qquad\qquad H(1, x) = \gamma(1).$$

We can think of the interval I as time. For every $t \in I$, $H(\cdot, t)$ defines a map $X \to Y$. Therefore we can think about homotopies as continuous transformations of maps with respect to time t. Additionally, homotopy between paths (as paths) requires the endpoints to be fixed, as otherwise, any two paths will be homotopic.

Example. Any path can be viewed as a homotopy, with X being the single-point space.

Definition 1.21. *If we have two paths γ_1 from x_0 to x_1; and γ_2 from x_1 to x_2, we define their* join *$\gamma_1 \cdot \gamma_2$ to be*

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Definition 1.22. *Suppose that $\pi : Y \to X$ is a map between topological spaces Y and X such that each $x \in X$ has an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets of Y each of which is mapped homeomorphically onto U by π. Then π is said to be a* covering projection *and Y is said to be a* covering space *for X.*

Definition 1.23. *Let $\pi : Y \to X$ be a covering projection and A be any topological space. We say that a map $f : A \to X$* lifts *if there exists a map $F : A \to Y$ such that $f = \pi \circ F$. The map F is then called a* lift *of f.*

When we have defined lifting, it is natural to ask whether lifts exist and if they are unique. The following lemma states that in the case of A connected, the lift of f is determined by any single point.

Lemma 1.24. *Suppose that X, Y and A are topological spaces and $\pi : Y \to X$ is a covering projection. Let $F_1, F_2 : A \to Y$ be two lifts of $f : A \to X$. Then*

$$S = \{a \in A : F_1(a) = F_2(a)\}$$

is clopen (both closed and open).

Proof. See Wilton [2019], Chapter 2, Section 2. □

The existence of lifts is not always guaranteed, but homotopies can always be lifted employing the following theorem. Since paths can be expressed as homotopies, we get path lifting as an easy corollary (again by taking the single-point space as A), albeit it being a crucial step in the proof.

Theorem 1.25. *Let $\pi : Y \to X$ be a covering projection, $H : A \times I \to X$ be a homotopy from f to g. Let F be a lift of f. Then there exists a unique homotopy $\tilde{H} : A \times I \to Y$ which satisfies*

- *$\tilde{H}(\cdot, 0) = F$; and*
- *$\tilde{H}$ is a lift of H.*

Proof. See Wilton [2019], Chapter 2, Section 2. □

This looks nice, but not everything we will work with can be viewed as a homotopy. The following theorem will provide us with a sufficient condition for the existence of a lift based only on the properties of the space A.

Definition 1.26. *A topological space X is* simply connected *if any two paths $\gamma, \delta : I \to X$ with common endpoints are homotopic as paths.*

Example. Any convex subset X of $\mathbb{R}^n$ is simply connected. Indeed, $H : I \times I \to X$ given by

$$H(x, t) = (1 - t) \cdot \gamma(x) + t\delta(x)$$

is clearly a homotopy from γ to δ for any paths γ, δ in X. This X can be all of $\triangle$, $\triangle^o$, $\triangle^{\setminus V}$, I and $(0, 1)$.

Theorem 1.27. *Let $\pi : Y \to X$ be a covering projection and $f : A \to X$ be a map. Suppose that A is simply connected, path-connected and locally path-connected (i. e. every $y \in A$ contains arbitrarily small path-connected neighborhoods in A). Then:*

(i) The map f lifts uniquely (in the sense of Lemma 1.24) to $F : A \to Y$.

(ii) If moreover $f : A \to f(A)$ is a homeomorphism, then F is a homeomorphism onto a connected component of $\pi^{-1}(f(A))$.

Proof. (i) Suppose that $a \in A$ and $y \in Y$ are points chosen so that $f(a) = \pi(y)$ and $F(a) = y$. We will prove that $F(b)$ is then determined for any $b \in A$.

Since Y is path-connected, we can take any two paths $\gamma, \delta : I \to A$ such that $\gamma(0) = a$, $\delta(0) = a$, $\gamma(1) = b$, $\delta(1) = b$. Let Γ be a lift of γ, i. e. a map $\Gamma : I \to Y$ satisfying $\pi \circ \Gamma = f \circ \gamma$. Similarly, let Δ denote a lift of δ.

Because A is simply connected, there exists a homotopy $H : I \times I \to A$ from γ to δ as paths. By Theorem 1.25, H lifts to $\tilde{H} : I \times I \to Y$ such that $\pi \circ \tilde{H} = f \circ H$.

Now, $\tilde{H}(1, \cdot)$ is again a lift of $H(1, \cdot)$. But $H(1, \cdot)$ is a constant path in A which by Lemma 1.24 lifts uniquely to a constant path in Y. Therefore $\Gamma(1) = \tilde{H}(1, 0) = \tilde{H}(1, 1) = \Delta(1)$, meaning that there is a well-defined map $F : A \to Y$ defined by $F(b) = \Gamma(1)$, where Γ is chosen as above. Then

$$\pi(F(b)) = \pi(\Gamma(1)) = f(\gamma(1)) = f(b),$$

hence F is a lift of f. Uniqueness will follow from Lemma 1.24 once we show that F is continuous at any given $b \in A$.

Let W be an open neighborhood of Y containing $F(b)$. By π being a covering projection, W contains an open neighborhood V of $F(b)$ which is mapped homeomorphically by π onto an open neighborhood U of $\pi(F(b)) = f(b)$ in X. Then $f^{-1}(U)$ is an open neighborhood of b in A, therefore it contains a path-connected open neighborhood T of b, as A is locally path-connected. For each $c \in T$ there exists a path γ_c from b to c in T.

Let $\phi : U \to V$ be the inverse map to the homeomorphism $\pi : V \to U$. By the construction of T, $\phi \circ f \circ \gamma_c$ is a continuous map whose range is a subset of V, thus it is the unique lift Γ_c of γ_c satisfying $\Gamma_c(0) = F(b) = \Gamma(1)$. Then we have

$$F(c) = (\Gamma \cdot \Gamma_c)(1) = \Gamma_c(1) = (\pi^{-1} \circ f \circ \gamma_c)(1) \in V \subset W,$$

where the first equality comes again from Lemma 1.24. Thus $F(T) \subset W$, meaning that F is continuous.

(ii) Now suppose that $f : A \to X$ is a homeomorphism onto its image. As the restriction of π to $\pi^{-1}(f(A))$ is again a covering projection, we may WLOG assume that $f(A) = X$. Note that since A is connected, so are $X = f(A)$ and $F(A)$.

Any $x \in X$ has a connected open neighborhood U_x in X such that $\pi^{-1}(U_x)$ is a disjoint union of open subsets of Y. Each of these subsets is mapped homeomorphically by π onto U_x. Since f is a homeomorphism, $F(f^{-1}(U_x))$ is a connected subset of $\pi^{-1}(U_x)$. Therefore it is contained in one of these open subsets, say V_x.

Let $\phi : U_x \to V_x$ be the inverse map to π restricted in its range to V_x. Since ϕ is a homeomorphism, it also holds that $F = \phi \circ f$ on $f^{-1}(U_x)$. Hence

$$F(f^{-1}(U_x)) = \phi(U_x) = V_x.$$

The set $F(A)$ contains exactly one preimage in π of each $x \in X$, in particular if $x \in U_x$, then this preimage lies in V_x. Hence also

$$F(A) \cap \pi^{-1}(U_x) = V_x = F(f^{-1}(U_x)).$$

Take any $y \in F(A)$ and let $x = \pi(y)$. Then V_x is an open neigborhood of y in Y contained in $F(A)$. Hence $F(A)$ is open in Y. Similarly, take any $y \in Y \setminus F(A)$ and let $x = \pi(y)$. Then $\pi^{-1}(U_x) \setminus V_x$ is an open neighborhood of y in Y contained in $Y \setminus F(A)$. It follows that $Y \setminus F(A)$ is open in Y. Therefore $F(A)$ is a clopen connected subset of Y and hence a connected component of Y.

We already showed that $F(A)$ contains precisely one preimage of each $x \in X$, hence F is injective. Therefore $F : A \to F(A)$ is a continuous bijection whose inverse is the continuous map $f^{-1} \circ \pi : F(A) \to A$. Thus F is a homeomorphism onto its image.

□

So far, we have only dealt with perfect covering spaces. However, in our situation, there will be a finite set of points (we will call them ramification points) violating our definition. The following theorem will tell us that even triangles whose vertices are ramification points can be uniquely lifted.

Theorem 1.28. *Let $\pi : Y \to X$ be a map and suppose that every $x \in X$ has an open neighborhood U in X such that every connected component of $\pi^{-1}(U)$ contains at most one point of $\pi^{-1}(x)$. Suppose that Y is compact, Hausdorff and that V is an open subset of X such that $\pi : \pi^{-1}(V) \to V$ is a covering projection. If $f : \triangle \to X$ is continuous and $f^{-1}(V)$ contains $\triangle^{\setminus V}$, then given $\tau \in \triangle^{\setminus V}$ and $y \in Y$ such that $\pi(y) = f(\tau)$ there exists a unique map $F : \triangle \to Y$ such that $F(\tau) = y$ and $\pi \circ F = f$.*

In this case, we also say that F is a lift of f.

Proof. By Theorem 1.27 there is a unique lift F of f defined on $\triangle^{\setminus V}$.

It suffices to extend F onto the vertices of $\triangle$. To do this, we will prove that F tends to a unique limit at each of the vertices. We will deal only with $(0, 0)$ since the arguments are analogous.

To start with, $f(0, 0) \in X$ has an open neighborhood U in X such that each connected component of $\pi^{-1}(U)$ contains at most one point of $\pi^{-1}(f(0, 0)))$. Now we can choose $\delta > 0$ sufficiently small so that

$$f(B(0, \delta)) \subset U.$$

Hence the deleted neighborhood $P(0, \delta) = B(0, \delta) \setminus \{(0, 0)\}$ satisfies

$$F(P(0, \delta)) \subset \pi^{-1}(U).$$

This way, $F(P(0, \delta))$ is connected and therefore contained in a component of $\pi^{-1}(U)$, say W.

Let $t_1, t_2, \ldots$ be a sequence in $P(0, \delta)$ which tends to $(0, 0)$. Since Y is compact, there is a subsequence

$$t_{n_1}, t_{n_2}, \ldots$$

such that $F(t_{n_k})$ converges to some $p \in Y$ as k goes to infinity. Recall that the limit p is unique since Y is Hausdorff. Then

$$\pi(p) = \lim \pi \circ F(t_{n_k}) = \lim f(t_{n_k}) = f(0, 0)$$

so $p \in \pi^{-1}(f(0, 0))$. But since $F(t_n) \in W$ for every n and W is an open and closed subset of $\pi^{-1}(U)$, the limit p must lie in W as well, therefore p must be the unique point of $\pi^{-1}(f(0, 0))$ lying in W.

For a contradiction, suppose that $F(t_k)$ does not converge to p as $k \to \infty$. Then there exists a subsequence

$$t_{m_1}, t_{m_2}, \ldots$$

and a neighborhood Z of p in y such that all elements of $F(t_{m_k})$ lie outside Z. Following the same argument, there exists a limit of some subsequence of $F(t_{m_k})$, which again must be p. But that is impossible by the construction of $F(t_{m_k})$.

We proved that the limit of F at $(0, 0)$ is unique and satisfies $\pi(F(0, 0)) = f(0, 0)$, hence there exists a unique extension and the proof is complete. $\square$

Remark 1.29. The theorem above holds with $(0, 1)$ instead of $\triangle^{\setminus V}$ and with I instead of $\triangle$ as well, the proof is exactly the same (just replace $(0, 0)$ with 0).

2. Branched covers of $\mathbb{P}^1$

Notation. Let C be a nonsingular algebraic curve of degree d in $\mathbb{P}^2$ defined by the polynomial $P(x, y, z)$. Throughout the rest of the thesis, $\phi : C \to \mathbb{P}^1$ will denote the map given by

$$\phi\,[x, y, z] = [x, z]\,.$$

It is easy to see that ϕ is well-defined if and only if $[0, 1, 0] \notin C$. Since we are only interested in topological properties of C, we can WLOG assume that $[0, 1, 0] \notin C$ (otherwise we can map C by some homeomorphic projective transformation).

Also, note that the coefficient of y^d in the homogeneous polynomial is nonzero, as it the only nonzero term in $P(0, 1, 0) \neq 0$.

So far, we did not assume that C is irreducible. However, this is fine as the following corollary of Theorem 1.13 tells us that this assumption can be safely omitted.

Corollary 2.1. *A nonsingular algebraic curve C in $\mathbb{P}^2$ is irreducible.*

Proof. Suppose that

$$C = \Big\{[x, y, z] \in \mathbb{P}^2 : P(x, y, z) \cdot Q(x, y, z) = 0\Big\}$$

is reducible. Then by Theorem 1.13, the projective curves defined by polynomials $P(x, y, z)$ and $Q(x, y, z)$ have either at least one common component or intersect in precisely $\deg P(x, y, z) \cdot \deg Q(x, y, z)$ points (counting multiplicities). Either way, there exists a point $[a, b, c] \in \mathbb{P}^2$ such that

$$P(a, b, c) = Q(a, b, c) = 0.$$

Then for each $w \in \{x, y, z\}$ it holds that

$$\frac{\partial\left(P(x, y, z) \cdot Q(x, y, z)\right)}{\partial w} = \frac{\partial P(x, y, z)}{\partial w} \cdot Q(x, y, z) + \frac{\partial Q(x, y, z)}{\partial w} \cdot P(x, y, z) = 0$$

showing that C is singular. □

Definition 2.2. *The* ramification index $\nu_\phi\,[a, b, c]$ *of ϕ at a point $[a, b, c] \in C$ is the order of the zero of the polynomial $P(a, y, c)$ in y at $y = b$. The point $[a, b, c]$ is called a* ramification point *of ϕ if $\nu_\phi\,[a, b, c] > 1$.*

Let R be the set of ramification points of ϕ. The image $\phi(R)$ is said to be the branch locus *of ϕ, and $\phi : C \to \mathbb{P}^1$ is called a* branched cover *of $\mathbb{P}^1$.*

In this section, we want to prove that there exists a projective transformation mapping C to D such that the number of ramification points of $\phi_D : D \to \mathbb{P}^1$ is exactly $d(d - 1)$. We will assume $d > 1$ while the case $d = 1$ will be considered by the remark at the end of the chapter.

Lemma 2.3. *The preimage $\phi^{-1}([a, c])$ of any $[a, c]$ in $\mathbb{P}^1$ contains exactly*

$$d - \sum_{p \in \phi^{-1}([a,c])} (\nu_\phi(p) - 1)$$

points. In particular, $\phi^{-1}([a, c])$ contains d points if and only if $[a, c]$ does not belong to the branch locus of ϕ.

Proof. A point of C lies in $\phi^{-1}([a,c])$ if and only if it is of a form $[a,b,c]$ where $P(a,b,c)=0$. The single-variable polynomial $P(a,y,c)$ of degree d can be factored as

$$P(a,y,c)=\lambda\prod_{1\le i\le r}(y-b_i)^{m_i}$$

where $\lambda\in\mathbb{C}\setminus\{0\}$, $b_1,\dots,b_r$ are distinct complex numbers and $m_1,\dots,m_r$ are positive integers satisfying

$$m_1+\cdots+m_r=d.$$

Hence

$$\phi^{-1}([a,c])=\{[a,b_i,c]:1\le i\le r\}$$

and the ramification index of ϕ at

$$\nu_\phi\,[a,b_i,c]=m_i.$$

Easy computation then finishes the proof. □

Lemma 2.4. *The map ϕ has at most $d(d-1)$ ramification points. Moreover, if $\nu_\phi\,[a,b,c]\le 2$ for all $[a,b,c]\in C$ then C has exactly $d(d-1)$ ramification points.*

Proof. Consider the homogeneous polynomial $Q(x,y,z)=\frac{\partial P}{\partial y}(x,y,z)$. The coefficient of y^d is nonzero, therefore $Q(x,y,z)$ has degree exactly $d-1>0$. By Corollary 2.1, $P(x,y,z)$ is irreducible, thus coprime with $Q(x,y,z)$. This means that the projective curve D defined by $Q(x,y,z)$ has no common component with C. The first claim then follows from Theorem 1.13, because the set R of ramification points is the intersection of C and D.

Moreover, suppose that

$$\nu_\phi\,[a,b,c]\le 2$$

for all $[a,b,c]\in C$. By Theorem 1.13, it is enough to prove that if $[a,b,c]$ lies in $R=C\cap D$ then $[a,b,c]$ is a nonsingular point of D such that the tangent lines to C and D at $[a,b,c]$ are distinct. Suppose the contrary. Then

$$P(a,b,c)=0=\frac{\partial P(a,b,c)}{\partial y},$$

as $[a,b,c]$ lies in both C and D, and the vector describing the tangent line to D

$$\left(\frac{\partial^2P(a,b,c)}{\partial x\partial y},\frac{\partial^2P(a,b,c)}{\partial y\partial y},\frac{\partial^2P(a,b,c)}{\partial z\partial y}\right)$$

is either zero or a scalar multiple of the vector describing the tangent line to C

$$\left(\frac{\partial P(a,b,c)}{\partial x},\frac{\partial P(a,b,c)}{\partial y},\frac{\partial P(a,b,c)}{\partial z}\right).$$

Either way,

$$P(a,b,c)=0=\frac{\partial P(a,b,c)}{\partial y}=\frac{\partial^2P(a,b,c)}{\partial y\partial y},$$

showing that the order of zero of $P(a,y,c)$ in y at $P(a,b,c)$ is at least 3 (i. e. $\nu_\phi\,[a,b,c]>2$). This is a contradiction with our assumption. □

The following lemma states that the most general case where there are precisely $d(d-1)$ ramification points can always be reached. The proof follows an intuitively appealing general position argument, where it suffices to find a projective transformation mapping $[0,1,0]$ anywhere outside the union of C and tangents to C at finitely many points of inflection of C (the points of inflection can be seen as points where the tangent to C has at least triple intersection with C). The detailed proof can be found in Kirwan [1992], Sections 3.3 and 4.2.

Lemma 2.5. *By applying a suitable projective transformation to C we may assume that*

$$\nu_\phi\,[a,b,c] \leq 2$$

for all $[a,b,c] \in C$.

Remark 2.6. If $d=1$ then always $\frac{\partial P}{\partial y} \neq 0$ (provided that $[0,1,0] \notin C$), hence there are no ramification points.

Moreover, each projective line is defined by two points. Hence, by Lemma 1.6 there is a projective transformation taking C to the line defined by the polynomial $y=0$. Then it easy to see that $\phi : C \to \mathbb{P}^1$ given by

$$[x,0,z] \mapsto [x,z]$$

is a homeomorphism.

3. The proof of the degree-genus formula

In this chapter, we are finally ready to prove the degree-genus formula. Our strategy will be to produce a triangulation of C and compute the number of vertices, edges and faces. First, we will need a technical but precise definition of triangulation.

Definition 3.1. *Let C be a nonsingular complex projective curve in $\mathbb{P}^2$. A* triangulation *of C is a triple of nonempty finite sets (V, E, F) where:*

(a) A set V of vertices consists of points.

(b) A set E of edges consists of continuous maps $e : I \to C$.

(c) A set F of faces consists of continuous maps $f : \triangle \to C$.

These sets have to satisfy:

(i) $V = \{e(0) : e \in E\} \cup \{e(1) : e \in E\}$ i.e. the vertices are the endpoints of the edges;

(ii) if $e \in E$ then the restriction of e to the open interval $(0, 1)$ is a homeomorphism onto its image in C which contains no points in V or in the image of any other edge $e' \in E$;

(iii) if $f \in F$ then the restriction of f to $\triangle^o$ is a homeomorphism onto a connected component K_f of $C \setminus \Gamma$ where

$$\Gamma = \bigcup_{e \in E} e\,(I)$$

is the union of the images of the edges. Moreover, if $r : I \to I$ and $\sigma_i : I \to \triangle$ for $1 \leq i \leq 3$ is defined by

$$r(t) = 1 - t, \quad \sigma_1(t) = (t, 0), \quad \sigma_2(t) = (1 - t, t), \quad \sigma_3(t) = (0, 1 - t)$$

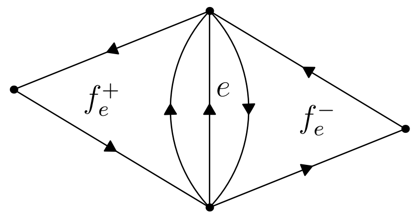
then either $f \circ \sigma_i$ or $f \circ \sigma_i \circ r$ is an edge e^i_f for $1 \leq i \leq 3$;

(iv) the mapping $f \mapsto K_f$ from F to the set of connected components is a bijection;

(v) for every $e \in E$ there is exactly one face $f^+_e \in F$ such that $e = f^+_e \circ \sigma_i$ for some $i \in \{1, 2, 3\}$ and exactly one face $f^-_e \in F$ such that $e = f^-_e \circ \sigma_i \circ r$ for some $i \in \{1, 2, 3\}$.

Note that our definition of triangulation additionally requires that each face is oriented, and the orientations of the two faces intersecting at $e \in E$ disagree on e (see Figure 3.1 for better understanding). Moreover, it is natural to identify the maps in E and F with their images in C.

Figure 3.1: The orientation of faces at $e \in E$.



Definition 3.2. *The* Euler number *of a triangulation is defined by*

$$\chi = \#V - \#E + \#F$$

where the symbol $\#S$ denotes the number of elements of a finite set S.

The following theorem and remark show us why the Euler number is essential.

Theorem 3.3. *The Euler number χ of a triangulation of C depends only on C, not on the triangulation.*

Proof. The proof is long and involved, see Wilton [2019], Chapter 6, Section 4. □

Definition 3.4. *The* genus *of a nonsingular projective curve C is*

$$g = \frac{1}{2}(2 - \chi)$$

where χ is the Euler number of C.

Remark 3.5. Suppose that $h : C \to D$ is a homeomorphism between nonsingular projective curves C and D. Let (V, E, F) be a triangulation of C. Then, easy but tedious verification shows that

$$(h(V), \{h \circ e : e \in E\}, \{h \circ f : f \in F\})$$

is a triangulation of D.

It follows that homeomorphic curves have the same Euler number and genus, which justifies using a projective transformation in the sense of Lemma 2.5. In other words, the Euler number and genus of a nonsingular projective curve are *topological properties*, meaning that they depend only on the topology of the curve.

In the rest of the thesis, we will work on finding a suitable triangulation of C. The proof will make use of the results from algebraic topology. However, we must first make sure their assumptions are satisfied. As a byproduct, we will also get that C is a surface.

Definition 3.6. *A* surface *is a Hausdorff topological space such that every point has a neighbourhood U that is homeomorphic to $\mathbb{R}^2$.*

Claim 3.7. *Let C be a nonsingular algebraic curve in $\mathbb{P}^2$ defined by the homogeneous polynomial $P(x, y, z)$ of degree d.*

(i) The map $\varphi : C \setminus \phi^{-1}(\phi(R)) \to \mathbb{P}^1 \setminus \phi(R)$ defined as a restriction of $\phi : C \to \mathbb{P}^1$ is a covering projection.

(ii) The curve C is a compact surface.

Proof. It holds for the natural projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ that $\pi^{-1}(C) = P^{-1}(\{0\}) \setminus \{0\}$, hence $\pi^{-1}(C)$ is closed in $\mathbb{C}^3 \setminus \{0\}$. By definition of $\mathbb{P}^2$, C is then closed in $\mathbb{P}^2$. By Claim 1.7, C is then Hausdorff and compact as a closed subspace of a Hausdorff space.

Suppose that $[a, b, c] \in C$. The curve C is nonsingular so at least one partial derivative is nonzero. Assume for now that

$$\left(\frac{\partial P}{\partial y}\right)(a, b, c) \neq 0.$$

Applying Lemma 1.14 to $P(a, b, c)$ yields

$$a\frac{\partial P}{\partial x}(a, b, c) + b\frac{\partial P}{\partial y}(a, b, c) + c\frac{\partial P}{\partial z}(a, b, c) = dP(a, b, c) = 0.$$

By our assumption, $a = c = 0$ is therefore impossible as it would require $b = 0$. So assume that $c \neq 0$. Then by the homogeneity of P (and its partial derivatives)

$$\frac{\partial P}{\partial y}\left(\frac{a}{c}, \frac{b}{c}, 1\right) = c^{-(d-1)}\frac{\partial P}{\partial y}(a, b, c) \neq 0.$$

We can thus apply the complex implicit function theorem[1] to the polynomial $P(x, y, 1)$ in x and y. This tells us that there exist open neighborhoods V and W of $\frac{a}{c}$ and $\frac{b}{c}$ respectively in $\mathbb{C}$ and a holomorphic function $g : V \to W$ such that if $x \in V$ and $y \in W$ then

$$P(x, y, 1) = 0 \Leftrightarrow y = g(x).$$

Define U by

$$\begin{aligned} U &= \left\{[x, y, z] \in C : z \neq 0, \frac{x}{z} \in V, \frac{y}{z} \in W\right\} \\ &= \{[x, y, 1] \in C : x \in V, y \in W\}. \end{aligned}$$

It is easy to see that U contains $[a, b, c]$ and that U is open in $\mathbb{C}^3$ when viewed as a subset of $\mathbb{C}^3$. Thus U is open in C. The projection map $\varphi : U \to V$ defined by

$$\varphi[x, y, z] = \frac{x}{z}$$

is continuous and its inverse is given by

$$w \mapsto [w, g(w), 1].$$

Hence U is mapped homeomorphically to V.

Take any $[a, b, c] \in C$. Following the argument above (potentially with x or z instead of y), we get a neighborhood U homeomorphically mapped to V which is homeomorphic to $\mathbb{C} \simeq \mathbb{R}^2$ which proves (ii).

[1]Kirwan [1992], Appendix B

Take any $[a, c] \in \mathbb{P}^1 \setminus \phi(R)$ and let $[a, b_i, c]$ for $1 \leq i \leq d$ be all the preimages of $[a, c]$ in ϕ.

By definition of $\phi(R)$ it holds that $\left(\frac{\partial P}{\partial y}\right)(a, b_i, c) \neq 0$ for all i. Thus the argument above can be performed with y. Moreover, a and c are the same for all preimages so we may WLOG assume that $c \neq 0$ for all i. Following the argument, we find a homeomorphism $\varphi_i : U_i \to V_i$ for all i where we may assume that U_i are pairwise disjoint. Now define

$$V = \bigcap_{1 \leq i \leq d} V_i.$$

Then V is an open neighborhood of $\frac{a}{c}$ in $\mathbb{C}$ homeomorphic to some $U_i' \subset U_i$ for each i.

Consider the embedding $\theta : V \to \mathbb{P}^1$ given by

$$\theta(\frac{x}{z}) = \left[\frac{x}{z}, 1\right] = [x, z].$$

By Remark 1.3 θ is a homeomorphism onto its image. Consequently, $\phi : U_i' \to \mathbb{P}^1$ is a homeomorphism onto its image as it is the composition of φ_i and θ restricted onto U_i'. Thus (i) is now proved.

□

Remark 3.8. Let $\phi : C \to \mathbb{P}^1$ be a branched cover of $\mathbb{P}^1$ and suppose that $[0, 1, 0] \notin C$. Any given $[a, b, c] \in C$ satisfies $a \neq 0$ or $c \neq 0$, let us assume the latter. The polynomial $P(x, y, 1)$ in y is of degree n, so we can multiply it by a scalar so that it is monic of degree n in y.

Consider the map $\phi' : C \cap U_2 \to U_1$ defined on its domain by $\phi' = \theta_1 \circ \varphi \circ \theta_2^{-1}$ where U_1, U_2, θ_1, θ_2 are defined as in Remark 1.3 and φ is defined as in the statement of Corollary 1.18. The individual steps of ϕ' map any $[a, b, 1] \in C \cap U_2$ to

$$[a, b, 1] \mapsto (a, b) \mapsto a \mapsto [a, 1],$$

showing that ϕ' is a restriction of ϕ.

By Corollary 1.18, any $z_0 \in \mathbb{C}$ has an open neighborhood G in $\mathbb{C}$ such that each connected component of $\varphi^{-1}(G)$ contains at most one point of $\varphi^{-1}(z_0)$. Since θ_1 and θ_2 are homeomorphisms onto their images (see Remark 1.3) and ϕ' is a restriction of ϕ, any $[z_0, 1] \in U_1$ has an open neighborhood G in U_1 such that each connected component of $\phi^{-1}(G)$ contains at most one point of $\phi^{-1}[z_0, 1]$. It is easy to see that G is open also in $\mathbb{P}^1$.

Similarly, if $a \neq 0$ then we get the same result with $\mathbb{P}^1 \setminus \{[0, 1]\}$ instead of U_1. It follows that the first assumption of Theorem 1.28 is satisfied for the branched cover $\phi : C \to \mathbb{P}^1$ of $\mathbb{P}^1$.

To find a triangulation, we will first find a triangulation of $\mathbb{P}^1$, and then we will lift it to C.

Lemma 3.9. *Let $\{p_1, \ldots, p_r\}$ be any set of at least three points in $\mathbb{P}^1$. Then there is a triangulation of $\mathbb{P}^1$ with $p_1, \ldots, p_r$ as its vertices and with $3r - 6$ edges and $2r - 4$ faces.*

Proof. We will use induction on $r \geq 3$. When $r = 3$, Lemma 1.6 tells us that there is a projective transformation taking p_1 to 1, p_2 to $e^{\frac{2\pi i}{3}}$ and p_3 to $e^{\frac{4\pi i}{3}}$ (here we identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$). We can then join these three points by segments of the unit circle in $\mathbb{C}$.

The exterior of the unit circle is mapped to the interior of the unit circle by the projective transformation $z \mapsto \frac{1}{z}$ (defined by the linear transformation $(x, y) \mapsto (y, x)$). Since there is a homeomorphism

$$\triangle \to \{z \in \mathbb{C} : |z| \leq 1\}$$

sending the vertices of $\triangle$ to 1, $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$, there exists a triangulation of $\mathbb{P}^1$ with three edges and two faces when $r = 3$.

Now suppose that $r > 3$ and let (V, E, F) be a triangulation with vertices $p_1, \dots, p_{r-1}$, $3r - 9$ edges and $2r - 6$ faces. We now consider two cases based on the position of p_r.

Suppose that p_r lies in the interior of some $f \in F$. Then we can subdivide f by connecting the vertices of f with p_r, adding two more faces and three more edges.

On the other hand, if p_r lies in the interior of some $e \in E$, we can subdivide both f_e^+ and f_e^- by connecting p_r with their vertices not lying on e. This argument gives a triangulation with two more faces, two new edges, and e subdivided into two edges.

Either way, the resulting triangulation has a correct number of edges and faces, so we are done. □

Theorem 3.10. *Let $\phi : C \to \mathbb{P}^1$ be a branched cover of $\mathbb{P}^1$. Suppose that (V, E, F) is a triangulation of $\mathbb{P}^1$ such that the set V contains the branch locus $\phi(R)$ of ϕ. Then there is a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C such that*

$$\tilde{V} = \phi^{-1}(V),$$

$$\tilde{E} = \{\tilde{e} : I \to C : \tilde{e} \text{ continuous}, \phi \circ \tilde{e} \in E\}$$

and

$$\tilde{F} = \left\{\tilde{f} : \triangle \to C : \tilde{f} \text{ continuous}, \phi \circ \tilde{f} \in F\right\}.$$

Moreover, if $\nu_\phi(p)$ is the ramification index of ϕ at p and d is the degree of C then

$$\#\tilde{V} = d\#V - \sum_{p \in R} (\nu_\phi(p) - 1),$$

$$\#\tilde{E} = d\#E$$

and

$$\#\tilde{F} = d\#F.$$

Proof. First, Claim 3.7 and Remark 3.8 tell us that the assumptions for Theorem 1.28 are satisfied for the map ϕ. We must show that $(\tilde{V}, \tilde{E}, \tilde{F})$ satisfy the definition of the triangulation and the formulas for $\#\tilde{V}$, $\#\tilde{E}$ and $\#\tilde{F}$ are correct.

By Theorem 1.28, if $f \in F$, $t \in \triangle^{\setminus V}$ and $p \in \phi^{-1}(f(t))$, then there is a unique continuous lift $\tilde{f} : \triangle \to C$ of f such that $\tilde{f}(t) = p$. By Lemma 2.3, $\phi^{-1}(f(t))$ consists of exactly d points of C (since $f(t)$ does not belong to the branch locus

$\phi(R)$), therefore there are exactly d continuous lifts of f. Since $\tilde{F}$ is defined as the set of all lifts of all $f \in F$, we have

$$\#\tilde{F} = d\#F.$$

It is easy to see that

$$\begin{aligned} C \setminus \phi^{-1}(V) &= \phi^{-1}\left\{f(t) : f \in F, t \in \triangle^{\setminus V}\right\} \\ &= \left\{\tilde{f}(t) : \tilde{f} \in \tilde{F}, t \in \triangle^{\setminus V}\right\}. \end{aligned}$$

In particular

$$G = \bigcup_{\tilde{f} \in \tilde{F}} \tilde{f}(\triangle)$$

contains $C \setminus \phi^{-1}(V)$, and because $\phi^{-1}(V)$ is finite by Lemma 2.3, $C \setminus G$ is also finite. However, $\triangle$ is compact so $\tilde{f}(\triangle)$ is compact for every $\tilde{f} \in \tilde{F}$ so G is compact so G is closed in C (here we use Claim 3.7). But then $C \setminus G$ is open in C, hence it is empty by Claim 3.7. It follows that

$$\phi^{-1}(V) = \left\{\tilde{f}(t) : \tilde{f} \in \tilde{F}, t \in \{(0,0),(1,0),(0,1)\}\right\}.$$

By Remark 1.29, if $e \in E$, $t \in (0,1)$ and $p \in \phi^{-1}(t)$, then there is a unique continuous lift $\tilde{e}$ of e such that $\tilde{e}(t) = p$. Moreover by Theorem 1.27 the restriction of $\tilde{e}$ to $(0,1)$ is a homeomorphism onto its image in C. Therefore (ii) of Definition 3.1 easily follows using the lifting property of members of $\tilde{E}$. Furthermore if $t \in (0,1)$ and $e \in E$ then $\phi^{-1}(e(t))$ consists of exactly d points of C (again because $e(t) \notin \phi(R)$). It follows that there are exactly d continuous lifts of e, hence

$$\#\tilde{E} = d\#E.$$

If $\tilde{f} \in \tilde{F}$ then $\phi \circ \tilde{f} \in F$ which means either $\phi \circ \tilde{f} \circ \sigma_i \in E$ or $\phi \circ \tilde{f} \circ \sigma_i \circ r \in E$ for every $1 \leq i \leq 3$ where σ_1, σ_2, σ_3 are defined as in Definition 3.1 (iii). Thus either $\tilde{f} \circ \sigma_i \in \tilde{E}$ or $\tilde{f} \circ \sigma_i \circ r \in \tilde{E}$ (proving the second half of the condition (iii) of Definition 3.1) so

$$\tilde{f}(t) \in \left\{\tilde{e}(0) : \tilde{e} \in \tilde{E}\right\} \cup \left\{\tilde{e}(1) : \tilde{e} \in \tilde{E}\right\}$$

if $t \in \{(0,0),(1,0),(0,1)\}$. Consequently

$$\left\{\tilde{e}(0) : \tilde{e} \in \tilde{E}\right\} \cup \left\{\tilde{e}(1) : \tilde{e} \in \tilde{E}\right\} \supset \phi^{-1}(V).$$

The opposite inclusion easily follows from the lifting property of each $\tilde{e} \in \tilde{E}$, so in fact, we have equality which is exactly the condition (i) of Definition 3.1.

It also follows that

$$\phi^{-1}\{e(t) : e \in E, t \in (0,1)\} = \left\{\tilde{e}(t) : \tilde{e} \in \tilde{E}, t \in (0,1).\right\}$$

Therefore if we define Γ as in the condition (iii) of Definition 3.1 then

$$\phi^{-1}(\Gamma) = \phi^{-1}(V) \cup \left\{\tilde{e}(t) : \tilde{e} \in \tilde{E}, t \in (0,1)\right\} = \tilde{\Gamma}$$

where we define

$$\tilde{\Gamma} = \bigcup_{\tilde{e} \in \tilde{E}} \tilde{e}\,(I)\,.$$

By Theorem 1.27 if $\tilde{f} \in \tilde{F}$ then the restriction of $\tilde{f}$ to $\triangle^o$ is a homeomorphism onto its image which is a connected component of $\phi^{-1}(f(\triangle^o))$.

Now suppose that there exists a connected set U such that $\tilde{f}(\triangle^o) \subsetneq U \subset C \setminus \tilde{\Gamma}$. Then it follows from the previous paragraph that there exists an $u \in U \setminus \phi^{-1}(f(\triangle^o))$. But then

$$\phi(U) \supset f(\triangle^o) \cup \phi(u) \supsetneq f(\triangle^o)$$

is also connected contradicting the fact that $f(\triangle^o)$ is a connected component of $\mathbb{P}^1 \setminus \Gamma$. This proves the first half of the condition (iii) of Definition 3.1 i.e. that $\tilde{f}(\triangle^o)$ is a connected component of $C \setminus \tilde{\Gamma}$. It follows easily that these are the only connected components of $C \setminus \tilde{\Gamma}$ proving (iv) of Definition 3.1.

If $\tilde{e} \in \tilde{E}$ then $\phi \circ \tilde{e} \in E$. Therefore for some $1 \leq i \leq 3$ it holds that $\phi \circ \tilde{f}_e^+ \circ \sigma_i = f_e^+ \circ \sigma_i = \phi \circ \tilde{e}$ where $\tilde{f}_e^+$ is a lift of f which satisfies

$$\tilde{f}_e^+(\sigma_i(t)) = \tilde{e}(t)$$

for some $t \in (0,1)$ (again using Theorem 1.28). By Remark 1.29 $\tilde{f}_e^+ \circ \sigma_i$ and $\tilde{e}$ agree on all $t \in I$. We can then perform the same argument with f_e^- instead of f_e^+ which together prove the condition (v) of Definition 3.1.

Finally, we want to show that

$$\#\tilde{V} = d\#V - \sum_{p \in R} (\nu_\phi(p) - 1)\,.$$

This follows immediately from Lemma 2.3 since V contains $\phi(R)$. □

Remark 3.11. It follows from the formulas for $\tilde{V}$, $\tilde{E}$, $\tilde{F}$ that the Euler number $\chi(C)$ of C is given by

$$\chi(C) = \#\tilde{V} - \#\tilde{E} + \#\tilde{F} = d\chi(\mathbb{P}^1) - \sum_{p \in R} (\nu_\phi(p) - 1)\,.$$

This formula is called the *Riemann-Hurwitz formula* for the branched cover $\phi : C \to \mathbb{P}^1$.

We have already done all the work needed to prove the degree-genus formula. All that remains is to put everything together.

Theorem 3.12. (The degree-genus formula) *Let C be a nonsingular algebraic curve of degree d in $\mathbb{P}^2$. The Euler number χ and genus g of C are given by*

$$\chi = d(3-d)$$

and

$$g = \frac{1}{2}(d-1)(d-2)$$

.

Proof. By Lemma 2.5, we can assume that after applying a suitable projective transformation to C the map $\phi : C \to \mathbb{P}^1$ defined by

$$\phi\,[x, y, z] = [x, z]$$

is well-defined and the ramification index $\nu_\phi\,[a, b, c]$ of ϕ at every $[a, b, c] \in C$ satisfies

$$\nu_\phi\,[a, b, c] \leq 2.$$

Then by Lemma 2.4 ϕ has exactly $d(d-1)$ ramification points i. e. $\#R = d(d-1)$.

By Lemma 3.9 if $r \geq \max(3, d(d-1))$ then we can choose a triangulation (V, E, F) of $\mathbb{P}^1$ such that $V \supset \phi(R)$ and $\#V = r$, $\#E = 3r - 6$ and $\#F = 2r - 4$. By Theorem 3.10 there is a triangulation $(\tilde{V}, \tilde{E}, \tilde{F})$ of C with

$$\#\tilde{E} = d\#E = 3(r-2)d,$$

$$\#\tilde{F} = d\#F = 2(r-2)d$$

and

$$\#\tilde{V} = d\#V - \sum_{p \in R} (\nu_\phi(p) - 1)\,.$$

Since $\#R = d(d-1)$ and $\nu_\phi(p) = 2$ for every $p \in R$ we have

$$\#\tilde{V} = rd - d(d-1).$$

A straightforward calculation then shows that

$$\chi = d(3-d)$$

and

$$g = \frac{1}{2}(d-1)(d-2),$$

as required. □

Corollary 3.13. *Let C and D be nonsingular* homeomorphic *projective curves of degrees n and m in $\mathbb{P}^2$. Then either $n = m$ or $\{n, m\} = \{1, 2\}$.*

Proof. By Theorem 3.12 the Euler number of such curve of degree d is given by $\chi = d(3-d)$ which is an almost injective function on $\mathbb{N}$ (apart from 1 and 2). By Remark 3.5 C and D have the same Euler number. Thus either $n = m$ or $\{n, m\} = \{1, 2\}$. □

The proof of the degree-genus formula used the definition of genus as a linear function of the Euler number. Nevertheless, it remains to show that this is consistent with the definition of genus as a number of handles attached to the sphere.

First, a theorem states that a triangulable compact connected surface is determined up to homeomorphism by the number of handles.

Theorem 3.14. *Any compact connected surface C with triangulation in the sense of Definition 3.1 is homeomorphic to a sphere with g handles.*

Proof. See Kirwan [1992], Appendix C.3. □

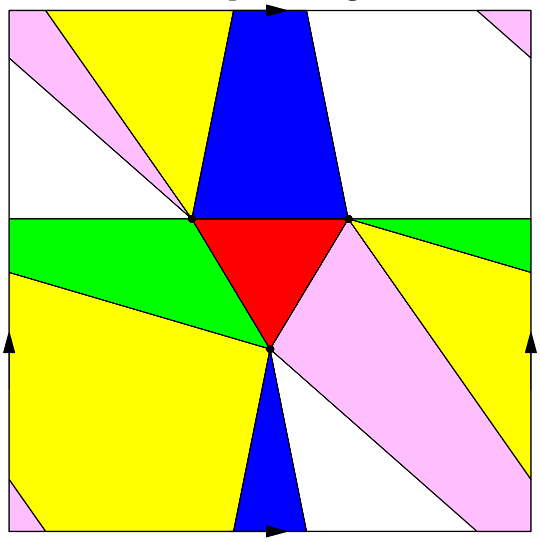
Remark 3.15. We want to show that the two definitions of genus match (at least when C is connected). We will present an informal argument here; the reader should refer to Wilton [2019], Chapter 6, Section 4 for rigorous proof of both Theorem 3.3 and this Remark.

Let Γ_g be a sphere with g handles. We can homeomorphically shrink each handle so that each handle influences a triangulation only locally. Hence the Euler characteristic of a surface is a linear function of a number of handles.

If $g = 0$ then Γ_g has a triangulation with 3 vertices on the equator, 2 faces and 3 edges. Thus $\chi = 2$, as we wished.

If $g = 1$ then there is a triangulation (shown in Figure 3.2) of Γ_g with 3 vertices, 6 faces colored each colored by a respective color and 9 edges. Hence in this case $\chi = 0$, again as we wished.

Figure 3.2: An example triangulation of a torus.



Both of the definitions of genus are linear in the Euler number and they coincide at two different values $g = 0$ and $g = 1$. Hence both of them are equivalent (and $g = \frac{1}{2}(2 - \chi)$ holds).

It follows that if C is a connected projective curve in $\mathbb{P}^2$ of degree d then C is homeomorphic to a sphere with $\frac{1}{2}(d-1)(d-2)$ handles.

Last but not least, we want to show that the assumption of nonsingularity is indeed warranted.

Remark 3.16. Let C be a projective curve defined by the polynomial $P(x, y, z) = y^2z - x^3$. It is easy to show that C is irreducible. Then

$$\frac{\partial P}{\partial x} = 3x^2, \qquad \frac{\partial P}{\partial y} = 2yz, \qquad \frac{\partial P}{\partial z} = y^2$$

vanish simultaneously at a single singular point $[0, 0, 1] \in C$. Thus C is nonsingular.

Consider a map $f : \mathbb{P}^1 \to C$ defined by $f\,[s, t] = [s^2t, s^3, t^3]$. We will show that f has an inverse $g : C \to \mathbb{P}^1$ given by

$$g\,[x, y, z] = \begin{cases} [0, 1] & \text{if } x = y = 0, \\ [y, x] & \text{otherwise.} \end{cases}$$

First, f and g map $[0,1]$ and $[0,0,1]$ to each other. Otherwise, both s and y are nonzero. Then

$$g \circ f\,[s,t] = g\left[s^2 t, s^3, t^3\right] = \left[s^3, s^2 t\right] = [s,t]$$

and

$$f \circ g\,[x,y,z] = f\,[y,x] = \left[y^2 x, y^3, x^3\right] = \left[y^2 x, y^3, y^2 z\right] = [x,y,z]$$

proving that f and g are inverse maps.

Choose any open set G in C. Then there exists an open set $G' \subset \mathbb{C}^3 \setminus \{0\}$ such that $\pi(G') \cap C = G$. Its preimage in f (considered as an affine map) is again open, therefore $f^{-1}(G)$ is also open. Thus f is continuous.

The converse of the previous idea shows that g is continuous at each $c \in C \setminus [0,0,1]$. Consider a neighborhood U of $[0,1]$ in $\mathbb{P}^1$. Then there exists $M \in \mathbb{R}^+$ such that if $[s,t]$ satisfies $|t| > |s| \cdot M$ then $[s,t] \in U$. Thus the preimage of U in g consists at least of all points $[x,y,z]$ of C satisfying $|z| > \max(|x|, |y|) \cdot M^2$ which is a neighborhood of $[0,0,1]$ in C. Thus g is continuous.

It follows that C is homeomorphic to $\mathbb{P}^1$ so $\chi(C) = \chi(\mathbb{P}^1) = 2$. This does not agree with the degree-genus formula. Hence, it cannot be applied to singular curves.

Conclusion

In the thesis, we managed to prove the classical degree-genus formula for nonsingular projective curves in $\mathbb{P}^2$ by producing a suitable triangulation of the curve C. Simply put, we showed that the topology of a projective nonsingular curve in $\mathbb{P}^2$ depends only on the degree of the polynomial defining it. In the proof, we introduced important topological invariants such as the Euler number and genus. Then we briefly sketched how our result relates to the other definition of genus as a number of handles attached to a sphere. Finally, we showed that the degree-genus formula does not hold in general for singular curves.

Our proof utilized techniques from general topology, algebraic topology, where we developed some theory around covering spaces, and complex analysis. We analyzed the branched cover $\phi : C \to \mathbb{P}^1$ of $\mathbb{P}^1$ and showed that it acts as a near-perfect covering projection (excluding a finite set of ramification points).

Several generalizations of the degree-genus formula are also known. If a curve C is irreducible (and possibly singular) then after performing a projective transformation in the sense of Lemma 2.5 it holds that

$$g = \frac{1}{2}(d-1)(d-2) - \frac{1}{2} \sum_{p \in Sing(C)} \left(I_p \left(P, \frac{\partial P}{\partial y} \right) - \nu_\phi(p) + \#\pi^{-1}\{p\} \right)$$

where $Sing(C)$ is the finite set of singular points of C, $I_p(Q, R)$ is the intersection number of Q and R at p and $\pi : \tilde{C} \to C$ is the map from the resolution of singularities of C. This equation is called Noether's formula. In particular, there exists a way to compute the genus despite it not depending solely on the degree.

Another generalization comes from the study of curves in higher dimensions. It turns out that the formula for the arithmetic genus of a nonsingular hypersurface H in $\mathbb{P}^n$ defined by a homogeneous polynomial of degree d is a natural extension of the degree-genus formula. Precisely,

$$g = \binom{d-1}{n}.$$

Further work could be done to prove any of the two generalizations. In addition to that, one could modify the topological arguments from this thesis to prove the general form of the Riemann-Hurwitz formula for a general map between projective algebraic curves. Here, a more general theory of ramification indices needs to be developed.

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