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Quotients in Algebraic Geometry

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Abstract: This thesis is concerned with the existence of pushouts in two different settings of algebraic geometry. At first, we study the pushouts in the category of affine algebraic sets over an infinite field. We show that this can be regarded as an instance of much general problem whether the pullback of finitely generated algebras over a commutative Noetherian ring is finitely generated. We give a partial solution to this problem and study some examples. Secondly, we examine the existence of pushouts in the category of schemes with an emphasis on diagrams of affine schemes. We use the methods of Ferrand [2003] and Schwede [2004] and generalise some of their results. We conclude by giving some examples and suggest another approach to the problem.

Keywords: algebraic geometry, commutative algebra, pullbacks, pushouts

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Introduction

An effort to form quotients was and still is one of the main driving forces in the development of algebraic geometry. While it is very easy and natural to perform quotient constructions in both algebra and topology, doing in it in the context of algebraic geometry is seldom so. Due to the compatibility requirements on their algebraic and geometric structures, objects in algebraic geometry are remarkably rigid. In taking quotients in the setting of algebraic geometry, one must manipulate both algebraic and geometric parts of objects at the same time to ensure that they remain compatible which is often impossible. Therefore one often has to decide whether to forgo the desired degree of compatibility or the universal property.

Most of these sought after quotients originated from group actions on algebro-geometric objects. It is worth noting that David Hilbert laid foundations of algebraic geometry as a field of mathematics in investigating the rings of invariants of classical group actions. Non-existence of many group quotients in classical algebraic geometry was one the crucial motivations for relaxing compatibility requirements on algebro-geometric objects and introduction of very abstract objects, like algebraic spaces or stacks, that play a pivotal role in contemporary algebraic geometry.[1]

However, in this thesis, we shall deal with pushouts which are another class of quotient constructions. In contrast to group quotients, which have more non-equivalent definitions, pushout (or amalgamated sum) is a well defined categorial notion. *Pushout* is the colimit of a pair of morphisms with common domain or given a diagram:

$$\begin{array}{ccccc} & & X & & \\ & \overset{\varphi}{\swarrow} & & \overset{\psi}{\searrow} & \\ Y & & & & Z \end{array}$$

then an object W together with morphisms $\iota_1 : Y \to W$ and $\iota_2 : Z \to W$ such that $\iota_1\varphi = \iota_2\psi$ is the pushout of this diagram if for any object W' and morphisms $\eta_1 : Y \to W$ and $\eta_2 : Z \to W$ such that $\eta_1\varphi = \eta_2\psi$ there exists a unique morphism $\pi : W \to W'$ such that $\eta_i = \pi\iota_i$ for $i = 1, 2$.

The dual notion to pushout is *pullback* which is the limit of a pair of morphisms with a common codomain. Due to presence of many contravariant functors in algebraic geometry, we will also deal with pullbacks very often.

Compared to group quotients which are a vibrant of study, the topic of existence of pushouts in categories of algebro-geometric objects has attracted relatively little attention so far. The main article is *Conducteur, descent, et pincement* (in French) by Daniel Ferrand published in the Bulletin of French Mathematical Society in 2003 which is a corrected version of author's dissertation from 1970's. Ferrand [2003] deals, among other things, with special pushouts in the category of schemes (one of the morphisms in the pushout datum is required to be a closed

[1] For a more detailed discussion, we refer the reader to Sections 1.3 and 1.7 in Eisenbud [1995].

immersion and the other is required to be affine). His strategy is to give sufficient conditions when the pushout, which always exists, of a diagram of schemes in the category of ringed spaces is also a scheme, and thus a pushout in the category of schemes.

This approach was also taken by Karl Schwede in an article *Gluing schemes and a scheme without closed points* published in 2004. Schwede [2004] independently discovered some of the results given by Ferrand [2003] and worked some interesting examples and counterexamples.

Kezheng Li, in his article *Push-outs of schemes* published in 2007, studied the existence of pushouts of schemes under some strict conditions on the morphisms of the pushout datum (e.g. flatness or finiteness). Under these strong assumptions, Li [2007] was able to give some sufficient and equivalent conditions for the existence of pushout and, moreover, grupoid quotients in the category of schemes.

Most recently, Michael Temkin and Ilya Tyomkin asked about the existence of pushouts investigated by Ferrand [2003] in the category of algebraic spaces in their article *Ferrand pushouts for algebraic spaces* from 2016. Temkin and Tyomkin [2016] managed to extend Ferrand's results to the category of algebraic spaces and pointed out the connections of the existence of pushouts with several other problems in the study of algebraic spaces.

In this thesis, we follow up on the articles of Ferrand and Schwede in examining the existence of puhsouts in the category of schemes and analyse the existence of pushouts in the context of classical algebraic geometry, specifically in the category of affine algebraic sets, as well. The latter problem does not appear to be treated in the literature at all, both Schwede [2004] and Temkin and Tyomkin [2016] point to its instances in some examples, though.

This work is thus divided into two chapters. The first chapter opens with a discussion of the existence of pushouts in the category of affine algebraic sets over an infinite field. We use the contravariant equivalence between the category of affine algebraic sets and the category of coordinate rings to equivalently reformulate the problem in a purely algebraic way − we ask whether the pullback a diagram of coordinate rings, which always exists in the category of algebras over the ground field, is a finitely generated algebra.

It turns out that this can be conveniently dealt with in a more general setting of finitely generated algebras over a commutative Noetherian ring. We provide a complete solution of the problem of existence of some pullbacks in such a category and give a partial solution to other instances. We conclude the chapter by considering intersections of finitely generated algebras and by analysing local properties of some quotients of affine algebraic sets.

In the second chapter, we mainly elaborate on the results of Ferrand [2003] and Schwede [2004] about the existence of pushouts in the category of schemes, particularly of the diagrams of affine schemes. We begin by characterising when the pushout of a diagram of affine schemes in the category of ringed spaces is the affine schemes that corresponds to the pullback of the corresponding diagram of global sections. The result is that we can limit ourselves to diagrams of affine schemes where all morphisms correspond to inclusions of commutative rings. We give some sufficient conditions and examples of the spectrum of intersection of two commutative subrings of a commutative ring being equal to the amalgamated sum of their respective spectra.

Subsequently, we try to generalise and allow the pushouts of diagrams of affine schemes to be merely schemes. Using a result related to the Ferrand's main theorem, we are able to give some sufficient conditions for their existence. Finally, we abandon the, as it emerges, overly restrictive approach of trying to prove that the pushout of a diagram of schemes in the category of ringed spaces is also a scheme itself. We prove that it suffices for a scheme to be in some sense close to that general pushout to be the pushout of the diagram in the category of schemes and show that such conditions are natural for diagrams affine schemes.

Main references for standard results and definitions are David Eisenbud's *Commutative Algebra with a View towards Algebraic Geometry* for commutative algebra and *Algebraic Geometry I: Schemes With Examples and Exercises* by Ulrich Görtz and Torsten Wedhorn for algebraic geometry.

The results of the first chapter were presented at the Prague Algebra Seminar in November 2017. An earlier version of this work was submitted to SVOČ (a Czech and Slovak competition in mathematical research for university students) in May 2018 and it won a shared first place in the M5+M6 category. Parts of this thesis will be sent for publication, the author gratefully acknowledges support from grant GACR 17-23112S of the Czech Science Foundation to this end.

1. Pullbacks of finitely generated algebras

In this chapter, we will be motivated by investigation of the existence of pushouts in the category of affine algebraic sets over an infinite field. Throughout, we will make free use of basic results and definitions of affine algebraic geometry as described for example in Chapter 1 of Görtz and Wedhorn [2010].

1.1 Pushouts of affine algebraic sets

Let K be an infinite field, consider, for example, these diagrams of algebraic sets with natural maps:

$$\begin{array}{ccc} & \mathbb{A}^1_K & \\ \swarrow & & \searrow \\ \mathbb{A}^0_K & & \mathbb{A}^2_K \end{array}$$

the pushout of which would correspond to contracting a line in a plane into a single point or:

$$\begin{array}{ccc} & \mathbb{A}^0_K \sqcup \mathbb{A}^0_K & \\ \swarrow & & \searrow \\ \mathbb{A}^0_K & & \mathbb{A}^1_K \end{array}$$

the pushout of which would correspond to identifying two points of on affine line. It is folklore knowledge that the first pushout does not exists in the category of affine algebraic sets (see Example 3.5 on page 7 in Schwede [2004]), but the second one does. In the following text, we will try to characterise under which circumstances we can form a pushout of affine algebraic sets or glue two affine algebraic sets via another algebraic set that maps to them in the most general way possible.

The existence of pushouts P and Q respectively, would, through contravariant equivalence of the category of algebraic sets and the category of coordinate rings, mean that $K[P]$ and $K[Q]$ are pullbacks in the category of coordinate rings of the respective diagrams:

$$\begin{array}{ccc} & K[P] & \\ \swarrow & & \searrow \\ K & & K[x,y] \\ \searrow & & \swarrow \\ & K[x] & \end{array} \qquad \begin{array}{ccc} & K[Q] & \\ \swarrow & & \searrow \\ K & & K[x] \\ \searrow & & \swarrow \\ & K^2 & \end{array}$$

Moreover, $K[P]$ and $K[Q]$, which are finitely generated as algebras over K, need to be the pullbacks of the respective diagrams in the category of $K-$algebras[1].

[1] Let A, B, C be $K-$algebras and let $\varphi : B \to A$ a $\psi : C \to A$ be their homomorphisms. Then the pullback of the corresponding diagram is isomorphic to $\{(b, c) \in B \times C \mid \varphi(b) = \psi(c)\}$.

This is due to the following lemmas:

Lemma 1. *Let X be a non-empty $K-$algebraic set, then any finitely generated $K-$subalgebra of $K[X]$ is a coordinate ring of some $K-$algebraic set.*

Proof. Let B be a finitely generated $K-$subalgebra of $K[X]$ with generators $\Phi_1, \ldots, \Phi_n$. There exists a unique homomorphism $\varphi : K[x_1, \ldots, x_n] \to B$ such that $x_i \mapsto \Phi_i$ for all $i \in \{1, \ldots, n\}$. However, φ can be thought of as a homomorphism from $K[x_1, \ldots, x_n]$ to X. Then φ gives rise to a polynomial map $\Phi = (\Phi_1, \ldots, \Phi_n) : X \to \mathbb{A}^n_K$. Suppose that a polynomial $f \in K[x_1, \ldots, x_n]$ vanishes on $\Phi(X)$, that means for all $x \in X$ we have $f(\Phi_1(x), \ldots, \Phi_n(x)) = 0$, for this we need that K is an infinite field. Let us have $f \in \operatorname{Ker} \varphi$, then:

$$0 = f(\Phi_1, \ldots, \Phi_n)(x) = f(\Phi_1(x), \ldots, \Phi_n(x))$$

for all $x \in X$, hence f vanishes on $\Phi(X)$, we established that $\operatorname{Ker} \varphi = I(\Phi(X))$. Furthermore, noting that $I(V(I(\Phi(X)) = I(\Phi(X))$, we observe that:

$$B \cong K[x_1, \ldots, x_n]/I(\Phi(X))$$

by the first isomorphism theorem, it is therefore the coordinate ring of $K-$algebraic set $V(I(\Phi(X))$. □

Lemma 2. *Suppose $\varphi : B \to A$ a $\psi : C \to A$ are homomorphisms of coordinate rings of affine algebraic sets over K, then D with morphisms θ_B, θ_C is the pullback of the corresponding diagram in the category of coordinate rings if and only if D is also the pullback of corresponding diagram in the category of all $K-$algebras.*

Proof. We know that the pullback of the corresponding diagram in the category of all $K-$algebras is of form $P = \{(b, c) \in B \times C \mid \varphi(b) = \psi(c)\}$ with η_B, η_C projections. However, B and C are coordinate rings of $K-$algebraic varieties X and Y respectively. This implies that $B \times C$ is a coordinate ring of $X \sqcup Y$. Therefore P is a finitely generated subalgebra is a coordinate ring.

($\Leftarrow$) Provided that P is finitely generated as an algebra over K, it is also a coordinate ring by the Lemma 1 and $D \cong P$ by the universal property of pullback.

($\Rightarrow$) Suppose that D is not pullback of corresponding diagram in the category of all $K-$algebras. If P is finitely generated, proceed by the previous paragraph. Assume that P is not finitely generated and D is the pullback in the category of coordinate rings. By virtue of P being pullback of the diagram in the category of all $K-$algebras, there is a homomorphism $\varrho : D \to P$ so that projections from D to B and C factor through that. Therefore $\theta_B = \eta_B \rho$ and $\theta_C = \eta_C \rho$. Clearly, this means that $\varrho(D)$ with projections defined as restrictions of η_B and η_C is also the pullback in the category of finitely generated $K-$algebras. This is due to the universal property of the pullbacks D and P in respective categories.

By virtue of P not being finitely generated, thus, $\operatorname{Im} \eta_B$ or $\operatorname{Im} \eta_C$ is not finitely generated by Proposition 6. Without loss of generality, assume that $\operatorname{Im} \eta_B$ is not finitely generated and find a finitely generated subalgebra D' of P such that

$\eta_B(\varrho(D))$ is strictly smaller than $\eta_B(D')$. We infer by Lemma 1 that D' is a coordinate ring.

This yields a contradiction as restriction of η_B to D' clearly does not factor through η_B restricted to $\varrho(D)$ − the assumed pullback of the diagram in the category of coordinate rings. □

Therefore we have successfully translated a geometric question about the existence of pushouts of $K-$algebraic sets to the question whether ring-theoretic pullbacks of induced diagrams of their coordinate rings are finitely generated.

It turns out, however, that the question can be conveniently generalised to a more general setting of finitely generated algebras over an arbitrary Noetherian ring and to establish strong finiteness conditions on the existence of finitely generated pullbacks we will need to use only very general results of commutative algebra.

1.2 Preliminary results

At first, we recall some two theorems of commutative algebra, of which we shall make an extensive use, and set some conventions for this chapter.

Theorem 3 (Hilbert basis theorem; Theorem 1.2 and Corollary 1.3 in Eisenbud [1995], pages 27 and 28)**.** *If a ring R is Noetherian, then the polynomial ring $R[x]$ is Noetherian. Furthermore any finitely generated algebra over R is Noetherian.*

We shall also often refer to the fact that a finitely generated module over a Noetherian ring is Noetherian, which is expressed in Eisenbud [1995] by Proposition 1.4, page 28. We will also use all three isomorphisms theorems quite frequently.

Theorem 4 (Artin-Tate lemma; Theorem in Exercise 4.32 in Eisenbud [1995], page 143)**.** *Suppose R is a Noetherian ring and S is a finitely generated R-algebra. If $T \subseteq S$ is a $R-$algebra such that S is a finitely generated $T-$module, then T is a finitely generated $R-$algebra.*

Throughout this chapter, R will denote a Noetherian commutative unital ring. We note that for $R-$algebras $S \subseteq B$ and ideal $I \subseteq B$, the quotient B/I has a natural structure of an $S-$module given by multiplication by elements $s + I$ for $s \in S$.

We begin by establishing some general results that will allow us to simplify our problem even further.

Proposition 5. *Suppose S is a $R-$algebra with ideals $I, J \subseteq S$ such that both S/I and S/J are finitely generated algebras over R, then $S/I \cap J$ is a finitely generated $R-$algebra.*

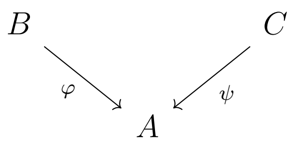
Proof. Without loss of generality, we can assume that $I \cap J = \{0\}$, otherwise we set $\tilde{S} = S/I \cap J$, $\tilde{I} = I/I \cap J$, and $\tilde{J} = J/I \cap J$.

Let us denote the canonical projections by $\pi_I : S \to S/I$ and $\pi_J : S \to S/J$. If we lift finitely many generators of S/I and S/J, we obtain S_I and S_J finitely generated $R-$subalgebras of S such that $\pi_I(S_I) = S/I$ and $\pi_J(S_J) = S/J$ respectively.

Additionally, we have that $I + J/J \cong I/I \cap J \cong I$ as $I \cap J = 0$. Since S/J is Noetherian by the Hilbert basis theorem, it can be viewed as a Noetherian S_J-module, and I is isomorphic to its submodule, we conclude that I is a Noetherian, hence finitely generated S_J-module.

Choose an arbitrary $s \in S$, as $\pi_I(S_I) = S/I$, there is an $s_I \in S_I$ such that $s - s_I \in I$. However, any element $i \in I$ can be expressed as $i = \sum_{k=1}^{n} \iota_k s_{k,J}$ for fixed $\iota_1, \dots, \iota_k \in I$ and some $s_{1,J}, \dots, s_{k,J} \in S_J$. Consequently, S is generated as a $R-$algebra by finitely many generators of S_I and S_J together with finitely many generators of I as a S_J-module. ☐

Proposition 6. *Let A, B, C be finitely generated $R-$algebras and let $\varphi : B \to A$ a $\psi : C \to A$ be their homomorphisms. Then the pullback of the corresponding diagram:*



is finitely generated $R-$algebra if and only if both $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ and $\psi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ are finitely generated as algebras over R.

Proof. Pullback of the diagram above exists in the category of commutative rings and can be expressed as $P = \{(y, z) \in B \times C, \varphi(y) = \psi(z)\}$ with projections $\pi_1 : P \to B, (y, z) \to y$ and $\pi_2 : P \to C, (y, z) \to z$. The ring P can be naturally equipped with $R-$algebra structure such that π_1 and π_2 become $R-$algebra homomorphisms. It is clear that $\pi_1(P) = \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ and $\pi_2(P) = \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ as well as $\operatorname{Ker}\pi_1 = (0, \operatorname{Ker}\psi)$ and $\operatorname{Ker}\pi_2 = (\operatorname{Ker}\varphi, 0)$.

($\Rightarrow$) We observe that $\operatorname{Ker}\pi_1 \cap \operatorname{Ker}\pi_2 = \{(0, 0)\}$. Under the assumption that $\pi_1(P) = \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ and $\pi_2(P) = \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ are finitely generated $R-$algebras, P is easily finitely generated as an algebra over R from Proposition 5.

($\Leftarrow$) If P is a finitely generated $R-$algebra, then both $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ and $\psi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ are finitely generated as its homomorphic images under π_1 and π_2 respectively. ☐

Remark (Special cases of two monomorphisms and two surjective homomorphisms). Let us retain the notation from the proposition above. Assume that φ and ψ are surjective homomorphisms, then the proposition above gives an affirmative answer to our question – as $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi) = B$ and $\psi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi) = C$ are finitely generated $R-$algebras, so is the pullback P. However, if both homomorphisms are monomorphisms, without loss of generality inclusions, the proposition is tautological. Since $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi) = \psi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi) = B \cap C$, which is also equal to the pullback P.

1.3 Subalgebras containing an ideal

Proposition 6 allows to proceed from asking whether the pullback of finitely generated algebras is finitely generated to asking if some subalgebras of finitely generated $R-$algebras are themselves finitely generated. For now, let us set aside the case when φ and ψ are both monomorphisms, using the notation of the proposition above, and suppose that φ is not a monomorphism. It follows that $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ contains an ideal of B, namely $\operatorname{Ker}\varphi \subseteq \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$.

As the ensuing discussion illustrates, this is a crucial fact, which makes determining whether $\varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$ is a finitely generated $R-$algebra much easier.

Lemma 7. *Let $S \subseteq B$ be $R-$algebras such that B is finitely generated as an algebra over R and there exists an ideal I of B which lies in S. If S is finitely generated over as an algebra R, then $B/\operatorname{Ann}_B(f)$ is a finitely generated $S-$module for each non-zero $f \in I$.*

Proof. By the Hilbert basis theorem, S is Noetherian. For each $f \in I$, we have $(f)_B \subseteq I \subseteq S$. As an ideal of S, $(f)_B$ has to be finitely generated. So there are $b_1, \dots, b_k \in B$ such that $(f)_B = (fb_1, \dots, fb_k)_S$. Choose an arbitrary $b \in B$, we can find $s_1, \dots, s_k \in S$ so that $fb = \sum_{i=1}^k fb_i s_i$. We deduce that $b - \sum_{i=1}^k b_i s_i \in \operatorname{Ann}_B(f)$ and thus $b_1 + \operatorname{Ann}_B(f), \dots, b_k + \operatorname{Ann}_B(f)$ generate $B/\operatorname{Ann}_B(f)$ as an $S-$module. □

We shall show that the result of the previous lemma can be strengthened considerably under additional assumptions, specifically if the ring is coprimary as module over itself.

Definition 8 (Prime ideals associated to a module and coprimary module; defined on pages 89 and 94 in Eisenbud [1995])**.** *Let M be an $R-$module. A prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ is associated to M if $\mathfrak{p}$ is the annihilator of an element of M. A submodule N of M is primary if only one prime is associated to M/N. We say that M is coprimary module if its zero submodule is primary.*

Lemma 9. *Suppose B is a coprimary and finitely generated $R-$algebra, then all elements of the only associated prime of B are nilpotent.*

Proof. This lemma follows easily from Proposition 3.9 on page 94 in Eisenbud [1995]. To be consistent with the notation in Eisenbud [1995], let us denote P the the only associated prime of B. Then B is $P-$coprimary as a module over itself. By b of the proposition, all elements of $B - P$ are not zero divisors. However, by c of the said proposition, a power of P annihilates B, as all elements of $B - P$ are not zero divisors, this power has to be zero. Thus, all elements of P are nilpotent. □

Proposition 10. *Suppose $S \subseteq B$ are $R-$algebras such that there exists a finitely generated ideal $I \subseteq B$ with the property that B/I is a finitely generated $S-$module and that all its elements are nilpotent, in effect B is a finitely generated $S-$module.*

Proof. Suppose $a_1, \dots, a_n \in I$ generate it as $B-$module and $a_i^m = 0$ for all $1 \leq i \leq n$. Then every element of I^{mn} is a $B-$linear combination of elements from $\{a_\varphi(1) \dots a_\varphi(mn); \text{for all functions}\, \varphi : \{1, \dots, mn\} \to \{1, \dots, n\}\}$, but for each $\varphi : \{1, \dots, mn\} \to \{1, \dots, n\}$ there is an index n_0 such that $|\varphi^{-1}(n_0)|$ is greater or equal than m, so $a_\varphi(1) \dots a_\varphi(mn) = 0$. Therefore $I^{mn} = \{0\}$.

Let us denote by $P_i = \{p_1, \ldots, p_{k_i}\}$ a finite set of generators of I^i as a $B-$module for each $1 \leq i \leq mn-1$, and by $S_B = \{s_1, \ldots, s_\ell\}$ a finite subset of B such that its image under the canonical projection generates B/I as an $S-$module. We shall show that the finite set $\bigcup_{i=1}^{mn-1} P_i S_B$ is a set of generators of an $S-$module I, where $P_i S_B = \{ab; a \in P_i, b \in S_B\}$ for all i.

If we denote $\tilde{S}_B$ the $S-$module generated by the finite set S_B, for any $b \in B$, there is a $b^S \in \tilde{S}_B$ for which $b - b^S \in I$. Choose a $p \in I^i$, there are $b_1, \ldots, b_{k_i} \in B$ such that $p = \sum_{j=1}^{k_i} p_j b_j$. It is simple to deduce that $p - \sum_{j=1}^{k_i} p_j b_j^S \in I^{i+1}$. As the element $p_j b_j^S$ can be written as an $S-$linear combination of $p_j s_1, \ldots, p_j s_\ell$ for each $1 \leq j \leq k_i$.

We can hence inductively approximate an arbitrary $p \in I$ by $S-$linear combinations of $P_i S_B$ denoted p_i such that $p - p_1 - \ldots - p_j \in I^{j+1}$. This approximation stops after $mn-1$ steps because $I^{mn} = \{0\}$. Consequently, I is contained in the $S-$module generated by the finite set $\bigcup_{i=1}^{mn-1} P_i S_B$.

It is clear now that B is a finitely generated $S-$module, since both I and B/I are finitely generated $S-$modules. □

The results of this section up to this point can be neatly put together:

Theorem 11. *Let $S \subseteq B$ be $R-$algebras such that B is finitely generated over R and coprimary as a module over itself and there exists an non-zero ideal I of B which lies in S. Then S is finitely generated over R if and only if B is a finitely generated $S-$module.*

Proof. ($\Leftarrow$) This implication follows trivially from the Artin-Tate lemma.

($\Rightarrow$) Let $\mathfrak{p} \in \operatorname{Spec} B$ be the only associated prime of B; we know that $\operatorname{Ann}_B(b) \subseteq \mathfrak{p}$ for each $b \in B$.

Since I contains a non-zero element f, by Lemma 7, $B/\operatorname{Ann}_B(f)$ is a finitely generated $S-$module. It follows obviously that $B/\mathfrak{p}$ is also a finitely generated $S-$module.

By Hilbert basis theorem, $\mathfrak{p}$ is a finitely generated $B-$module and by Lemma 9 all its elements are nilpotent. We can now directly apply Proposition 10 to obtain that B is a finitely generated $S-$module. □

It is possible to extend the result of the previous theorem to an arbitrary finitely generated algebras by using the primary decomposition of its zero ideal.

Theorem 12 (Lasker-Noether theorem or primary decomposition; Theorem 3.10 in Eisenbud [1995], page 95)**.** *Let M be a finitely generated $R-$module. Any proper submodule M' of M is the* [finite] *intersection of primary submodules.*

Theorem 13. *Assume that $S \subseteq B$ are $R-$algebras such that B is finitely generated as an algebra over R and there exists an ideal I of B which lies in S. Let $P_1, \ldots, P_n$ be a primary decomposition of the zero ideal in B such that $I \subseteq P_i$ for all $1 \leq i \leq n'$ and that $I \nsubseteq P_j$ for every $n'+1 \leq j \leq n$. Then S is finitely generated as algebra over R if and only if S/I is a finitely generated $R-$algebra and B/P_j is a finitely generated $S + P_j/P_j-$module for every $n'+1 \leq j \leq n$.*

Proof. ($\Leftarrow$) This implication follows immediately as S/I is finitely generated $R-$algebra as a homomorphic image of finitely generated $R-$algebra S and so are $S+P_j/P_j$ for every $n'+1 \leq j \leq n$. Since $I \nsubseteq P_j$ for every $n'+1 \leq j \leq n$ then $I+P_j/P_j$ is non-zero for all j. Therefore, using Theorem 11, B/P_j is a finitely generated $S+P_j/P_j-$module for all j.

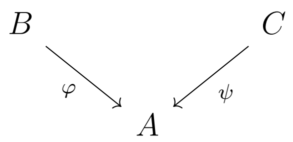
($\Rightarrow$) We know that $I \subseteq P_i$ for all $1 \leq i \leq n'$ and S/I is a finitely generated algebra over R. Thus $S+P_i/P_i \cong S/P_i \cap I$ is also finitely generated $R-$algebras for all i. As well, we have that B/P_j is a finitely generated $S+P_j/P_j-$module for every $n'+1 \leq j \leq n$, by Artin-Tate lemma or Theorem 11 $S+P_j/P_j \cong S/P_j \cap I$ is a finitely generated $R-$algebra. By the choice of $P_1, \ldots, P_n$ we have that $\bigcap_{k=1}^n P_k = \{0\}$, specifically, we get $\bigcap_{k=1}^n (I \cap P_k) = \{0\}$. If we inductively apply Proposition 5 we get that S is a finitely generated $R-$algebra. □

1.4 Intersections of finitely generated algebras

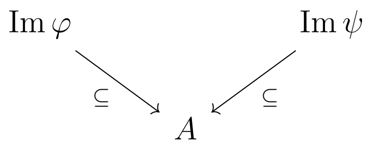
In the preceding part, we dealt with the problem whether a subalgebra of a finitely generated algebra containing its ideal is finitely generated and formulated a strong general result in Theorem 13. However, the strength of this theorem is limited − generally, we can replace the condition of subalgebra being finitely generated algebra by the condition of the algebra being module-finite over the subalgebra only some of the primary components of the algebra which forces us to assume that some quotient of the algebra in question is finitely generated.

To illustrate this point, let us go back to the pullback discussed in the context of Proposition 6. We have finitely generated $R-$algebras A, B, C and $\varphi : B \to A$ a $\psi : C \to A$ is finitely generated and ask whether the pullback of this diagram is finitely generated. To this end, we would like to know it about $S = \varphi^{-1}(\operatorname{Im}\varphi \cap \operatorname{Im}\psi)$. If we were to use Theorem 13, we would encounter a problem since, typically, we need to have no knowledge of $S/I = \operatorname{Im}\varphi \cap \operatorname{Im}\psi$ where $I = \operatorname{Ker}\varphi$ required in the theorem.

Therefore determining if the pullback of this diagram:



is finitely generated using the tools of previous part generally requires knowing that the pullback of this diagram:



which is $\operatorname{Im}\varphi \cap \operatorname{Im}\psi$ is also finitely generated algebra over K. However, we do not need to consider the second diagram if either B or C is coprimary and one of the homomorphisms is surjective due to Theorem 11 (see Theorem 20 in the next section), for example. The second diagram may as well be trivial if B and C are $K-$algebras and either $\operatorname{Im}\varphi$ or $\operatorname{Im}\psi$ is equal to A − that is the case of both motivational examples, which we will revisit in the next section.

In this section we will follow up on the preceding section and the remarks above and deal with the problem when an intersection of two finitely generated algebras over a commutative Noetherian ring is also finitely generated. This is, in general, a very difficult problem as shown by Pinaki Mondal in his article *When is the Intersection of Two Finitely Generated Subalgebras of a Polynomial Ring Also Finitely Generated?*, for example.

In the introduction Mondal [2017] lists an easy example of two finitely generated subalgebras $\mathbb{C}[x^2, y-x]$ and $\mathbb{C}[x^2, x^3, y]$ of $\mathbb{C}[x, y]$ whose intersection is not finitely generated, which is derived from another example due to Neena Gupta and Wilberd van der Kallen.

On the other hand, somewhat similarly looking finitely generated algebras $K[y, x^2y]$ and $K[x, xy]$ have a finitely generated intersection for K an arbitrary field, namely $K[x^2y, x^2y^2]$. In this section, we will show to sufficient conditions for an intersection of two finitely generated algebras to be finitely generated $-$ one will rely on the results of the preceding section, mainly Theorem 13, the other will be more of a combinatorial nature and a general instance of the example we have just offered.

Lemma 14. *Assume, that $S \subseteq B$ are $R-$algebras, $I \subseteq B$ is an ideal of B which is contained in S, then B is a finitely generated $S-$module if and only if B/I is a finitely generated $S-$module.*

Proof. ($\Rightarrow$) This implication is trivially valid.

($\Leftarrow$) Let us take a finitely generated $S-$module $B' \subseteq B$ such that its image under the canonical projection is equal to B/I. generate B/I as an $S-$module. Choose $b \in B$ then there exists $b_S \in B'$ which differs from b by an element of I. As $I \subseteq S$, we get $B' + S = B$. However both of these $S-$modules are finitely generated, B is a finitely generated S-module as well. □

Lemma 15. *Suppose that M is an $R-$module with submodules $M_1, \ldots, M_n$ such that M/M_i is Noetherian for all $1 \leq i \leq n$, then $M/M_1 \cap \cdots \cap M_n$ is also Noetherian.*

Proof. Let us denote $\pi_i : M \to M/M_i$ the canonical projection for each i. Consider a homomorphism $\pi : M \to \prod_{i=1}^{n} M/M_i$ given in $i-$th coordinate by π_i. We easily observe that $\operatorname{Ker} \pi = M_1 \cap \cdots \cap M_n$, hence $M/M_1 \cap \cdots \cap M_n \cong \operatorname{Im} \pi$, which is Noetherian as a submodule of a clearly Noetherian module $\prod_{i=1}^{n} M/M_i$. □

Lemma 16. *Let B be a commutative ring and I, J, P its ideals, then:*

$$\sqrt{(I \cap J) + P} = \sqrt{(I + P) \cap (J + P)}.$$

Proof. Firstly, we note some basic facts: for any two ideals of B, intersection of their radicals is the radical of their intersection and sum of their radicals is below the radical of their sum. Furthermore, we recall that the lattice of radical ideals[2] of B is distributive. Finally, we also observe that we always have that $(I \cap J) + P \subseteq (I + P) \cap (J + P)$.

[2]For R_1, R_2 radical ideals of B, we put $R_1 \wedge R_2 = R_1 \cap R_2$ and $R_1 \vee R_2 = \sqrt{R_1 + R_2}$. The fact that this lattice is distributive means that for any radical ideals R_1, R_2, R_3 of B the equality $\sqrt{R_1 + \sqrt{R_2 \cap R_3}} = \sqrt{R_1 + R_2} \cap \sqrt{R_1 + R_3}$ holds.

Now, let us gradually apply the above stated facts to obtain what we seek:

$$\sqrt{(I \cap J) + P} = \sqrt{\sqrt{I \cap J} + \sqrt{P}} = \sqrt{\sqrt{I} \cap \sqrt{J} + \sqrt{P}} =$$

at this point we use the distributivity of the lattice of radical ideals of B = $\sqrt{\sqrt{I} + \sqrt{P}} \cap \sqrt{\sqrt{J} + \sqrt{P}} = \sqrt{I + P} \cap \sqrt{J + P} = \sqrt{(I + P) \cap (J + P)}$. □

Remark. This proof can be simplified and seen geometrically, if rewritten using properties of Zariski topology on $\operatorname{Spec} B$. Simply write $V((I \cap J) + P) = V(I \cap J) \cap V(P) = (V(I) \cup V(J)) \cap V(P) = (V(I) \cap V(P)) \cup (V(J) \cap V(P)) = V(I + P) \cup V(J + P) = V((I + P) \cup (J + P))$. See Section 2.1 for details.

Theorem 17. *Let $S \subseteq B$ be $R-$algebras such that B is finitely generated as an algebra over R and $I, J \subseteq B$ be ideals of B. Let $P_1, \dots, P_n$ be a primary decomposition of the zero ideal in B such that $I \cap J \subseteq P_i$ for all $1 \leq i \leq n'$ and that $I \cap J \nsubseteq P_j$ for every $n' + 1 \leq j \leq n$. Then*

1. *if B is a finitely generated $S + I-$module and $S + J-$module, B is a finitely generated $S + (I + J)-$module, $S + (I \cap J)-$module, and $(S + I) \cap (S + J)-$module, furthermore, all of these $R-$algebras are finitely generated,*

2. *if both $S + I$ and $S + J$ are finitely generated $R-$algebras, $S + (I + J)$ is a finitely generated algebra over R,*

3. *if both $S + I$ and $S + J$ are finitely generated $R-$algebras, then $S + (I \cap J)$ and $(S + I) \cap (S + J)$ are finitely generated $R-$algebras.*

Proof. (*1.*) Since $S + I \subseteq S + (I + J)$ and $S + (I \cap J) \subseteq (S + I) \cap (S + J)$, it suffices to prove that B is a finitely generated $S + (I \cap J)-$module if it is a finitely generated $S + I-$module and $S + J-$module.

Suppose I and J are $S + (I \cap J)-$modules. However, by Lemma 14, we know that B/I and B/J are finitely generated modules over $S+I$ and $S+J$ respectively. As $S + (I \cap J) + I/I \cong S + I/I$, $S + (I \cap J)$ acts on B/I and B/J in the same way, hence B/I and B/J are also finitely generated $S + (I \cap J)-$modules. By Lemma 15, $B/I \cap J$ is a finitely generated $S + (I \cap J)-$module. Another usage of Lemma 15 consequently establishes that B is a finitely generated $S + (I \cap J)-$module.

All of the said $R-$algebras are obviously finitely generated using Artin-Tate lemma.

(*2.*) The $R-$algebra $S + (I + J)$ is generated by finitely many generators of $S + I$ and $S + J$ as algebras over R.

(*3.*) We know that $S + I/I$ and $S + J/J$ are finitely generated $R-$algebras, that $S + (I \cap J) + I = (S + I) \cap (S + J) + I = S + I$ and analogically for adding J. By Proposition 5, we consequently have that $S + (I \cap J)/I \cap J$ and $(S + I) \cap (S + J)/I \cap J$ are finitely generated algebras over R.

Without loss of generality, we can assume that $P_1, \dots, P_n$ form a primary decomposition of zero and that for all $1 \leq i \leq n'$ we have $I \cap J \subseteq P_i$ and for $n' + 1 \leq j \leq n$ we have that $I \cap J \nsubseteq P_J$. We shall show that B/P_j is a finitely generated $S + (I \cap J)-$module for each $n' + 1 \leq j \leq n$. By Lemma 14, it suffices to prove that $B/(I \cap J) + P_j$ is a finitely generated module.

We know that the nilradical of $B/(I \cap J) + P_j$ is $\sqrt{(I \cap J) + P_j}/(I \cap J) + P_j$. However, as $(I \cap J) + P_j \subseteq (I + P_j) \cap (J + P_j)$ and $\sqrt{(I + P_j) \cap (J + P_j)} = \sqrt{(I \cap J) + P_j}$, which we proved in Lemma 16, $(I + P_j) \cap (J + P_j)/(I \cap J) + P_j$ is specifically finitely generated nilpotent ideal of $B/(I \cap J) + P_j$.

From Theorem 13 and Lemma 14 and the fact that both $S + I$ and $S + J$ are finitely generated $R-$algebras, we deduce that $B/(I + P_j)$ and $B/(J + P_j)$ are finitely generated $S-$modules. Whereas, an application of Proposition 5 gives us that $B/(I + P_j) \cap (J + P_j)$ is a finitely generated module over S. Finally, this means that $B/(I \cap J) + P_j$ is a finitely generated as $S-$module by Proposition 10.

Now, we have that B/P_j is a finitely generated $S + (I \cap J)-$module. As $S + (I \cap J) \subseteq (S + I) \cap (S + J)$, B/P_j is *a fortiori* a finitely generated as a module over $(S + I) \cap (S + J)$. Invoking Theorem 13 once more, we obtain the desired result $-$ the $R-$algebras $S + (I \cap J)$ and $(S + I) \cap (S + J)$ are finitely generated. □

Remark. Notice that part (*3.*) the theorem above solves a very specific instance of the problem whether an intersection of two finitely generated subalgebra of an $R-$algebra are finitely generated as we set out above.

Proposition 18. *Let K be a field and $S_1, S_2 \subseteq B$ be finitely generated subalgebras of $K-$algebra B which is a domain. Then $S_1 \cap S_2$ is a finitely generated $K-$algebra if and only if there are $g_1, \dots, g_m$ and $h_1, \dots, h_m$ generators of S_1 and S_2 respectively such that M generates $S_1 \cap S_2$ as a vector space over K with M denoting the intersection of multiplicative sets generated by $g_1, \dots, g_m$ and $h_1, \dots, h_m$ respectively.*

Proof. ($\Rightarrow$) Assume that $S_1 \cap S_2$ is a finitely generated $K-$algebra with generators $f_1, \dots, f_k$ and that $g_1, \dots, g_m$ and $h_1, \dots, h_m$ are generators of S_1 and S_2 respectively. Then the claim clearly holds for the intersection of multiplicative sets generated by $1, f_1, \dots, f_k, g_1, \dots, g_m$ and $1, f_1, \dots, f_k, h_1, \dots, h_m$ respectively.

($\Leftarrow$) Suppose that $g_1, \dots, g_m$ and $h_1, \dots, h_m$ are generators of S_1 and S_2 respectively and that $M \cap (S_1 \cap S_2)$ generates $S_1 \cap S_2$ a vector space over K. Where M denotes the intersection of multiplicative sets generated by $g_1, \dots, g_m$ and $h_1, \dots, h_m$ respectively. This means that every element of $S_1 \cap S_2$ is a $K-$linear combination of elements of M.

To prove that $S_1 \cap S_2$ is finitely generated, it suffices to show that there are finitely many elements of M in $S_1 \cap S_2$ such that any element of M in $S_1 \cap S_2$ is a product of their powers.

The set M is in bijection with the following set:

$$\mathfrak{I} = \{(a_1, \dots, a_n, b_1, \dots, b_m) \in \mathbb{N}_0^{n+m} : g_1^{a_1} \dots, g_m^{a_m} = h_1^{b_1} \dots h_n^{b_n}\}$$

we will consider $\mathbb{N}_0^{n+m}$ to be equipped with a natural partial ordering $\mathbf{a} \geq \mathbf{b}$ if $a_i \geq b_i$ for all $1 \leq i \leq n + m$. Suppose that $(a_1, \dots, a_m, b_1, \dots, b_n) \in \mathfrak{I}$ and $(c_1, \dots, c_m, d_1, \dots, d_n) \in \mathfrak{I}$ such that:

$$(a_1, \dots, a_m, b_1, \dots, b_n) > (c_1, \dots, c_m, d_1, \dots, d_n),$$

we will show that $(a_1 - c_1, \dots, a_m - c_m, b_1 - d_1, \dots, b_n - d_n) \in \mathfrak{I}$. That follows easily from:

$$p \cdot g_1^{a_1 - c_1} \dots, g_m^{a_m - c_m} = g_1^{a_1} \dots, g_m^{a_m} = h_1^{b_1} \dots h_n^{b_n} = h_1^{b_1 - d_1} \dots h_n^{b_n - d_n} \cdot p$$

where all $a_i - c_i, b_j - d_j \geq 0$ (at least one such inequality is strict) and $p = g_1^{c_1} \dots, g_m^{c_m} = h_1^{d_1} \dots h_n^{d_n}$. Cancelling p out, we obtain:

$$g_1^{a_1 - c_1} \dots, g_m^{a_m - c_m} = h_1^{b_1 - d_1} \dots h_n^{b_n - d_n}$$

what we wanted to prove.

By Dickson's lemma which says that every subset of $\mathbb{N}_0^{n+m}$ has finitely many minimal elements with respect to the natural partial order, see Theorem 5 on page 71 in Cox et al. [2007] for an equivalent formulation in related terms of monomial ideals, there are finitely many elements $\mathbf{i}_1, \dots, \mathbf{i}_k$ in $\mathfrak{I}$ such that for each $\mathbf{a} \in \mathfrak{I}$ at least one of those elements lies beneath it. By induction on $||\mathbf{a}||_1 = \sum_{i=1}^{m+n} a_i$, we will prove that for each $\mathbf{a} \in \mathfrak{I}$ there are $c_1, \dots, c_k \in \mathbb{N}_0$ such that $\mathbf{a} = c_1\mathbf{i}_1 + \dots + c_k\mathbf{i}_k$.

Let $\mathbf{a}$ have minimal norm $||\mathbf{a}||_1$ over $\mathfrak{I}$, then, clearly, $\mathbf{a}$ is one of $\mathbf{i}_1, \dots, \mathbf{i}_k$ as there is no element of $\mathfrak{I}$ strictly beneath it (any $\mathbf{b}$ with $\mathbf{a} > \mathbf{b}$ has to have strictly smaller norm). The induction arguments goes as follows: let $\mathbf{a} \in \mathfrak{I}$, then either $\mathbf{a}$ is one of $\mathbf{i}_1, \dots, \mathbf{i}_k$ or one of $\mathbf{i}_1, \dots, \mathbf{i}_k$ is strictly beneath $\mathbf{a}$, thus $\mathbf{a} - \mathbf{i}_j \in \mathfrak{I}$ for some $1 \leq j \leq k$. However, $||\mathbf{a}||_1 > ||\mathbf{a} - \mathbf{i}_j||_1$, which enables us to apply the inductive assumption.

This means, by the correspondence of M and $\mathfrak{I}$ that each element of M is a product of powers of $g_1^{i_1^1} \dots, g_m^{i_m^1}, \dots, g_1^{i_1^k} \dots, g_m^{i_m^k}$. These elements generate $S_1 \cap S_2$ as $K-$algebra. □

While the proposition above may seem almost tautological, it can actually be used to prove that a large class of subalgebras arising as intersections of two finitely generated algebras are finitely generated:

Theorem 19. *Let $S_1, S_2 \subseteq K[x_1, \dots, x_n]$ be subalgebras generated by monomials, then $S_1 \cap S_2$ is finitely generated.*

Proof. At first, we will show that for $i = 1, 2$ if $f \in S_i$, then all monomials whose sum f belong S_i. However, all elements of S_i are $K-$linear combination of products of powers of its generators. Such products are indeed monomials if we assume S_i is generated by monomials and clearly belong to S_i. Our claim then follows from the uniqueness of expression of a polynomial in $K[x_1, \dots, x_n]$ a $K-$linear combination of monomials – they form a basis of $K[x_1, \dots, x_n]$ as a vector space over K.

Suppose that $f \in S_1 \cap S_2$ and $f = \sum_{i=1}^{m} c_i f_i$ where $c_1, \dots, c_m \in K$ and $f_1, \dots, f_m$ are monomials. We know that as $f \in S_i$, then $f_1, \dots, f_m \in S_i$ for $i = 1, 2$. This means that $g_1, \dots, g_m \in S_1 \cap S_2$ and that $S_1 \cap S_2$ is generated by monomials. The algebra $S_1 \cap S_2$ is generated by all its elements, however, any such element is a sum of monomials in $S_1 \cap S_2$, thus $S_1 \cap S_2$ is generated by its monomials.

To prove that $S_1 \cap S_2$ is finitely generated, we use the proposition above. Let $g_1, \dots, g_m$ and $h_1, \dots, h_m$ be monomial generators of S_1 and S_2 respectively.

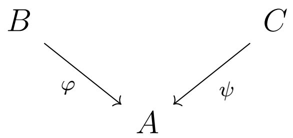
Take M to be the intersection of multiplicative sets M_1 and M_2 generated by $1, g_1, \dots, g_m$ and $1, h_1, \dots, h_m$ respectively. Suppose that $f \in S_1 \cap S_2$ is a monomial, then $f \in M_1$ and $f \in M_2$ by uniqueness of expression of elements of $K[x_1, \dots, x_n]$ as sums of monomials. Then $f \in M$, thus all monomials in $S_1 \cap S_2$ are in M and M generates generates $S_1 \cap S_2$ as a vector space over K. □

Remark. Note that Theorem 19 can be used to prove that finitely generated algebras $K[y, x^2y]$ and $K[x, xy]$ over K have a finitely generated intersection, as stated at the beginning of this section.

1.5 Examples and local properties

In this section, we will give a partial solution to the problem whether the pullback of a diagram of finitely generated algebras is finitely generated, revisit the motivational examples given at the beginning of this chapter, and investigate local properties of pushouts of algebraic sets and algebras arising as pullbacks. To these ends, we shall employ the tools we developed in all sections before.

Theorem 20. *Let A, B, C be finitely generated $R-$algebras, B is a domain, and let $\varphi : B \to A$ a $\psi : C \to A$ be their homomorphisms, φ is surjective with* $\operatorname{Ker}\varphi$ *non-zero. Then the pullback of the corresponding diagram:*



is finitely generated $R-$algebra if and only if A is a finitely generated module over $\operatorname{Im}\psi$*.*

Proof. Let P_B and P_C be images of the pullback of the diagram P with morphisms π_B, π_C under π_B, π_C respectively. Cleary, as $\operatorname{Ker}\varphi$ is non-zero, P_B contains it. Combining Theorem 11 and Lemma 14, we obtain that P_B is finitely generated $R-$algebra if and only if A is a finitely generated module over $\varphi(P_B) = \operatorname{Im}\psi$, as φ is onto. Provided that P is a finitely generated $R-$algebra, P_B as its image needs to be so as well, this gives us that A is a finitely generated module over $\operatorname{Im}\psi$.

See the second remark below for the other implication. □

Remark. Retain the notation of the theorem, provided that $\operatorname{Ker}\varphi = 0$, the pullback of the corresponding diagram is finitely generated automatically, since it is C with morphisms $\psi : C \to A$ and $\mathrm{id} : C \to C$ about which we assume that it is a finitely generated algebra over R. If P is finitely generated.

Remark. The if part trivially holds even if A is not a domain. Use Lemma 14 to show that B is finitely generated module over P_B, the same for C and P_C. By Artin-Tate lemma, P_B and P_C are finitely generated $R-$algebras, using proposition 6, the pullback is also a finitely generated $R-$algebra. This result is included as Lemma 15.5.1. in Part 1, Chapter 15 *More on algebra* in Stacks project [Stacks project authors, 2018, Tag 00IT].

Example (Contracting a line in $\mathbb{A}^2_K$; included without proof as Example 3.5 in Schwede [2004], page 7)*.* Consider this diagram:

$$\begin{array}{ccc} K[x,y] & & K \\ & \searrow^{\pi} \quad \swarrow^{\iota} & \\ & K[x,y]/(y) & \end{array}$$

with ι and π being canonical inclusion and projection, respectively. The pullback of this diagram is $K + (y)$.

Theorem 20 gives $K + (y)$ is finitely generated $K-$algebra if and only if $K[x,y]$ is a finitely generated $K + (y)-$module. By Lemma 14, this is equivalent to $K[x] \cong K[x,y]/(y)$ being finitely generated $K + (y)-$module, that clearly does not hold, as $K[x]$ would have to be a finite dimensional vector space over $K \cong K + (y)/(y)$.

Therefore $K + (y)$ is not a finitely generated $K-$algebra and it is impossible to contract a line in $\mathbb{A}^2_K$ into a point in the category of algebraic sets over K. △

Remark. We can use the reasoning from the example above also in treating Example 3.2 in Temkin and Tyomkin [2016].

Example (Contracting a finite number of points on algebraic varieties over K; a special case given as Example 3.6 in Schwede [2004], page 7)*.* Let L be a finitely generated $K-$algebra, and I an intersection of finitely many maximal ideals of L. Consider this diagram:

$$\begin{array}{ccc} L & & K \\ & \searrow^{\pi} \quad \swarrow^{\iota} & \\ & L/I & \end{array}$$

with ι and π being canonical inclusion and projection, respectively. The pullback of this diagram is $K + I$ as in the previous example. This situation can be view as contracting a finite number of points to a single one.

Suppose $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are maximal ideals of L such that $I = \bigcap_{i=1}^n \mathfrak{m}_i$. However, as by Theorem 4.19 on page 132 in Eisenbud [1995], $K \subseteq L/\mathfrak{m}_i$ is a finite field extension, we have by Lemma 14 that L is a finitely generated $K + \mathfrak{m}_i-$module for all i. Using Theorem 17 inductively, we get that L is a finitely generated $K + \bigcap_{i=1}^n \mathfrak{m}_i-$module, thus $K + I$ is a finitely generated $K-$algebra by the Artin-Tate lemma. △

Example (Contracting two points on $\mathbb{A}^1_{\mathbb{C}}$ results in a singularity)*.* Let $k_1, k_2 \in K$ be two its distinct elements. We know that $K[x]$ is a principal ideal domain, so the intersection of maximal ideals $(x - k_1), (x - k_2) \subseteq K[x]$ is equal to $I = ((x - k_1)(x - k_2)) \subseteq K[x]$.

In the penultimate example of this section, we show that $K + I$ is generated by $(x - k_1)(x - k_2)$ and $x(x - k_1)(x - k_2)$ as an algebra over K.

If we were to identify points -1 and 1 on in the complex $\mathbb{A}^1_{\mathbb{R}}$, we get a plane curve $\mathbb{R}[y,z]/(y^3 - z^2 + y^2)$ as $(x^2 - 1)^3 - (x(x^2 - 1))^2 + (x^2 - 1)^2 = 0$ and we put $y = x(x - 1)$ and $z = x^2(x - 1)$.

Observing the real locus of the resulting plane curve $\mathbb{R}[y,z]/(y^3 - z^2 + y^2)$ we see that has a singularity of order 2 at the origin. △

For the rest of this section, we will concern ourselves with singularities arising from forming pushouts of algebraic sets or, more generally, from forming pullbacks of finitely generated subalgebras. At first, we discuss local properties in a classical case of pullback of two surjective homomorphisms which corresponds to gluing two algebraic sets via their common closed subset. Then we prove a result that in special case shows that if we glue points on an algebraic variety, we get a singularity at least of order of dimension of the variety times the number of identified points, speaking in rather loose terms. This section is concluded by a discussion on finer properties of algebras and singularities arising in some cases of gluing points in affine spaces.

Proposition 21. *Let us have two surjective homomorphisms of finitely generated $R-$algebras $\pi : B \to A$ and $\varrho : C \to A$. Then the pullback of the corresponding diagram $P = B \times_A C = \{(b, c); \pi(b) = \varrho(c), b \in B, c \in C\}$ together with maps $\tau_1 : B \times_A C \to B$ and $\tau_2 : B \times_A C \to C$ is a finitely generated $R-$algebra. Suppose that $\mathfrak{m} \subseteq B \times_A C$ is a maximal ideal*

1. *if* $(\operatorname{Ker}\pi, 0) \subseteq \mathfrak{m}$ *and* $(0, \operatorname{Ker}\varrho) \nsubseteq \mathfrak{m}$*, then:*

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \cong \tau_2(\mathfrak{m})_{\tau_2(\mathfrak{m})}/\tau_2(\mathfrak{m})^2_{\tau_2(\mathfrak{m})},$$

2. *if* $(\operatorname{Ker}\pi, 0) \nsubseteq \mathfrak{m}$ *and* $(0, \operatorname{Ker}\varrho) \subseteq \mathfrak{m}$*, then:*

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \cong \tau_1(\mathfrak{m})_{\tau_1(\mathfrak{m})}/\tau_1(\mathfrak{m})^2_{\tau_1(\mathfrak{m})},$$

3. *if* $(\operatorname{Ker}\pi, 0), (0, \operatorname{Ker}\varrho) \subseteq \mathfrak{m}$*, then we have that:*

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \cong (\operatorname{Ker}\pi, 0)_\mathfrak{m} + \mathfrak{m}_\mathfrak{m}^2/\mathfrak{m}_\mathfrak{m}^2 \oplus \tau_2(\mathfrak{m})_{\tau_2(\mathfrak{m})}/\tau_2(\mathfrak{m})^2_{\tau_2(\mathfrak{m})}$$

   and

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \cong (0, \operatorname{Ker}\varrho)_\mathfrak{m} + \mathfrak{m}_\mathfrak{m}^2/\mathfrak{m}_\mathfrak{m}^2 \oplus \tau_1(\mathfrak{m})_{\tau_1(\mathfrak{m})}/\tau_1(\mathfrak{m})^2_{\tau_1(\mathfrak{m})},$$

   as vector spaces over $P/\mathfrak{m} \cong B/\tau_1(\mathfrak{m})$ a $P/\mathfrak{m} \cong C/\tau_2(\mathfrak{m})$.

Proof. The fact that the pullback P is finitely generated follows immediately from Proposition 6, as $\operatorname{Im}\pi \cap \operatorname{Im}\varrho = A$ and so $\pi^{-1}(\operatorname{Im}\pi \cap \operatorname{Im}\varrho) = B$ and $\varrho^{-1}(\operatorname{Im}\pi \cap \operatorname{Im}\varrho) = C$ which we assume are finitely generated algebras over R.

Since $(\operatorname{Ker}\pi, 0) \cdot (0, \operatorname{Ker}\varrho) = \{0\}$ and $\mathfrak{m} \subseteq B \times_A C$ is, *a fortiori*, a prime ideal, we get that $(\operatorname{Ker}\pi, 0) \subseteq \mathfrak{m}$ or $(\operatorname{Ker}\pi, 0) \subseteq \mathfrak{m}$. This shows that our proposition is correctly formulated.

At first, let us deal with *1.* Suppose that we have $(\operatorname{Ker}\pi, 0) \subseteq \mathfrak{m}$ and $(0, \operatorname{Ker}\varrho) \nsubseteq \mathfrak{m}$. Localise P in $\mathfrak{m}$ via $u : P \to P_\mathfrak{m}$. There is an element of $(0, \operatorname{Ker}\varrho)$ that becomes invertible as $(0, \operatorname{Ker}\varrho) \nsubseteq \mathfrak{m}$, this gives us that $(\operatorname{Ker}\pi, 0) \subseteq \operatorname{Ker} u$. Since τ_2 is a onto and projection with kernel $(\operatorname{Ker}\pi, 0)$, $\tau_2(\mathfrak{m})$ is a maximal ideal of C. By the homomorphism theorem there is a unique homomorphism $u : C \to P_\mathfrak{m}$, such that $u = u'\tau_2$. However, we observe that u' is the localisation of C in $\tau_2(\mathfrak{m})$. From that we deduce that $\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 \cong \tau_2(\mathfrak{m})_{\tau_2(\mathfrak{m})}/\tau_2(\mathfrak{m})^2_{\tau_2(\mathfrak{m})}$. Part *2.* can be proven along the same lines.

Now let us assume that $(\operatorname{Ker}\pi, 0), (0, \operatorname{Ker}\varrho) \subseteq \mathfrak{m}$. It follows that $\tau_1(\mathfrak{m})$ is a maximal ideal of B and $\tau_2(\mathfrak{m})$ is a maximal ideal of C, since both τ_1 and τ_2 are onto. Using the third isomorphism theorem, we show that $P/\mathfrak{m} \cong B/\tau_1(\mathfrak{m})$ and $P/\mathfrak{m} \cong C/\tau_2(\mathfrak{m})$. We can write:

$$(\mathfrak{m}_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m})/((\operatorname{Ker}\pi, 0)_\mathfrak{m} + \mathfrak{m}^2_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m}) \cong \mathfrak{m}_\mathfrak{m}/(\operatorname{Ker}\pi, 0)_\mathfrak{m} + \mathfrak{m}^2_\mathfrak{m} \cong$$

$$(\mathfrak{m}/(\operatorname{Ker}\pi, 0) + \mathfrak{m}^2)_\mathfrak{m} \cong [(\mathfrak{m}/(\operatorname{Ker}\pi, 0))/(\mathfrak{m}^2 + (\operatorname{Ker}\pi, 0)/(\operatorname{Ker}\pi, 0))]_\mathfrak{m}.$$

However, we have $\mathfrak{m}^2 + (\operatorname{Ker}\pi, 0)/(\operatorname{Ker}\pi, 0) \cong \mathfrak{m}^2/\mathfrak{m}^2 \cap (\operatorname{Ker}\pi, 0)$. Now, let us show that $(\mathfrak{m}/(\operatorname{Ker}\pi, 0))^2 \cong \mathfrak{m}^2/\mathfrak{m}^2 \cap (\operatorname{Ker}\pi, 0)$. Because:

$$(a + (\operatorname{Ker}\pi, 0))(b + (\operatorname{Ker}\pi, 0)) = ab + (\operatorname{Ker}\pi, 0)$$

for all $a, b \in \mathfrak{m}$, we know to $(\mathfrak{m}/(\operatorname{Ker}\pi, 0))^2$ is an image of $\mathfrak{m}^2$ under the projection with kernel $(\operatorname{Ker}\pi, 0)$ which is $\mathfrak{m}^2/\mathfrak{m}^2 \cap (\operatorname{Ker}\pi, 0)$. Putting it all together, we get that:

$$\mathfrak{m}_\mathfrak{m}/(\operatorname{Ker}\pi, 0)_\mathfrak{m} + \mathfrak{m}^2_\mathfrak{m} \cong \tau_2(\mathfrak{m})_\mathfrak{m}/\tau_2(\mathfrak{m})^2_\mathfrak{m} \cong \tau_2(\mathfrak{m})_{\tau_2(\mathfrak{m})}/\tau_2(\mathfrak{m})^2_{\tau_2(\mathfrak{m})},$$

the last isomorphism is due to τ_2 being surjective. We already have:

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m} \cong (\operatorname{Ker}\pi, 0)_\mathfrak{m} + \mathfrak{m}^2_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m} \oplus \tau_2(\mathfrak{m})_{\tau_2(\mathfrak{m})}/\tau_2(\mathfrak{m})^2_{\tau_2(\mathfrak{m})}$$

as vector spaces over $P/\mathfrak{m} \cong C/\tau_2(\mathfrak{m})$ which is what we wanted to prove. The second part can be proven it the same way. □

Proposition 22. *Suppose B is an integral domain finitely generated as an algebra over R, $S \subseteq B$ is its subalgebra and $I \subseteq S$ is an ideal of B which can be written as an intersection of prime ideals, $I = \bigcap_{a\in A} \mathfrak{p}_a$. Assume, moreover, that $h_1 + I, \ldots, h_k + I$ form a free basis of an $S/I-$module B/I. Let $\{i_1, \ldots, i_n\} \subseteq I$ be linearly independent in $\mathfrak{p}_{a\mathfrak{p}_a}/\mathfrak{p}^2_{a\mathfrak{p}_a}$ for each $a \in A$, then $\{i_\ell h_1, \ldots, i_\ell h_k; \ell = 1, \ldots, n\}$ is linearly independent in I_I/I_I^2, we think of I_I as an ideal of S_I.*

Proof. Assume there exist coefficients $s_{\ell j} + I_I \in S_I/I_I$, without loss of generality $s_{\ell j} \in S$, such that for all $1 \leq \ell \leq n$ and $1 \leq j \leq k$ we have that $\sum_{j,\ell} i_\ell h_j s_{\ell j} \in I_I^2$. Taking $s_{\ell j} \in S$ can be justified by thinking about them as elements of the respective quotient field $Q(B)$, we can then cancel out all their denominators by multiplying with elements of $S - I$ under which is S closed.

Denote $h'_\ell = \sum_j h_j s_{\ell j}$ for all ℓ. Since we assume that $I \subseteq \mathfrak{p}_a$ for all $a \in A$, it follows that $I_I^2 \subseteq \mathfrak{p}^2_{a\mathfrak{p}_a}$ and hence $i_1 h'_1 + \cdots + i_n h'_n \in \mathfrak{p}^2_{a\mathfrak{p}_a}$ for every $a \in A$. However, we supposed furthermore that $i_1 + \mathfrak{p}^2_{a\mathfrak{p}_a}, \ldots, i_n + \mathfrak{p}^2_{a\mathfrak{p}_a}$ are linearly independent, therefore for every $1 \leq \ell \leq n$ the element h'_ℓ has to be in $\mathfrak{p}_{a\mathfrak{p}_a}$. Moreover as each $h'_\ell \in B$, then $h'_\ell \in \mathfrak{p}_a$ for all $1 \leq \ell \leq n$ and $a \in A$.

This means that $h'_\ell \in \bigcap_{a\in A} \mathfrak{p}_a = I$ for every $1 \leq \ell \leq n$. Take any such ℓ, we've shown that $h'_\ell + I = 0 + I$, let us expand that to $\sum_{j=1}^k h_j s_j + I = 0 + I$. Since we supposed that $h_1, \ldots, h_\ell$ form a free basis of B/I as an $S/I-$module, then all $s_{\ell j} + I$ need to be zero for all possible ℓ and j. Therefore, the set $\{i_\ell h_1, \ldots, i_\ell h_k; \ell = 1, \ldots, n\}$ is linearly independent in I_I/I_I^2. □

Remark. The proposition above can conveniently applied in the case where B is a coordinate ring of a $K-$algebraic variety X, suppose furthermore that K is algebraically closed, and $S = K + I$ where I is the ideal of B such that $I = \bigcap_{i=1}^n \mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of B. Then $K + I$ can be thought of as a coordinate ring of an algebraic variety X with finitely many points corresponding to the ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are identified.

We can observe that B/I is a finite dimensional vector space as in the second example of this section, hence free module, over $S/I \cong K$ and that I is maximal, thus prime, ideal of $K + I$. Let us now suppose that $\mathfrak{m}_i = (x_1 - a_{1i}, \dots, x_m - a_{mi})/I(X)$ for all $1 \le i \le n$ with all $a_{ji} \in K$. For simplicity, let us also assume that $a_{ji} \neq a_{j'i}$ iff $j \neq j'$. This guarantees that $x_j - a_{ji} \notin \mathfrak{m}_{i'}/I(X)$ for $i' \neq i$.

It is clear that $(x_1 - a_{1i}) + \mathfrak{m}^2_{i\,\mathfrak{m}_i}, \dots, (x_m - a_{mi}) + \mathfrak{m}^2_{i\,\mathfrak{m}_i}$ generate $\mathfrak{m}_{i\,\mathfrak{m}_i}/\mathfrak{m}^2_{i\,\mathfrak{m}_i}$ for each $1 \le i \le n$. Thus, there is a set of indices J_i such that $(x_1 - a_{j_1^i i}) + \mathfrak{m}^2_{i\,\mathfrak{m}_i}, \dots, (x_{j_i^i} - a_{j_i^i i}) + \mathfrak{m}^2_{i\,\mathfrak{m}_i}$ form a basis of $\mathfrak{m}_{i\,\mathfrak{m}_i}/\mathfrak{m}^2_{i\,\mathfrak{m}_i}$. Without loss of generality, assume that $|J_1| = k$ is the smallest of such indices. We observe that $(x_{j_1^1} - a_{j_1^1 1})(x_{j_1^2} - a_{j_1^2 2}) \dots (x_{j_1^n} - a_{j_1^n n}) + I(X), \dots, (x_{j_k^1} - a_{j_k^1 1})(x_{j_k^2} - a_{j_k^2 2}) \dots (x_{j_k^n} - a_{j_k^n n}) + I(X)$ belong to $\mathfrak{m}_1 \dots \mathfrak{m}_n/I(X) \subseteq I$.

Now, notice that these k elements of are linearly independent in all $\mathfrak{m}_{i\,\mathfrak{m}_i}/\mathfrak{m}^2_{i\,\mathfrak{m}_i}$ for each $1 \le i \le n$, since $(x_{j_\ell^1} - a_{j_\ell^1 1})(x_{j_\ell^2} - a_{j_\ell^2 2}) \dots (x_{j_\ell^n} - a_{j_\ell^n n}) +$ and $(x_{j_\ell^i} - a_{j_\ell^i i})$ span the same subspace of $\mathfrak{m}_{i\,\mathfrak{m}_i}/\mathfrak{m}^2_{i\,\mathfrak{m}_i}$. This is due to the fact that for all i and j we have that $x_j - a_{ji} \notin \mathfrak{m}_{i'}/I(X)$ for $i' \neq i$.

By the proposition above, the S_I/I_I-dimension of I_I/I_I^2 in S_I is at least $k \dim_K B/I$, so the smallest dimension of a tangent space among identified points $(a_{11}, \dots, a_{1m}), \dots, (a_{n1}, \dots, a_{nm})$ times the number of points which is equal to $\dim_K B/I$. Therefore, the order of the arising singularity is proportional to the number of identified points.

The singularities arising from gluing finitely many points on affine algebraic sets can be studied even more closely by looking at rings of formal power series in them. In treating the following two examples, we will assume that $\operatorname{char} K = 0$.

Example (Gluing finitely many points on $\mathbb{A}^1_K$). Let $a_1, \dots, a_n \in K$ be distinct, then the ideal I defining $\{a_1, \dots, a_n\}$ is generated by $\varphi_0(x) = (x - a_1) \dots (x - a_n)$. We will try to describe $K + I$.

We know that $K + I$ is a finitely generated algebra over K. Now, we will find its generators and relations between them. We say that $K + I$ is generated by $\varphi_0(x), \dots, \varphi_{n-1}(x)$ where $\varphi_{i+1}(x) = x\varphi_i(x)$ for $0 \le i \le n-2$. The proof this claim goes by induction on degree of the non-zero polynomial $f \in S$, it suffices to assume that $f \in I$.

As $a_1, \dots, a_n \in K$ are distinct, there are no polynomials of degree less than n in I and there is, up to a multiple by an element of K, only one polynomial of degree n, $\varphi_0(x)$.

Let $f \in I$ be of degree $m > n$ and we know that all polynomials of smaller degree belong to $K[\varphi_0(x), \dots, \varphi_{n-1}(x)]$. Write $m = kn + r$, where $1 \le k$ and $0 \le r \le n-1$ and denote ℓ the leading coefficient of f. Then $f - \ell\varphi_0^{k-1}\varphi_r$ is of strictly smaller degree. We can conclude the proof by pointing out that both $f - \ell\varphi_0^{k-1}\varphi_r$ and $\ell\varphi_0^{k-1}\varphi_r$ belong to $K[\varphi_0(x), \dots, \varphi_{n-1}(x)]$.

We know that $K + I \cong K[x_0, \dots, x_{n-1}]/J$ with J an ideal of $K[x_0, \dots, x_{n-1}]$ by $x_i \mapsto \varphi_i$ for $0 \le i \le n-1$. We will show that J is generated by two types of relations: $x_i x_j = x_k x_\ell$ for all $i + j = k + \ell$ and $x_i x_j x_k = \sum_{\ell=0}^{n} b_{n-\ell}[n-\ell]_2$ where $i + j + k \le n - 2$, $b_{n-\ell}$ are coefficients of φ_0 and $[n-\ell]_2$ denotes a product of $x_a x_b$ such that $a + b = i + j + k - \ell$.

Our strategy will be to proceed by induction of degree of an element of J divide an element of J into a sum of two elements, one whose image under the $x_i \mapsto \varphi_i$ is a sum of monomials of degrees high enough so that they need to annihilate each other and that elements belongs to J, thus the other elements belongs to J as well and we will reduce its degree by applying the relations of the second type.

At first, we need prove that if $i_1 + \cdots + i_m = j_1 + \cdots + j_m$, then $x_{i_1} \ldots x_{i_m} - x_{j_1} \ldots x_{j_m} \in J$, by induction on m. The claim clearly holds for degrees 1 by definition and 2 it is given by relations of the first type. Assume the degree is $k \geq 1$ and the claim holds for all smaller degrees. Without loss of generality, suppose that $i_1 \geq \cdots \geq i_k$ and $j_1 \geq \cdots \geq j_k$ and $i_1 \geq j_1$. If $i_1 = j_1$, the claim holds by inductive assumption. If $i_1 > j_1$, then $j_1 \geq i_k$. Otherwise, we would have $i_1 + \cdots + i_k \geq i_1 + i_k + \cdots + i_k > j_1 + \cdots + j_1 \geq j_1 + \cdots + j_k$ which yields a contradiction. This gives us that $i_1 - j_1 \leq i_1 - i_k$ or $i_k + (i_1 - j_1) \leq i_1$. We can therefore rewrite $x_{i_1} x_{i_k} = x_{i_1-(i_1-j_1)} x_{i_k+(i_1-j_1)}$ using one of the relations of the first type. After this, both monomials contain x_{j_1} and we can use the inductive assumption, hence completing the proof.

Let $F(x_0, \ldots, x_{n-1}) \in J$ be a homogenous relation of degree m. Observe that it can be rewritten as a sum of homogenous relations of the same degree $F(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{m(n-1)} F_i(x_0, \ldots, x_{n-1})$ where $F_i(x_0, \ldots, x_{n-1})$ is a K-linear combination of monomials $x_{j_1} \ldots x_{j_m}$ where $i = j_1 + \cdots + j_m$ and whose coefficients add up to zero. Inductively, using the claim right above, we can show that all $F_i(x_0, \ldots, x_{n-1}) \in J$, and so $F(x_0, \ldots, x_{n-1}) \in J$.

At this point, suppose that $F(x_0, \ldots, x_{n-1}) \in J$ is a general relation of degree m. Denote $F_m(x_0, \ldots, x_{n-1})$ the $m-$th homogenous part of F. Now, let $F'_m(x_0, \ldots, x_{n-1})$ be a sum of monomials $x_{i_1} \ldots x_{i_m}$ of F_m such that $i_1 + \cdots + i_m > (m-2)(n-1)$. These are monomials for which we cannot use the relations of the second type to lower their degree. We show that $F'_m(x_0, \ldots, x_{n-1}) \in J$. Let $F''_m(x_0, \ldots, x_{n-1})$ denote the part of F'_m made of monomials with the highest sum of indices, $i_1 + \cdots + i_m$ for $x_{i_1} \ldots x_{i_m}$. Those monomials map to polynomials of degree $i_1 + \cdots + i_m + mn$ under $x_i \mapsto \varphi_i$, by virtue of $i_1 + \cdots + i_m$ being the highest among them, monomials in the other part of $F'_m(x_0, \ldots, x_{n-1})$ map to polynomials of strictly smaller degree. For any other monomial that is part of F, the degree of polynomial to which it is mapped is at most $(m-1)(n-1) + (m-1)n = (m-1)(2n-1)$ as it is of degree at most $m-1$. However, our assumption that $i_1 + \cdots + i_m > (m-2)(n-1)$ gives us that $_1 + \cdots + i_m + mn > (m-2)(n-1) + mn = (m-2)(n-1) + m(n-1) + m = (m-1)(2n-2) + m = (m-1)(2n-1) + 1$. As the image of F under $x_i \mapsto \varphi_i$ and no other parts of it map to elements of such high degree, the image of F''_m needs to be zero, thus the sum of coefficients in F''_m is zero. This allows us to use relations $x_{i_1} \ldots x_{i_m} - x_{j_1} \ldots x_{j_m} \in J$ for $i_1 + \cdots + i_m = j_1 + \cdots + j_m$ to prove that $F''_m \in J$, indeed. We can replace $F \in J$ by $F - F''_m \in J$ and proceed inductively.

Let $x_{i_1} \ldots x_{i_m}$ be a monomial of degree m such that $i_1 + \cdots + i_m \leq (m-2)(n-1)$. If $i_1 + \cdots + i_m = (m-2)(n-1)$, we can rewrite $x_{i_1} \ldots x_{i_m}$ to $x_{n-1}^{m-2} x_0^2$ and use the relation of the second type on to rewrite $x_{n-1} x_0^2$ to a sum of monomials of degree two, hence lowering the degree. Suppose that $i_1 + \cdots + i_m = k(n-1) + r$ where $r < n-1$ and $k \leq m-3$. Then, $x_{i_1} \ldots x_{i_m} - x_{n-1}^k x_r x_0^{m-k-1} \in J$ and $r \leq n-2$, so $x_r x_0^2$ can be rewritten as a sum of monomials of degree two by a rule of the second type. Therefore $F_m - F'_m$ can be rewritten as polynomial of degree $m-1$ at most. Using the inductive assumption, we conclude our proof.

Let us now examine two specific cases, glueing two and three points on $\mathbb{A}^2_K$. We shall work under the assumption that $\operatorname{char} K = 0$ and that K is algebraically closed. We will glue roots of unity in both cases.

Identifying roots of $x^2 - 1$ on $\mathbb{A}^1_K$, we get a variety V_2 with the following coordinate ring:

$$K[x_0, x_1]/(x_0^3 - x_1^2 + x_0^2)$$

with $x_0^3 - x_1^2 + x_0^2$ being an instance of the rule of the second type. To examine the resulting singularity at 0 closely, we move to the ring of formal power series of this variety at 0. We get $K[[x_0, x_1]]/(x_0^2(1 + x_0) - x_1^2)$. However, we can take $u \in K[[x_0, x_1]]$ a formal square root of $1 + x_0$ that is also invertible. The ring $K[[x_0, x_1]]/(x_0^3 - x_1^2 + x_0^2)$ is thus isomorphic to:

$$K[[y_0, y_1]]/((y_0 - y_1)(y_0 + y_1)).$$

This means that the resulting singularity locally looks like a pair of intersecting lines.

Identifying roots of $x^3 - 1$ on $\mathbb{A}^1_K$, we get a variety V_3 with the following coordinate ring:

$$K[x_0, x_1, x_2]/(x_0x_2 - x_1^2, x_0^3 - x_1x_2 + x_0^2, x_0^2x_1 - x_2^2 + x_0x_1)$$

by using our results above. The first relation is of the first type, the latter two are of the second type. As above, we will examine the corresponding ring of formal power series. Rewrite the latter two relations as $x_0^2(1+x_0)-x_1x_2$ and $x_0x_1(1+x_0)-x_2^2$. However, we can find $v \in K[[x_0, x_1, x_2]]$ such that $v^3 = 1+x_0$, furthermore, this v is invertible. Rewrite the relations as $x_0(x_2v^{-2})-(x_1v^{-1})^2$, $x_0^2-(x_1v^{-1})(x_2v^{-2})$, and $x_0(x_1v^{-1}) - (x_2v^{-2})^2$. This means that $K[[x_0, x_1, x_2]]/(x_0x_2 - x_1^2, x_0^3 - x_1x_2 + x_0^2, x_0^2x_1 - x_2^2 + x_0x_1)$ is isomorphic to:

$$K[[y_0, y_1, y_2]]/(y_0^2 - y_1y_2, y_1^2 - y_0y_2, y_2^2 - y_0y_1).$$

Take $K[y_0, y_1, y_2]/(y_0^2 - y_1y_2, y_1^2 - y_0y_2, y_2^2 - y_0y_1)$ and set $y_1 = a$ for non zero $a \in K$. This results in $a^2 = y_1y_2$, $y_1^2 = ay_2$, and $y_2^2 = ay_1$. Take $y_2 = \frac{a^2}{y_1}$, then both remaining equations can be written as $y_1^3 = a^3$. Denote ξ_1, ξ_2, ξ_3 three distinct roots of $x^3 - 1$ in K where $\xi_1 = 1$ and ξ_1 and ξ_2 are roots of $x^2 - x + 1$. We have three solutions (a, a, a), (a, ξ_1a, ξ_2a), and (a, ξ_2a, ξ_1a). Choosing $y_1 = 0$, we get $y_1 = y_2 = 0$. This means that $K[y_0, y_1, y_2]/(y_0^2 - y_1y_2, y_1^2 - y_0y_2, y_2^2 - y_0y_1)$ is the coordinate ring of three lines which span K^3 as a vector space[3].

The singularity of V_3 at 0 hence looks locally as three distinct lines intersecting in a single point. △

Example (Gluing two points on $\mathbb{A}^n_K$). Let $a_1, a_2 \in K^n$ be two distinct points. Because there always exists a linear bijection of $K^n \to K^n$ that maps any pair of distinct points to any other pair of distinct points, we can assume that $a_1 = (-1, 0, \ldots, 0)$ and $a_2 = (1, 0, \ldots, 0)$. We claim that the ideal I corresponding to the algebraic set $\{a_1, a_2\}$ is equal $((x_1 - 1)(x_1 + 1), x_2, \ldots, x_n)$. Clearly,

[3] Vectors $(1, 1, 1), (1, \xi_1, \xi_2)$, and $(1, \xi_2, \xi_1)$ are linearly independent by regularity of Vandermonde matrix, since $\xi_1^2 = \xi_2$ and $\xi_2^2 = \xi_1$.

$I = (x_1 - 1, x_2, \dots, x_n) \cap (x_1 + 1, x_2, \dots, x_n)$ and $((x_1 - 1)(x_1 + 1), x_2, \dots, x_n)$ is below this intersection. Take $f \in (x_1 - 1, x_2, \dots, x_n) \cap (x_1 + 1, x_2, \dots, x_n)$, clearly $f(x_1, 0, \dots, 0)$ is in the intersection as well as it is zero on $\{a_1, a_2\}$. The polynomial $f(x_1, \dots, x_n) - f(x_1, 0, \dots, 0) \in (x_2, \dots, x_n)$ as it is zero on the x_1-axis. Since $f(x_1, 0, \dots, 0)$ is in one variable and zero for $-1, 1$, then $f(x_1, 0, \dots, 0) \in ((x_1 - 1)(x_1 + 1))_{K[x_1]}$ where $K[x_1]$ is naturally a subring of $K[x_1, \dots, x_n]$. Then clearly $f(x_1, 0, \dots, 0) \in ((x_1 - 1)(x_1 + 1))$ and $f \in ((x_1 - 1)(x_1 + 1)) + (x_2, \dots, x_n)$ which is what we wanted to prove.

We will now show that as an $K-$algebra $K + I$ is generated by the following elements:

$$x_1^2 - 1, x_1(x_1^2 - 1), x_2, x_1x_2, \dots, x_n, x_1x_n.$$

Let $f \in I$ be of degree one, then $f = f - f(x_1, 0, \dots, 0)$ as $f(x_1, 0, \dots, 0)$ is of degree at least two. Therefore, f is a $K-$linear combination of $x_2, \dots, x_n$. Suppose $f \in I$ is of degree $d \geq 2$ or more, then $f = f_d + f'$ where f_d is the homogenous part of degree d and $\deg f' \leq d - 1$. Let $x_1^{c_1} \dots x_n^{c_n}$ be a monomial where $c_1 + \dots + c_n = d$ and $c_1, \dots, c_n \geq 0$. If $c_1 = 1$ then there is $2 \leq j \leq n$ such that $c_j \geq 1$, then $x_1^{c_1} \dots x_n^{c_n} = (x_1x_j)x_2^{c_2} \dots x_j^{c_j - 1} \dots x_n^{c_n}$ which clearly is generated by our elements. If $c_1 \geq 2$ then there are a, b non-negative integers such that $2a + 3b = c_1$, then $(x_1^2 - 1)^a(x_1(x_1^2 - 1))^b x_2^{c_2} \dots x_n^{c_n}$ is a product of our generators and $x_1^{c_1} \dots x_n^{c_n} - (x_1^2 - 1)^a(x_1(x_1^2 - 1))^b x_2^{c_2} \dots x_n^{c_n}$ is of degree at most $d - 1$. After this discussion, we can prove that f is in $K-$algebra generated by our elements using a simple induction on its degree, analogously as above.

Let us have a homomorphism $\psi : K[y_1, \dots, y_n, z_1, \dots, z_n] \to K + I$ such that for $y_1 \mapsto x_1^2 - 1, z_1 \mapsto x_1(x_1^2 - 1)$, and $y_i \mapsto x_i, z_i \mapsto x_1x_i$ for all $2 \leq i \leq n$. We claim that the kernel of this homomorphism is generated by the following relations $y_iy_jy_1 - z_iz_j - y_iy_j$ and $y_iz_j - y_jz_i$ for all $1 \leq i, j \leq n$.

We observe that no non-trivial relations of degree one are satisfied. Moreover, all relations of degree two need to be homogenous. Any relation F of degree 2 can be written as $F = F_2 - F_1$ where F_2 is the homogenous part of degree two and F_1 is the homogenous part of degree one. After plugging in, we get the equation $\psi(F_2) = \psi(F_1)$ in $K + I$. We infer that F_2 cannot contain a multiple of monomials $y_1^2, y_iz_1, z_iz_j, y_1z_i$, $1 \leq i, j \leq n$, as after applying ψ they are of degree at least 4, however, $\deg \psi(F_1) \leq 3$. It also cannot contain multiples of monomials y_iy_j, y_iz_j for $2 \leq i, j \leq n$ as they contain two variables aside of x_1 which does not hold for any of the generators.

This means that $F_1 = 0$. We show that $F_2 = \sum_{i<j} c_{i,j}(y_iz_j - y_jz_i)$ by discussion of possible cases. Suppose that F_2 contains a square y_i^2 or z_i^2, this cannot be the case since a square of each our generator can be written as a product of two generators in only one way. We observe the same about y_iy_j and z_iz_j. This leaves us with y_iz_j for $1 \leq i, j \leq n$, the situation when $i = j$ is trivial. We notice that $\psi(y_iz_j) = \psi(y_jz_i)$ and that the product $\psi(y_iz_j)$ can be expressed only in these two ways as a product of two of our generators. Thus if F_2 contains cy_iz_j, $i < k$, it needs to contain $c(y_iz_j - y_jz_i)$ for some $c \in K$.

Now, let us have a relation F of degree $d \geq 3$. We will proceed similarly as in the example above, we will isolate F_d, the homogenous part of F of degree d, if possible reduce degree using the prescribed relations and show that the rest needs to cancel out itself. We recognise three types of monomials whose $K-$linear combination is F_d — at first there are monomials which do not contain y_1 or z_1, then there are monomials that contain y_1 or z_1 but at most two y's, and finally monomials which contain y_1 or z_1 and contain at least three y's.

Monomials of the third type can be rewritten using relations of type $y_iz_j - y_jz_i$ to contain $y_iy_jy_1$ which in turn can be transformed by the other relations of type $y_iy_jy_1 - z_iz_j - y_iy_j$ to something of degree $d-1$.

Monomials of the second type contain at most two y's and thus at least $d-2$ z's. We notice that if we have a product P of m of our generators of $K+I$ which contains $m' \geq m-2$ generators of the second type, then any other product Q of at most m of our generators of $K+I$ having the same homogenous part in the highest degree as P has to contain at least m' of second type generators and actually $P = Q$. We will show that F'_d is in the kernel. We prove this by induction on m. It is clear for $m = 2$. In the induction step, it suffices to prove that such P and Q, where P is a product of $m+1$ generators, have a common factor of a second type generator. If P is a polynomial in x_1, this follows trivially, if P as a product contains x_1x_i for $2 \leq i \leq n$, then Q as a polynomial is divisible by x_1x_i as Q has the same same homogenous part in the highest degree as P and is a product of our generators of $K+I$.

After having shown this, we can easily deduce that monomials of the second type cancel themselves simply using relations of type $y_iz_j - y_jz_i$ as they are only different realisations of products from the claim above.

Monomials of the first type can be described by a, the number of z's in them, and $(a_i)_{i=2}^n$ where a_i are numbers of occurrences of y_i and z_i in the monomial for $2 \leq i \leq n$. Using relations of type $y_iz_j - y_jz_i$, one can transform two monomials of the first type an with a fixed a's and $(a_i)_{i=2}^n$'s to a common form and they need to cancel themselves out.

Therefore, we write $F_d = F_{d,1}+F_{d,2}+F_{d,3}$ as sum of monomials of first, second, and third type, where $F_{d,1}, F_{d,2}$ are in the kernel of ψ and there is G in the kernel of ψ such that $F_{d,3} - G$ is of degree at most $d-1$. Using our relations, we have thus replaced F by F' of strictly smaller degree. An inductive argument gives us that the kernel of ψ is generated by the following relations $y_iy_jy_1 - z_iz_j - y_iy_j$ and $y_iz_j - y_jz_i$ for all $1 \leq i, j \leq n$.

Identifying $(-1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ in K^n we get a variety W_n with coordinate ring:

$$K[y_1, \ldots, y_n, z_1, \ldots, z_n]/(y_iy_jy_1 - z_iz_j - y_iy_j, y_iz_j - y_jz_i \,\text{for}\, 1 \leq i, j \leq n).$$

As above, we will examine the ring of formal power series, we have:

$$K[[y_1, \ldots, y_n, z_1, \ldots, z_n]]/(y_iy_jy_1 - z_iz_j - y_iy_j, y_iz_j - y_jz_i \,\text{for}\, 1 \leq i, j \leq n)$$

but taking $u^{-1}y_i$ for each $1 \leq i \leq n$ where u is a square root of $1 - x_1$ which is invertible in $K[[y_1, \ldots, y_n, z_1, \ldots, z_n]]$ we get the following isomorphic ring:

$$K[[y'_1, \ldots, y'_n, z_1, \ldots, z_n]]/(y'_iy'_j - z_iz_j, y'_iz_j - y'_jz_i \,\text{for}\, 1 \leq i, j \leq n).$$

Thus around origin W_n looks *analytically* same as the variety W'_n with a coordinate ring:

$$K[y'_1, \ldots, y'_n, z_1, \ldots, z_n]/(y'_iy'_j - z_iz_j, y'_iz_j - y'_jz_i \,\text{for}\, 1 \leq i, j \leq n).$$

Our equations prescribe that $(y'_i - z_i)(y'_i + z_i) = 0$ for all $1 \leq i \leq n$. Suppose that a point with non-zero y'_i, y'_j and $z_i = y'_i$ and $z_j = -y'_j$ but we need to have $y'_i y'_j - z_i z_j = 2y'_i y'_j$ by another equation, we arrive to a contradiction. This means that W'_n looks as a pair of linear subspaces of dimension n intersecting at the origin. These linear subspaces are $(y'_1, \ldots, y'_n, y'_1, \ldots, y'_n)$ and $y'_1, \ldots, y'_n, -y'_1, \ldots, -y'_n)$ for $y'_1, \ldots, y'_n \in K$. △

2. Pushouts of affine schemes

In this chapter, we will focus on the existence of pushouts of affine schemes in the category of schemes. We will follow the the main reference articles Schwede [2004] and Ferrand [2003] and look for the possible pushout in the category of ringed spaces which is cocomplete. We generalise their results and show that the existence of pushouts of affine schemes in the category of schemes depends on the existence of pushouts of diagrams of affine schemes corresponding to inclusions of two subrings of a ring. For such diagrams, the approach of trying to prove that the pushout of a diagram in the category of ringed spaces is a scheme, moreover an affine scheme, will prove to be overly restrictive. We thus conclude this chapter by moving beyond it: we give a sufficient condition for a scheme to be a pushout of a diagram schemes and apply it to our case of interest – diagrams of affine schemes.

2.1 Basics of ringed spaces and schemes

At first, we recall the basic notions of the theory of ringed spaces and schemes from Chapters 2 and 3 of Görtz and Wedhorn [2010].

We begin by stating the definition of Zariski topology on prime spectra of commutative rings. It can be viewed as a vast generalisation of Zariski topology on affine algebraic sets over fields. Indeed, these two variants of Zariski topology share many key properties and, practically, coincide in case of algebraically closed fields.

Definition 23 (Spectrum of a ring defined as on page 41 in Görtz and Wedhorn [2010])**.** *Let A be a ring. We set* $\operatorname{Spec} A$ *to be the set of all prime ideals of A. For a set $I \subseteq A$, we set* $V(I) = \{\mathfrak{p} \in \operatorname{Spec} A : I \subseteq \mathfrak{p}\}$.

Lemma 24 (Lemma 2.1. on page 41 in Görtz and Wedhorn [2010])**.** *The map $I \mapsto V(I)$ is inclusion-reversing map from ideals of A to subsets of* $\operatorname{Spec} A$[1] *and also*

1. $V(0) = \operatorname{Spec} A$ *and* $V(1) = \emptyset$,
2. $V\left(\bigcup_{j\in J} I_j\right) = V\left(\sum_{j\in J} I_j\right) = \bigcap_{j\in J} V(I_j)$,
3. *for I, J ideals* $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

Definition 25 (Defintion 2.2 on page 41 in Görtz and Wedhorn [2010])**.** *Let A be a ring. The set* $\operatorname{Spec} A$ *of all prime ideals of A with the topology whose closed sets are the sets $V(I)$, where I runs through the set of ideals of A, is called the prime spectrum of A or simply the spectrum of A. The topology thus defined is called the Zariski topology on* $\operatorname{Spec} A$.

Remark. Open sets in Zariski topology are, of course, of form $\operatorname{Spec} A - V(I)$ for I and ideal. We define principal open sets as complements of principal ideals

[1] Clearly $V(I) = V((I)_A)$ where $(I)_A$ is the ideal generated by I in A, so we can without loss of generality assume to work only with ideals.

$D(f) = \operatorname{Spec} A - V(f)$ for $f \in A$. Note that we abuse the notation a little bit by writing $V(f)$ instead of $V(\{f\})$. Principal open sets $D(f)$ for $f \in A$ form a basis of the topology on $\operatorname{Spec} A$, see Proposition 2.5 on page 43 in Görtz and Wedhorn [2010].

Definition 26 ($I(-)$ defined as on page 42 in Görtz and Wedhorn [2010])**.** *Let A be a ring and $Y \subseteq \operatorname{Spec} A$ be a subset, we define:*

$$I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Proposition 27 (Part of Proposition 2.3. on page 42 in Görtz and Wedhorn [2010])**.** *Let A be a ring, $J \subseteq A$ an ideal, and Y a subset of* $\operatorname{Spec} A$*, then*

1. $\sqrt{I(Y)} = I(Y)$*,*
2. $I(V(J)) = \sqrt{J}$ *and $V(I(Y))$ is the closure of Y in* $\operatorname{Spec} A$ *with Zariski topology.*

Definition 28 (As defined on page 44 in Görtz and Wedhorn [2010])**.** *$A \mapsto \operatorname{Spec} A$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\varphi : A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a prime ideal of B, $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of A. Therefore we obtain a map ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ such that $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$.*

It turns out that there are natural maps of spectra continuous with respect to Zariski topology coming from homomorphisms of rings. These maps neatly correspond to polynomial maps between affine algebraic sets.

Proposition 29 (Part Proposition 2.10. and ensuing remarks on page 44 in Görtz and Wedhorn [2010])**.** *Let A be a ring and $\varphi : A \to B$ be a ring homomorphism, then for any M subset of A we have that ${}^a\varphi^{-1}(V(M)) = V(\varphi(M))$, hence ${}^a\varphi$ is continuous with respect to Zariski topologies on* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$*.*

While spectra of commutative rings share many properties with affine algebraic sets, it requires much more effort to equip spectra with an additional structure of functions over its open sets. For this end, we need to develop sophisticated machinery of sheafs.

Using sheaf, we will be able to define ringed spaces which are topological spaces with functions that respect the topology of the underlying space and schemes which are ringed spaces that locally look like affine schemes, speaking in loose terms.

Definition 30 (Modified Definition 2.17. on page 47 in Görtz and Wedhorn [2010])**.** *Let X be a topological space. A presheaf of rings $\mathcal{F}$ on X consists of the following data: a ring $\mathcal{F}(U)$ (or $\Gamma(U, \mathcal{F})$) for every $U \subseteq X$ open, and a ring homomorphism $\operatorname{res}^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ for each $U \subseteq V \subseteq X$ open. The data need to satisfy $\operatorname{res}^U_U = \operatorname{id}_U$ for each $U \subseteq X$ open and $\operatorname{res}^W_U = \operatorname{res}^V_U \circ \operatorname{res}^W_V$ for each $U \subseteq V \subseteq W$ open.*

Remark. For $s \in \mathcal{F}(V)$ and $U \subseteq V$, we will sometimes write simply $s|_U$ for $\operatorname{res}^V_U(s)$.

Definition 31 (Modified Definition 2.18. on page 48 in Görtz and Wedhorn [2010])**.** *Let X be a topological space. A presheaf of rings $\mathcal{F}$ on X is a sheaf if it satisfies for all U and all coverings $(U_i)_{i\in I}$ as above the following condition (sheaf property or axiom): the diagram*

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \underset{\sigma'}{\overset{\sigma}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $\sigma((s_i)_{i\in I}) = \sigma'((s_i)_{i\in I})$, where $\sigma : (s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j})_{i,j}$ and $\sigma' : (s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j})_{i,j}$.

Remark. If we know the value $\mathcal{F}(U)$ of a sheaf on every element U of some basis B of the topology on X, we can use the sheaf property to determine $\mathcal{F}(V)$ on an arbitrary open. This is made precise in the discussion after Examples 2.19. on page 49 in Görtz and Wedhorn [2010].

Definition 32 (Definition 2.21. and preceding remarks on page 50 in Görtz and Wedhorn [2010])**.** *Let X be a topological space, $\mathcal{F}$ be a presheaf on X, and let $x \in X$ be a point. The system $(\mathcal{F}(U), (\mathrm{res}_U^V)_{U\subseteq V})$ which is indexed by the set of open subsets $U \subseteq X$ with $x \in U$, ordered by containment, is a filtered inductive system. Then the inductive limit:*

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U)$$

is called the stalk of $\mathcal{F}$ in x.

Definition 33 (Modified Definition 2.29. on page 55 in Görtz and Wedhorn [2010])**.** *A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.*

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are ringed spaces, we define a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as a pair $(f, f^\flat)$, where $f : X \to Y$ is a continuous map and where $f^\flat$ is a collection of ring homomorphisms $f_U^\flat : \Gamma(U, \mathcal{O}_Y) \to \Gamma(f^{-1}(U), \mathcal{O}_X)$ form each $U \subseteq Y$ open and for every $U \subseteq V \subseteq Y$ open the following diagram commutes:

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_Y) & \xrightarrow{f_V^\flat} & \Gamma(f^{-1}(V), \mathcal{O}_X) \\ \downarrow{\scriptstyle \mathrm{res}_U^V} & & \downarrow{\scriptstyle \mathrm{res}_{f^{-1}(U)}^{f^{-1}(V)}} \\ \Gamma(U, \mathcal{O}_Y) & \xrightarrow{f_U^\flat} & \Gamma(f^{-1}(U), \mathcal{O}_X) \end{array}$$

A morphism of ringed spaces $(f, f^\flat) : X \to Y$ induces a morphism on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ for all $x \in X$. More details can be found above Definition 2.30. on pages 55 and 56 in Görtz and Wedhorn [2010].

Definition 34 (Definition 2.30. on page 56 in Görtz and Wedhorn [2010])**.** *A locally ringed space is a ringed space $\mathcal{O}_{X,x}$ such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.*

A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is is a morphism of ringed spaces $(f, f^\flat)$ such that for all $x \in X$ the induced homomorphism on stalks $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism that means that the only maximal ideal of $\mathcal{O}_{Y,f(x)}$ maps under the only maximal ideal of $\mathcal{O}_{X,x}$.

With the machinery of sheafs ready, we can equip spectra of commutative rings with functions.

Proposition 35 (Theorem 2.33. on page 57 and ensuing discussion in Görtz and Wedhorn [2010])**.** *Let A be a ring, then a presheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ given on the basis of principal open sets as $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ for all $f \in A$ with restriction morphisms given by universal property of the localisation (more details on the construction can be found above Theorem 2.33. on pages 57 and 58) is a sheaf and $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a locally ringed space with $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Using the additional structure on prime spectra, we get an analogue of one of the basic results of affine algebraic geometry − contravariant equivalence between categories of affine algebraic sets and coordinate rings.

Remark. Let $\varphi : A \to B$ be a ring homomorphism gives rise to a morphism of locally ringed spaces ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ gives rise where ${}^a\varphi^\flat_{D(f)} : A_f \mapsto B_{\varphi(f)}$ for all $f \in A$ determine ${}^a\varphi^\flat$ on all other $U \subseteq \operatorname{Spec} A$ open. For more information, see remarks after Definition 2.34. on page 59 in Görtz and Wedhorn [2010].

Definition 36 (Definition 2.34. on page 59 in Görtz and Wedhorn [2010])**.** *A locally ringed space $(X, \mathcal{O}_X)$ is called affine scheme, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.*

If $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of affine schemes, then it is uniquely determined by a ring homomorphism $f^\flat_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$, this gives rise to a contravariant equivalence of the category of rings and the category of affine schemes. See Theorem 2.35. and the discussion above it on page 59 in Görtz and Wedhorn [2010].

Definition 37 (Definition 3.1. on page 66 in Görtz and Wedhorn [2010])**.** *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open covering $X = \bigcup_{i \in I} U_i$ such that all locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are affine schemes. A morphism of schemes is a morphism of locally ringed spaces.*

Proposition 38 (Proposition and Definition 3.2. on page 67 in Görtz and Wedhorn [2010])**.** *Let X be a scheme, and $U \subseteq X$ an open subset. Then the locally ringed space $(U, \mathcal{O}_X|_U)$ is a scheme. We call U an open subscheme of X. If U is an affine scheme, then U is called an affine open subscheme. The affine open subschemes are a basis of the topology.*

Having defined all the necessary notions, we can commence our discussion on pushout of affine schemes. Our starting point is formed by the following definition and theorem:

Definition 39 (Definition and Proposition 2.1 on page 2 of Schwede [2004])**.** *Let $\varphi : X \to Y$ and $\psi : X \to Z$ be morphisms of ringed spaces. We define the pushout of these two morphisms as a ringed space $W = Y \sqcup Z / \sim$ with $\iota_1 : Y \to W$ and $\iota_2 : Z \to W$ where the equivalence $\sim$ is generated by pairs $(\varphi(z), \psi(z))$ for all $z \in Z$. The ring portion of W is given as follows: $\mathcal{O}_W(U)$ is the pullback of these two morphisms $\varphi^\flat_{\iota_1^{-1}(U)} : \mathcal{O}_Y(\iota_1^{-1}(U)) \to \mathcal{O}_X((\iota_1\varphi)^{-1}(U))$ and $\psi^\flat_{\iota_2^{-1}(U)} : \mathcal{O}_Z(\iota_2^{-1}(U)) \to \mathcal{O}_X((\iota_2\psi)^{-1}(U))$ for all $U \subseteq W$ open.*

Theorem 40 (Théorème 5.1. on page 568 of Ferrand [2003] and Theorem 3.4 on page 7 of Schwede [2004]). *Let $P = B \times_A C$ together with projections π_1 and π_2 be a pullback of the ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$. If φ or ψ is onto, then* $\operatorname{Spec} P$ *is the pushout of the induced diagram of schemes.*

We will try to generalise it by relaxing the assumptions made on φ, ψ that at least one of them is onto. We begin by generalising one step of proof of Theorem 40 in Schwede [2004].

Proposition 41. *Let $P = B \times_A C$ together with projections π_1 and π_2 be a pullback of the ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$ and $S \subseteq P$ be a multiplicative subset of P, then $S^{-1}P$ is the pullback of the following diagram of rings:*

$$\begin{array}{ccc} & & \pi_2(S)^{-1}C \\ & & \Big\downarrow {\scriptstyle \pi_2(S)^{-1}(\psi)} \\ \pi_1(S)^{-1}B & \xrightarrow{\pi_1(S)^{-1}(\varphi)} & (\varphi\pi_1)(S)^{-1}A \end{array}$$

where $\pi_1(S)^{-1}(\varphi)$ is the unique homomorphism extending:

$$B \xrightarrow{\varphi} A \longrightarrow (\varphi\pi_1)(S)^{-1}A$$

to a homomorphism from $\pi_1(S)^{-1}B$ to $(\varphi\pi_1)(S)^{-1}A$ which exists and is unique due to the universal property of localisation. The map $\pi_2(S)^{-1}(\psi)$ is defined analogously.

Proof. Suppose there are elements $\frac{b}{\pi_1(s_1)} \in \pi_1(S)^{-1}B$ and $\frac{c}{\pi_2(s_2)} \in \pi_2(S)^{-1}C$ such that $b \in B, c \in C$, $s_1, s_2 \in S$, and

$$\pi_1(S)^{-1}(\varphi)\left(\frac{b}{\pi_1(s_1)}\right) = \pi_2(S)^{-1}(\psi)\left(\frac{c}{\pi_2(s_2)}\right).$$

Expanding both sides of the equality above, we get:

$$\frac{\varphi(b)}{\varphi\pi_1(s_1)} = \frac{\psi(c)}{\psi\pi_2(s_2)}.$$

The resulting equality implies, that there exists $s \in S$ such that:

$$\varphi\pi_1(s)\psi(c)\varphi\pi_1(s_1) = \varphi\pi_1(s)\varphi(b)\psi\pi_2(s_2).$$

Since for all $p \in P$ we have $\varphi\pi_1(p) = \psi\pi_2(p)$, we can rewrite that as:

$$\psi(\pi_2(s)\pi_2(s_1)c) = \varphi(\pi_1(s)\pi_1(s_2)b).$$

This gives us that $(\pi_2(s)\pi_2(s_1)c, \pi_1(s)\pi_1(s_2)b) \in P$, as $P = B \times_A C$. Let us have:

$$s^{-1}s_1^{-1}s_2^{-1}(\pi_2(s)\pi_2(s_1)c, \pi_1(s)\pi_1(s_2)b) \in S^{-1}P.$$

However this element maps to $\frac{b}{\pi_1(s_1)} \in \pi_1(S)^{-1}B$ under $S^{-1}\pi_1$ and $\frac{c}{\pi_2(s_2)} \in \pi_1(S)^{-1}C$ under $S^{-1}\pi_2$. These maps are defined analogously to the map $\pi_1(S)^{-1}(\varphi)$.

Denote Q, ϱ_1, ϱ_2 the pullback of the localised diagram $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$ together with projections, then we trivially have that $\iota : S^{-1}P \hookrightarrow Q$. Suppose that $q \in Q$, then $\pi_1(S)^{-1}(\varphi)(\varrho_1(q)) = \pi_2(S)^{-1}(\psi)(\varrho_2(q))$. But we showed that there is $p \in S^{-1}P$ such that $S^{-1}\pi_1(p) = \varrho_1(q)$ and $S^{-1}\pi_2(p) = \varrho_2(q)$. Therefore $\iota(p) = q$, homomorphism ι is then onto and, in effect, an isomorphism. ☐

Remark. This result is included as Lemma 15.5.3. in Part 1, Chapter 15 *More on algebra* in Stacks project by [Stacks project authors, 2018, Tag 01Z8].

We can restate the Proposition 41 in a more geometric way in the following corollary.

Corollary. Let $P = B \times_A C$ together with projections π_1 and π_2 be a pullback of the ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$ and $f \in P$. Denote $X = \operatorname{Spec} P$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, $W = \operatorname{Spec} A$. Then $\mathcal{O}_X(D(f))$ together with maps ${}^a\pi_1{}^\flat_{D(f)}$ and ${}^a\pi_2{}^\flat_{D(f)}$ is the pullback of the induced diagram ${}^a\varphi^\flat_{D(\pi_1(f))} : \mathcal{O}_Y(D(\pi_1(f))) \to \mathcal{O}_W(D(\varphi\pi_1(f)))$ and ${}^a\psi^\flat_{D(\pi_2(f))} : \mathcal{O}_Z(D(\pi_2(f))) \to \mathcal{O}_W(D(\psi\pi_2(f)))$.

2.2 Pushout of the form of an affine scheme

We begin by trying to generalise the main result of Ferrand [2003] and Schwede [2004], Theorem 40. A straightforward generalisation commands us to look to the spectrum of the pullback of the induced map of rings when trying to find the pushout of a diagram of affine schemes in the category of schemes. This is supported by Proposition 41 which describes, in light of Definition 39, a part of structure sheaf of the possible pushout that looks like the structure sheaf on the spectrum of the pullback.

The following theorem will establish that the problem of the existence o f pushout of diagrams of affine schemes can be reduced to examining special types of diagrams of affine schemes that correspond two monomorphisms.

Theorem 42. *Let $P = B \times_A C$ together with projections π_1 and π_2 be a pullback of the ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$ and $U \subseteq \operatorname{Spec} P$ be an open set. Denote $X = \operatorname{Spec} P$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$, $W = \operatorname{Spec} A$. Then the following are equivalent*

1. *X is the pushout of the diagram ${}^a\varphi : W \to Y$ and ${}^a\psi : W \to Z$ in the category of ringed spaces,*
2. *X is the pushout of the diagram ${}^a\varphi : W \to Y$ and ${}^a\psi : W \to Z$ in the category of topological spaces,*
3. $\operatorname{Spec} \operatorname{Im} \varphi \cap \operatorname{Im} \psi$ *is the pushout of the induced diagram* $\operatorname{Spec} \operatorname{Im} \varphi \to W$ *and* $\operatorname{Spec} \operatorname{Im} \psi \to W$ *with natural maps in the category of topological spaces.*

Proof. (1) $\Leftrightarrow$ (2) This equivalence is established by the Theorem 40 and the definition of ringed space and pushout of ringed spaces in Definition 39. We know that $X = \operatorname{Spec} P$ has a basis of principal affine opens $\operatorname{Spec} P_f$ for all $f \in P$, simply localisations in all its elements. The structure sheaf of X is determined by its sections on basis open sets $\operatorname{Spec} P_f$ which are P_f. But the Proposition 41 and the Definition 39 of pushout of ringed spaces tell us what the sections should look just like that.

(2) $\Rightarrow$ (3) Denote $V = \operatorname{Spec} \operatorname{Im} \varphi \cap \operatorname{Im} \psi$. We observe that it is naturally homeomorphic to $V = V_P((\operatorname{Ker} \varphi, \operatorname{Ker} \psi))$. Clearly:

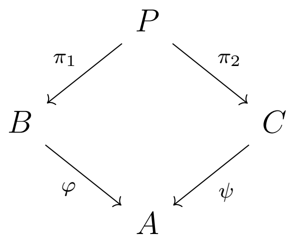
$$ {}^a\pi_1^{-1}(V_P((\operatorname{Ker} \varphi, \operatorname{Ker} \psi))) = V_B(\operatorname{Ker} \varphi) $$

and

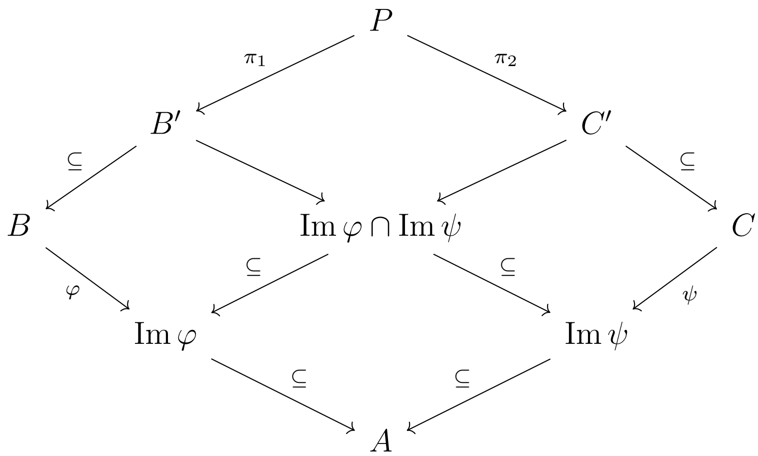
$$ {}^a\pi_2^{-1}(V_P((\operatorname{Ker}\varphi, \operatorname{Ker}\psi))) = V_C(\operatorname{Ker}\psi). $$

Moreover, it holds that ${}^a\varphi(W) \subseteq V_B(\operatorname{Ker}\varphi)$ and ${}^a\psi(W) \subseteq V_C(\operatorname{Ker}\psi)$. However, X being the pushout of the diagram ${}^a\varphi : W \to Y$ and ${}^a\psi : W \to Z$ means that $V \subseteq X$ is the pushout of the induced diagram of preimages, in our case ${}^a\varphi : W \to V_B(\operatorname{Ker}\varphi)$ and ${}^a\psi : W \to V_C(\operatorname{Ker}\psi)$.

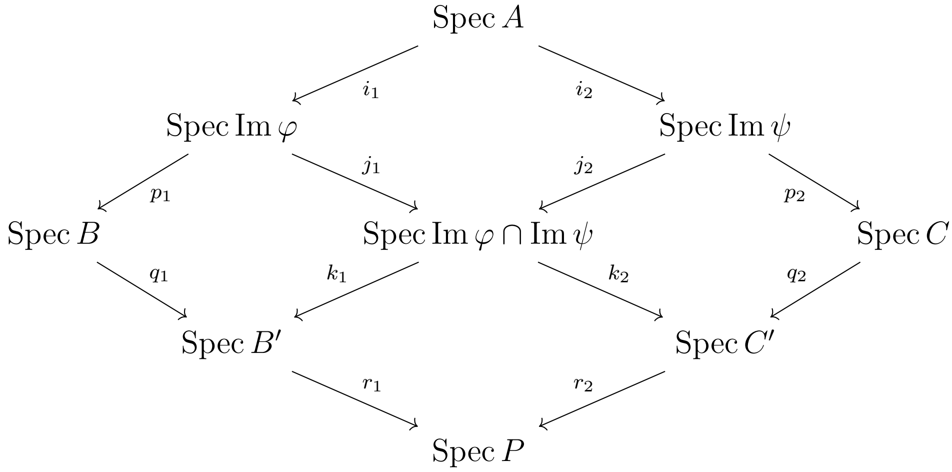
(3) ⇒ (2) At first, we expand the diagram:



into the following commutative diagram of rings:



It is easy to observe that all squares are commutative and that $\mathrm{Im}\pi_1 = B', \mathrm{Im}\pi_2 = C', P$, and $\operatorname{Im}\varphi \cap \operatorname{Im}\psi$ are pullbacks of their respective diagrams. This diagram transforms via the contravariant equivalence of categories into this diagram of schemes, or:



We know that $\operatorname{Spec} B'$, $\operatorname{Spec} C'$, and $\operatorname{Spec} P$ are pushouts of their respective diagrams by the previous theorem, as ring homomorphisms corresponding to p_1, p_2, k_1, k_2 are surjective. Also, we assume $\operatorname{Spec} \operatorname{Im} \varphi \cap \operatorname{Im} \psi$ to be the pushout of the respective diagram.

Suppose there is D with maps $d_1 : \operatorname{Spec} B \to D$ and $d_2 : \operatorname{Spec} C \to D$ such that $d_1 p_1 i_1 = d_2 p_2 i_2$. Then there exists a unique map $e : \operatorname{Spec} \operatorname{Im} \varphi \cap \operatorname{Im} \psi \to D$ such that $d_1 p_1 = e j_1$ and $d_2 p_2 = e j_2$. Subsequently, we get unique maps $f_1 : \operatorname{Spec} B' \to D$ and $f_2 : \operatorname{Spec} C' \to D$ with the property that $d_1 = f_1 q_1, e = f_1 k_1$ and $d_2 = f_2 q_2, e = f_2 k_2$ respectively. Finally, there exists a unique map $g : \operatorname{Spec} P \to D$ such that $f_1 = g r_1$ and $f_2 = g r_2$. The uniqueness of g is provided by its construction. Another demonstration of this claim proceed by multiple application of the so called pasting law for pushouts, see The nLab project authors for details. This abstract non-sense argument gives us that $\operatorname{Spec} P$ is the pushout of the diagram:

$$\begin{array}{ccccc} & & \operatorname{Spec} A & & \\ & \swarrow^{p_1 i_1} & & \searrow^{p_2 i_2} & \\ \operatorname{Spec} B & & & & \operatorname{Spec} C \\ & \searrow_{r_1 q_1} & & \swarrow_{r_2 q_2} & \\ & & \operatorname{Spec} P & & \end{array}$$

in the category of topological spaces. □

The Theorem 42 tells us basically that we can restrict ourselves to testing whether $\operatorname{Spec} P$ is the pushout of:

$$\begin{array}{ccccc} & & \operatorname{Spec} A & & \\ & \swarrow^{{}^a\varphi} & & \searrow^{{}^a\psi} & \\ \operatorname{Spec} B & & & & \operatorname{Spec} C \\ & \searrow_{{}^a\pi_1} & & \swarrow_{{}^a\pi_2} & \\ & & \operatorname{Spec} P & & \end{array}$$

in the category of schemes given that P is the pullback of the corresponding diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ B & & & & C \\ & \searrow_{\varphi} & & \swarrow_{\psi} & \\ & & A & & \end{array}$$

in the category of commutative rings where all morphisms are monomorphisms or, without loss of generality, inclusions. Notice that the very same situation was the most problematic in our discussion in the first chapter, see opening discussion in Section 1.4. We also know that it suffices to show that $\operatorname{Spec} P$ is the pushout of the diagram only in the category of topological spaces.

We know that if $\operatorname{Spec} P$ is the pushout of the diagram, then it has to be naturally a quotient of $\operatorname{Spec} B \sqcup \operatorname{Spec} C$ which is naturally homeomorphic to

Spec $B \times C$, where the quotient map is given by ${}^a\pi_1$ on Spec B and by ${}^a\pi_2$ on Spec C or ${}^a(\pi_1, \pi_2) : P \to B \times C$.

We will now thus investigate when an inclusion of rings $C \subseteq D$ induces a quotient map Spec $D \to$ Spec C.

Lemma 43. *Let $C \subseteq D$ be an extension of rings, we shall denote the inclusion ι. Then the following are equivalent:*

1. Spec(ι) *is surjective,*
2. *for each ideal I of C we have $\sqrt{(I)_D} \cap C = \sqrt{I}$,*
3. *if for any $c_1, \ldots, c_n \in C$ there are $x_1, \ldots, x_{n-1} \in D$ with the property that $\sum_{i=1}^{n-1} x_i c_i = c_n$ then $c_n \in \sqrt{(c_1, \ldots, c_{n-1})_C}$.*

Proof. (1) $\Rightarrow$ (2) Let I then be an ideal of C. We can write its radical as $\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \in \operatorname{Spec} C} \mathfrak{p}$, however for each such $\mathfrak{p}$ there is a prime $\mathfrak{q}$ of D lying over it. Hence, the radical of (I_D) is below the intersection of such $\mathfrak{q}$. This intersection, however, does not contain any elements of C not in $\sqrt{I}$. Therefore $\sqrt{(I)_D} \cap C \subseteq \sqrt{I}$, the other inclusion being obvious.

(2) $\Rightarrow$ (3) Let there be $x_1, \ldots, x_{n-1} \in D$ such that for $c_1, \ldots, c_n \in C$, we have $\sum_{i=1}^{n-1} x_i c_i = c_n$. Clearly, $c_n \in (c_1, \ldots, c_{n-1})_D$, c_n lies in the radical of $(c_1, \ldots, c_{n-1})_D$. By our assumption, we have that c_n is contained in the radical of $(c_1, \ldots, c_{n-1})_C$.

(3) $\Rightarrow$ (1) Let $\mathfrak{p}$ be a prime of C. Take the ideal $(\mathfrak{p})_D$. Suppose that there is $c \in C - \mathfrak{p}$, $c \in (\mathfrak{p})_D$. That would mean, however, that there are $c_1, \ldots, c_n \in \mathfrak{p}$ and $x_1, \ldots, x_n \in D$ such that $\sum_{i=1}^{n} c_i x_i = c$. We assume that c is the radical of $(c_1, \ldots, c_n)_C$ which is contained in $\mathfrak{p}$, this yields a contradiction. Therefore, $(\mathfrak{p})_D \cap C = \mathfrak{p}$. As $(\mathfrak{p})_D$ avoids the multiplicative set $C - \mathfrak{p}$, there has to be a prime $\mathfrak{q}$ of D by Theorem 46, such that $(\mathfrak{p})_D \subseteq \mathfrak{q}$ and $\mathfrak{q} \cap C - \mathfrak{p}$, thus $\mathfrak{q} \cap C = \mathfrak{p}$. □

Remark. Proposition 4.8 on page 96 in Görtz and Wedhorn [2010] deals with the same problem in Proposition 43 in an entirely different way.

While it is possible to provide a full characterisation of quotient maps of affine schemes arising from inclusions of rings, we prefer to give the following simple sufficient condition. This condition will be shown to generalise two important properties of morphisms of affine schemes − going up and going down. However, before stating the result, we need to introduce some notions and results.

Definition 44 (Definition on page 28 in Kaplansky [1974]). *Let $R \subseteq S$ be an extension of rings, we say that it has lying over if for every $\mathfrak{p}$ prime of R there is $\mathfrak{q}$ a prime of S such that $\mathfrak{q} \cap R = \mathfrak{p}$, we also say that $\mathfrak{q}$ lies over $\mathfrak{p}$. The extension is said to have going up if for every pair of primes $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ and every $\mathfrak{q}_1$ lying over $\mathfrak{p}_1$, there is $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ lying over $\mathfrak{p}_2$. Similarly, we say that the extension has going down if for pair every primes $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ and every $\mathfrak{q}_2$ lying over $\mathfrak{p}_2$, there is $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ lying over $\mathfrak{p}_1$. The extension is said to have incomparability property if any two primes of S that lie over the same prime of R are not comparable with respect to inclusion.*

Theorem 45 (Theorem 17 on page 11, Theorems 44 and 48 on page 29 in Kaplansky [1974]. Hereafter referred to as going up theorem.)**.** *Let $R \subseteq S$ such that S is integral over R, this is especially true if S is a finitely generated $R-$module. Then the extension has lying over, incomparability property, and going up.*

Theorem 46 (Proposition 2.11 on page 70 in Eisenbud [1995])**.** *Let R be a commutative ring, $U \subseteq R$ a multiplicative subset, and I a maximal ideal of R not meeting U, then I is a prime ideal.*

Lemma 47. *Let $C \subseteq D$ be an extension of rings, we shall denote the inclusion ι. If $\operatorname{Spec}(\iota)$ is surjective and for each primes $\mathfrak{p} \subseteq \mathfrak{p}'$ of C there are primes $\mathfrak{q} \subseteq \mathfrak{q}'$ of D such that $\mathfrak{q} \cap C = \mathfrak{p}$ and $\mathfrak{q}' \cap C = \mathfrak{p}'$, then $\operatorname{Spec}(\iota)$ is a quotient map.*

Proof. Assume X is a subset of $\operatorname{Spec} C$ such that its preimage X' under $\operatorname{Spec}(\iota)$ is closed in $\operatorname{Spec} D$. We shall show that $X = \overline{X}$. Let $\mathfrak{p}'$ be an element of $\overline{X}$. Then $C - \mathfrak{p}'$ is a multiplicative set. From surjectivity of the map $\operatorname{Spec}(\iota)$, we have that $I(X') \cap C = I(X) = I(\overline{X})$. Therefore, $I(X')$ avoids $C - \mathfrak{p}'$. There exists a prime $\tilde{\mathfrak{q}}$ of D by Theorem 46, in X' as it is closed, such that $\tilde{\mathfrak{q}}$ avoids $C - \mathfrak{p}'$. We obtain a prime $\mathfrak{p} = \tilde{\mathfrak{q}} \cap C$ in X that is below $\mathfrak{p}'$ since it avoids the complement of $C - \mathfrak{p}'$. However, by our presupposition, there are $\mathfrak{q} \subseteq \mathfrak{q}'$ of D such that $\mathfrak{q} \cap C = \mathfrak{p}$ and $\mathfrak{q}' \cap C = \mathfrak{p}'$. By definition, $\mathfrak{q} \in X'$, but so there is $\mathfrak{q}'$ as it is above $\mathfrak{q}$ and X' is closed. Then, $\mathfrak{p}'$ is in X. □

Corollary. Let $C \subseteq D$ be an extension of rings and the induced map of spectra be denoted $\operatorname{Spec}(\iota)$. Provided that $\operatorname{Spec}(\iota)$ is surjective and the extension has going up or going down, then $\operatorname{Spec}(\iota)$ is a quotient map.

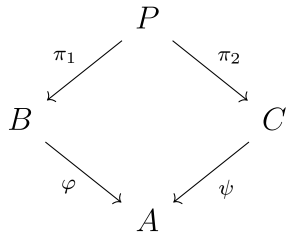
Proof. Suppose that $\mathfrak{p} \subseteq \mathfrak{p}'$ are primes of C. By surjectivity of the induced map, there are primes $\mathfrak{q}$ and $\mathfrak{r}'$ of D such that $\mathfrak{q} \cap C = \mathfrak{p}$ and $\mathfrak{r}' \cap C = \mathfrak{p}'$. If the extension has going down, then there is $\mathfrak{r} \subseteq \mathfrak{r}'$ a prime of D with the property that $\mathfrak{r} \cap C = \mathfrak{p}$. Provided that the extension has going up, we can find a prime $\mathfrak{q}'$ of D containing $\mathfrak{q}$ that maps to $\mathfrak{p}'$ under $\operatorname{Spec}(\iota)$. □

Corollary. Let $C \subseteq D$ be an extension of domains and the induced map of spectra be denoted $\operatorname{Spec}(\iota)$. Provided that $\operatorname{Spec}(\iota)$ is surjective, all non-zero prime ideals of C are of height one, then $\operatorname{Spec}(\iota)$ is a quotient map.

Proof. The only prime of height 0 of C lies generally under $\bigcap_{\mathfrak{p} \in \operatorname{Spec} C} \mathfrak{p}$ which is the nilradical, therefore $\sqrt{(0)_C}$ needs to be the only height 0 prime of C. Moreover, this ideal needs to be $(0)_C$ as C is a domain.

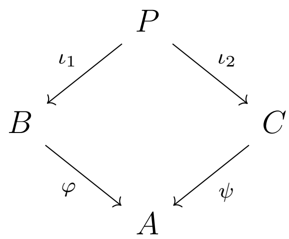
As D is also assumed to be a domain, $(0)_D$ is a prime and $\operatorname{Spec}(\iota)((0)_D) = (0)_C$. Suppose we have an inclusion of primes of C, the only non-trivial are $(0)_C \subseteq \mathfrak{p}$ for $\mathfrak{p}$ non-zero. However, we suppose that there is $\mathfrak{q}$ prime of D which maps to $\mathfrak{p}$ under $\operatorname{Spec}(\iota)$, but it has to contain $(0)_D$ which maps to $(0)_C$. By Lemma 47, $\operatorname{Spec}(\iota)$ is a quotient map. □

However, it does not suffice to show that $\operatorname{Spec} P$ is naturally a quotient of $\operatorname{Spec} B \times C$ while discussing whether $\operatorname{Spec} P$ is the pushout of the diagram of affine schemes in the category of schemes arising from a diagram of rings:

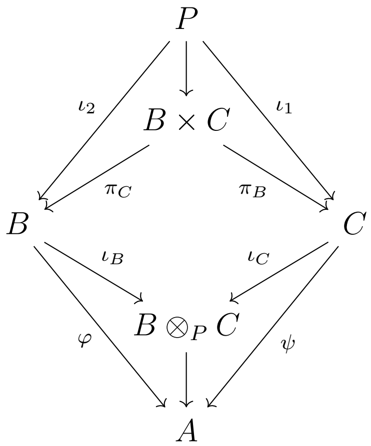


where all morphisms are monomorphisms and P is the pullback. We need to ensure that the equivalence induced by morphisms ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ and ${}^a\psi : \operatorname{Spec} A \to \operatorname{Spec} C$ merges all pairs primes of $\operatorname{Spec} B \times C$ which map to the same primes in $\operatorname{Spec} P$ under ${}^a(\pi_1, \pi_2)$ which is generally a daunting task. We thus strive to give at least some sufficient conditions.

Suppose we have the following commutative diagram of rings:



where $\iota_1, \iota_2, \varphi, \psi$ are monomorphisms and P is the pullback of the said diagram. We can enrich this diagram by adding two rings $B \times C$ and $B \otimes_P C$ with natural maps:



where the map from P to $B \times C$ is the induced (ι_1, ι_2) and the map from $B \times_P C$ is the induced map $\varphi \otimes_P \psi$. Passing to their spectra, we get a diagram of affine schemes with reversed arrows in the category of topological spaces. Let us assume that $\operatorname{Spec} P$ is the pushout of the diagram, then it has to be naturally homeomorphic to $\operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C$. Therefore, the induced map $\operatorname{Spec}(\iota_1, \iota_2)$ has to be a quotient map, as $\operatorname{Spec} B \times C$ is naturally homeomorphic to $\operatorname{Spec} B \sqcup \operatorname{Spec} C$.

Lemma 48. *Let $P \subseteq B$ and $P \subseteq C$ be extensions of rings, we shall denote the natural inclusion $P \subseteq B \times C$ by ι. If both extensions, $P \subseteq B$ and $P \subseteq C$, satisfy that given primes $\mathfrak{p} \subseteq \mathfrak{p}'$ of P such that there is a prime $\mathfrak{r}$ of B,C respectively* *lying over $\mathfrak{p}$, then there are primes $\mathfrak{q} \subseteq \mathfrak{q}'$ of B, C respectively lying over $\mathfrak{p}$ and $\mathfrak{p}'$ respectively, then so does the extension $P \subseteq B \times C$.*

Proof. The statement follows immediately from the fact that $\operatorname{Spec} B \times C \cong \operatorname{Spec} B \sqcup \operatorname{Spec} C$. □

Corollary. Suppose that $P \subseteq B$ and $P \subseteq C$ are extensions of rings and that the natural inclusion $P \subseteq B \times C$ is denoted ι. Provided that one of the extensions has going up and the other going up or going down, then the induced map $\operatorname{Spec} \iota$ is a quotient map of topological spaces. If $\operatorname{Spec} \iota$ is surjective and both extensions have going down, then $\operatorname{Spec} \iota$ is a quotient map of topological spaces.

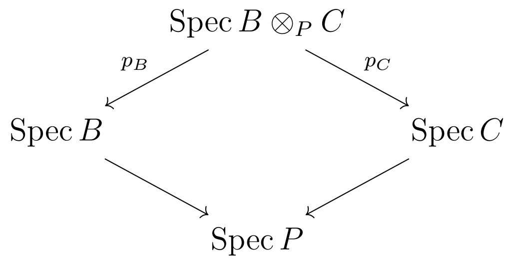
Proof. It easily follows from the previous lemma and the fact that going up implies lying over in this case. See Theorem 42 on page 29 of Kaplansky [1974]. □

Proposition 49. *Let $P \subseteq B$ and $P \subseteq C$ be extensions of rings, we shall denote the natural inclusion $P \subseteq B \times C$ by ι. Provided that*

1. *the induced map* $\operatorname{Spec}(\iota)$ *is a quotient map,*

2. *and we have that if more then one prime of either of rings lie over the same prime of P, then there is a prime of the other ring lying over the said prime of P,*

then $\operatorname{Spec} P$ *is the pushout of the following diagram in the category of topological spaces:*



where p_B and p_C are maps of spectra induced by natural inclusions of B and C into $B \otimes_P C$ respectively.

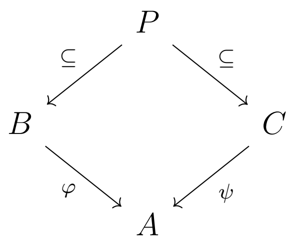
Proof. We assume that $\operatorname{Spec} P$ is a quotient of $\operatorname{Spec} B \times C$ which is naturally homeomorphic to $\operatorname{Spec} B \sqcup \operatorname{Spec} C$. Therefore, we only need to show that the kernel of $\operatorname{Spec}(\iota)$ is the right equivalence and, for that, it suffices to show that primes of $B \times C$ lying over the same prime of P are identified.

Suppose that primes $\mathfrak{q}_B$ of B and $\mathfrak{q}_C$ of C lie over same prime of P. We will show that there is a prime $\mathfrak{q}$ of $B \otimes_P C$ such that $p_B(\mathfrak{q}) = \mathfrak{q}_B$ and $p_C(\mathfrak{q}) = \mathfrak{q}_C$. The existence of such $\mathfrak{q}$ is given by Theorem 46, take a maximal ideal of $B \otimes_P C$ above the ideal $(\mathfrak{q}_B \otimes 1, 1 \otimes \mathfrak{q}_C)_{B \otimes_P C}$ avoiding the multiplicative subset $\{q_1 \otimes q_2, q_1 \in B - \mathfrak{q}_B, q_2 \in C - \mathfrak{q}_C\}$, which is prime.

Suppose that distinct primes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ of B lie over the same prime of P. By our second assumption, there is a prime $\mathfrak{q}_C$ of C lying over the same prime of P. We know that $(\mathfrak{q}_1, \mathfrak{q})$ and $(\mathfrak{q}_2, \mathfrak{q})$ are in tke kernel of $\operatorname{Spec}(\iota)$, therefore by symmetry and transitivity, so is $(\mathfrak{q}_1, \mathfrak{q}_2)$. For two primes of C lying over over the same prime of P, we proceed similarly due to the second assumption. □

Remark. Notice that the condition 2. in the previous proposition, which may appear technical, is very important. It allows us to exploit the existence of primes in $\operatorname{Spec} B \sqcup \operatorname{Spec} C$ corresponding to pairs of primes in $\operatorname{Spec} B$ and $\operatorname{Spec} C$ to show that we glue also pairs of primes of either B or C that map to the same prime of P. Without it, the proposition would not hold in general.

Corollary. Under the assumptions of the proposition above, suppose we have the following commutative diagram of rings:

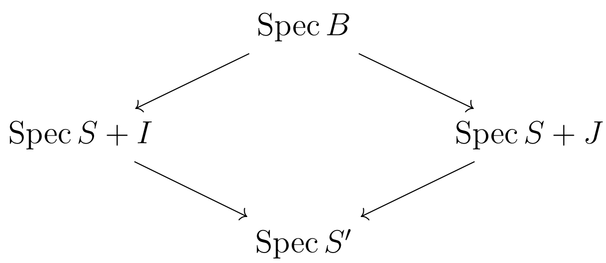


where P is the pullback of the commutative diagram and φ, ψ are monomorphisms. If the induced map of spectra $\operatorname{Spec} \varphi \otimes_P \psi$ is surjective, then $\operatorname{Spec} P$ is the pushout of the induced diagram in the category of topological spaces.

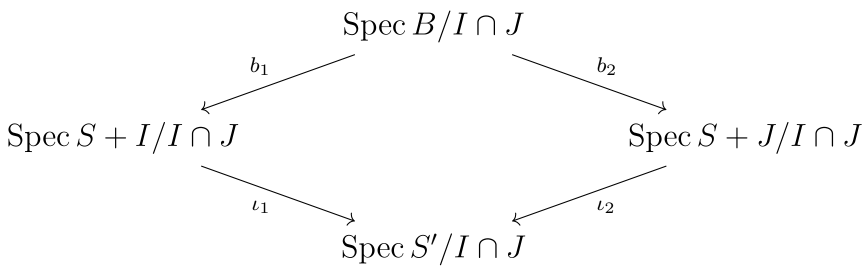
Proof. The proof goes along the same lines as the proof above. □

The following example represents a connection of this section and the previous chapter. We show that, under some additional assumptions, the pullback in the category of finitely generated algebras from Theorem 17 gives rise to a pushout in the from of an affine scheme in the category of schemes.

Example. Let B a finitely generated algebra and a domain over R, I, J its ideals and $S \subseteq B$ is $R-$subalgebra such that $(S \cap I) + (S \cap J) = S \cap (I + J)$. Suppose that $S + I$ and $S + J$ are finitely generated as $R-$algebras. This means that also their intersection S' is finitely generated by Theorem 17 and contains $I \cap J$ which, in turn, implies that $B, S + I, S + J$ are finitely generated $S'-$modules by Theorem 11. We get an induced diagram of spectra:



where all maps are closed an onto by the going up theorem. We will prove that $\operatorname{Spec} S'$ is the pushout of the corresponding diagram in the category of ringed spaces. It suffices to show that $\operatorname{Spec} S'/I \cap J$ is the pushout of the following diagram of spectra



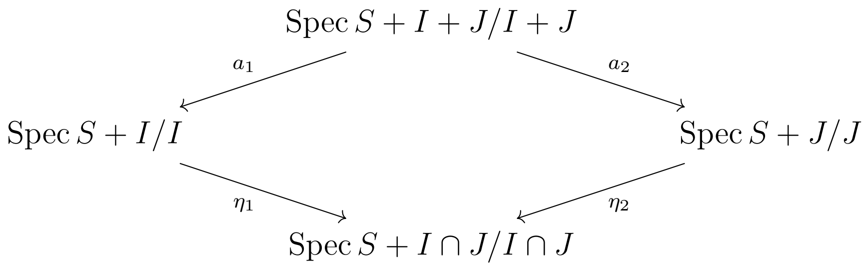
this due to the trick of localising in elements of $I \cap J$, a common ideal of all rings in the diagram, which yield a diagram of the same rings − it is made precise by Schwede [2004] in his proof of Theorem 40. We need to show that $\operatorname{Spec} S'/I \cap J$ is naturally homeomorphic to the quotient of $\operatorname{Spec} S + I/I \cap J \sqcup \operatorname{Spec} S + J/I \cap J$ by the equivalence θ generated by $(\mathfrak{p}, \mathfrak{q}) \in \operatorname{Spec} S + I/I \cap J \times \operatorname{Spec} S + J/I \cap J$ such that $b_1(\mathfrak{r}) = \mathfrak{p}$ and $b_2(\mathfrak{r}) = \mathfrak{q}$ for some $\mathfrak{r} \in \operatorname{Spec} B/I \cap J$.

As all maps in the diagram are closed and onto by the going up theorem, we know that $\operatorname{Spec} S'/I \cap J$ is naturally a quotient of $\operatorname{Spec} S + I/I \cap J \sqcup \operatorname{Spec} S + J/I \cap J$. We only need to ensure that for any $(\mathfrak{p}, \mathfrak{q}) \in \operatorname{Spec} S + I/I \cap J \times \operatorname{Spec} S + J/I \cap J$ such that $\iota_1(\mathfrak{p}) = \iota_2(\mathfrak{q})$, $(\mathfrak{p}, \mathfrak{q}) \in \theta$. Suppose we have $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} S + I/I \cap J$ such that $\iota_1(\mathfrak{p}_1) = \iota_1(\mathfrak{p}_2)$. Then there is $\mathfrak{r} \in \operatorname{Spec} B/I \cap J$ such that $b_1(\mathfrak{r}) = \mathfrak{p}_2$, clearly $\iota_1(\mathfrak{p}_1) = \iota_2(b_2(\mathfrak{r}))$ by commutativity of our diagram. If we have that $(\mathfrak{p}_1, b_2(\mathfrak{r})) \in \theta$, then we also know that $(\mathfrak{p}_1, \mathfrak{p}_2) \in \theta$ by its transitivity, symmetry, and the fact that $(b_1(\mathfrak{r}), b_2(\mathfrak{r})) \in \theta$ by definition.

Assume that there are $\mathfrak{p} \in \operatorname{Spec} S + I/I \cap J$ and $\mathfrak{q} \in \operatorname{Spec} S + J/I \cap J$ such that $\iota_1(\mathfrak{p}) = \iota_2(\mathfrak{q})$. We will discuss two possible cases, either $I/I \cap J \subseteq \mathfrak{p}$ and $J/I \cap J \subseteq \mathfrak{q}$ or $I/I \cap J \subseteq \mathfrak{p}$ and $J/I \cap J \nsubseteq \mathfrak{q}$. The case of $I/I \cap J \nsubseteq \mathfrak{p}$ and $J/I \cap J \subseteq \mathfrak{q}$ can be reduced to the second case by symmetry.

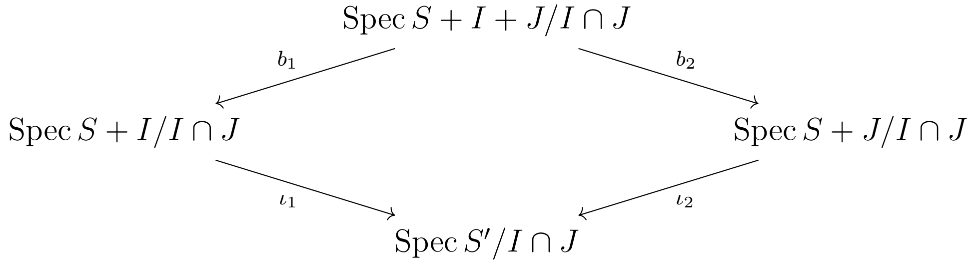
The remaining case, $I/I \cap J \nsubseteq \mathfrak{p}$ and $J/I \cap J \nsubseteq \mathfrak{q}$, can be reduced to the first case as we will show. take $\mathfrak{r}, \mathfrak{s} \in \operatorname{Spec} B/I \cap J$, they have to contain $I/I \cap J$ or $J/I \cap J$ since their product is zero in $B/I \cap J$, such that $b_1(\mathfrak{r}) = \mathfrak{p}$ and $b_2(\mathfrak{s}) = \mathfrak{q}$. Since $I/I \cap J \nsubseteq \mathfrak{p}$ and $J/I \cap J \subseteq \mathfrak{q}$, then $J/I \cap J \subseteq \mathfrak{r}$ and $I/I \cap J \subseteq \mathfrak{s}$, which permits us to deal with $I/I \cap J \subseteq b_1(\mathfrak{s})$ and $J/I \cap J \subseteq b_2(\mathfrak{r})$ as in the first case. Because if $(b_1(\mathfrak{s}), b_2(\mathfrak{r})) \in \theta$, then $(\mathfrak{p}, \mathfrak{q}) \in \theta$ by its transitivity and symmetry, since it contains $(\mathfrak{p}, b_2(\mathfrak{r}))$ and $(b_1(\mathfrak{s}), \mathfrak{q})$ by definition.

Let $I/I \cap J \subseteq \mathfrak{p}$ and $J/I \cap J \subseteq \mathfrak{q}$ such that $\iota_1(\mathfrak{p}) = \iota_2(\mathfrak{q})$. We know that $\operatorname{Spec} S + I \cap J/I \cap J$ is the pushout of the following commutative diagram:



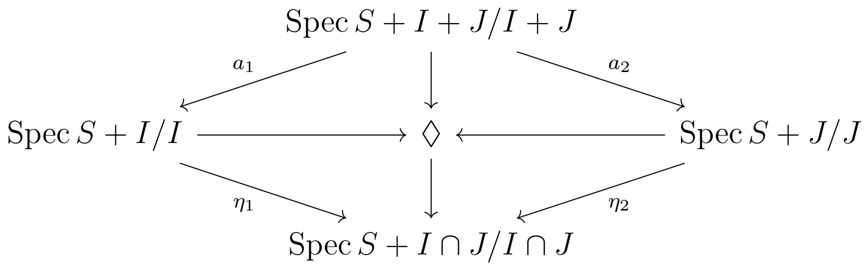
which actually only represents that $V(S \cap J) \cup V(S \cap I) = V(S \cap I \cap J)$ and $V(S \cap J) \cup V(S \cap I) = V(S \cap I \cap J)$ and $V(S \cap J) \cap V(S \cap I) = V((S \cap I) + (S \cap J)) = V(S \cap (I + J))$ by our technical assumption that $(S \cap I) + (S \cap J) = S \cap (I + J)$. This result is mentioned by Ferrand [2003] as *théorème chinois* on page 557 and $\operatorname{Spec} S + I \cap J/I \cap J$ also can be seen to be the pushout of the following diagram using Theorem 40 as both ring homomorphisms in question are surjective.

Into the diagram above, we can inscribe the following commutative diagram:

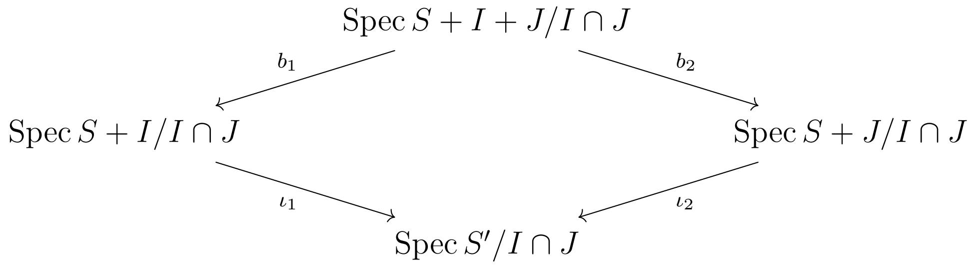


so that the resulting diagram is commutative, since we have natural closed immersions of affine schemes $\operatorname{Spec} S+I+J/I+J \hookrightarrow \operatorname{Spec} S+I+J/I\cap J, \operatorname{Spec} S+I/I \hookrightarrow \operatorname{Spec} S+I/I\cap J, \operatorname{Spec} S+J/J \hookrightarrow \operatorname{Spec} S+J/I\cap J$ which correspond to taking appropriate quotients of the respective rings. We will think of these morphisms of schemes as of inclusions. Also $S+I\cap J/I\cap J$ is naturally a subalgebra of $S'/I\cap J$, which gives us a morphism of schemes $\operatorname{Spec} S'/I\cap J \to \operatorname{Spec} S+I\cap J/I\cap J$.

The result is a commutative diagram that looks like this:



where $\Diamond$ represents the diagram:



Notice that, for this manoeuvre, we used the technical assumption.

Let us return to our $\mathfrak{p}$ and $\mathfrak{q}$, naturally $\mathfrak{p} \in \operatorname{Spec} S+I/I$ and $\mathfrak{q} \in \operatorname{Spec} S+J/J$. We know that $\eta_1(\mathfrak{p}) = \eta_2(\mathfrak{q})$ as $\iota_1(\mathfrak{p}) = \iota_2(\mathfrak{q})$. This means that there is $\mathfrak{r} \in \operatorname{Spec} S+I+J/I+J$ such that $a_1(\mathfrak{r}) = \mathfrak{p}$ and $a_2(\mathfrak{r}) = \mathfrak{q}$, this $\mathfrak{r}$ can be atken as a prime of $\operatorname{Spec} S+I+J/I\cap J$, naturally.

Clearly, we have a surjective morphism of affine schemes $\operatorname{Spec} B/I\cap J \to \operatorname{Spec} S+I+J/I\cap J$ given that $S+I+J/I\cap J$ is a subring of $B/I\cap J$. Thus, there is $\mathfrak{r}' \in \operatorname{Spec} B/I\cap J$ such that $\mathfrak{r}'$ maps to $\mathfrak{r}$. This gives us that $\mathfrak{p} = b_1(\mathfrak{r}')$ and $\mathfrak{q} = b_2(\mathfrak{r}')$, thus $(\mathfrak{p},\mathfrak{q}) \in \theta$.

Now, let $I/I\cap J \subseteq \mathfrak{p}$ and $J/I\cap J \nsubseteq \mathfrak{q}$. Our previous reasoning gives us $\mathfrak{r} \in \operatorname{Spec} B/I\cap J$ such that $b_2(\mathfrak{r}) = \mathfrak{q}$, since $J/I\cap J \nsubseteq \mathfrak{q}$, $I/I\cap J \subseteq \mathfrak{r}$. We can thus deal with two primes of $\operatorname{Spec} S+I/I\cap J$ such that $I/I\cap J \subseteq \mathfrak{p},\mathfrak{p}'$, where $\mathfrak{p}' = b_1(\mathfrak{r})$, of course.

We assume that $\iota_1(\mathfrak{p}) = \iota_1(\mathfrak{p}')$ which means that they map to the same prime of $\operatorname{Spec} S + I \cap J / I \cap J$. However, $\mathfrak{p}, \mathfrak{p}'$ are naturally in $\operatorname{Spec} S + I/I$, and them having the same image in $\operatorname{Spec} S + I \cap J / I \cap J$ means that they need to be equal, as $\operatorname{Spec} S + I/I$ is naturally a subset of $\operatorname{Spec} S + I \cap J / I \cap J$. Therefore, $(\mathfrak{p}, \mathfrak{p}') \in \theta$ trivially, as $\mathfrak{p} = \mathfrak{p}'$. △

We also include an example that shows it is possible to study group actions on ring spectra and the resulting quotients using the pushout formalism we work with.

Example. Let R be a commutative ring and $G = \{g_1, \dots, g_n\}$ a non-trivial finite subgroup of $\operatorname{Aut}(R)$ that acts freely (has no fixed points) on $\operatorname{Spec} R$, denote $g_1 = \mathrm{id}_R$ and R^G the pullback of the following diagram of commutative rings:

$$\begin{array}{ccc} R & & R \\ & {\scriptstyle (\mathrm{id}_R,\dots,\mathrm{id}_R)}\searrow \qquad \swarrow{\scriptstyle (g_1,\dots,g_n)} & \\ & R^n & \end{array}$$

clearly, R^G is the ring of $G-$invariants. We will show that $\operatorname{Spec} R^G$ is the pushout of the induced diagram of spectra in the category of ringed spaces.

At first, we will deal with the morphism $\operatorname{Spec} R \to \operatorname{Spec} R^G$ induced by inclusion. For this we will use some arguments used in proof of Lemma 11 on pages 352 and 353 of Cox et al. [2007]. We notice that R is integral over R^G as each $r \in R$ is a root of $(x - g_1(r)) \dots (x - g_n(r)) = a_n x^n + \dots + a_1 x + a_0$ which is easily seen to be polynomial in $R^G[x]$, as $(x - g_i g_1(r)) \dots (x - g_i g_n(r)) = g_i(a_n) x^n + \dots + g_i(a_1) x + g_i(a_0)$ for all $1 \leq i \leq n$ are the same polynomial, hence its coefficients are $G-$invariant. Suppose that $\mathfrak{p}$ is a prime of R, then $g_i(\mathfrak{p})$ are also primes as $g_i \in \operatorname{Aut}(R)$ for all $1 \leq i \leq n$. It is easy to deduce that all $\mathfrak{p} \cap R^G = g_i(\mathfrak{p}) \cap R^G$ for all $1 \leq i \leq n$.

Assume there is $\mathfrak{q}$ a prime of R such that $\mathfrak{p} \cap R^G = \mathfrak{q} \cap R^G$ and $\mathfrak{q} \neq g_i(\mathfrak{p})$ for any i. Let us have a radical ideal:

$$I = g_2(\mathfrak{p}) \cap \dots \cap g_n(\mathfrak{p}) \cap \mathfrak{q} \cap g_2(\mathfrak{q}) \cap \dots \cap g_n(\mathfrak{q}).$$

We prove that $I \not\subseteq \mathfrak{p}$. By going up theorem, we have that all primes:

$$g_2(\mathfrak{p}), \dots, g_n(\mathfrak{p}), \mathfrak{q}, g_2(\mathfrak{q}), \dots, g_n(\mathfrak{q})$$

are incomparable with $\mathfrak{p}$ as they map to the same prime of R^G and they are different due to the action of G being free on $\operatorname{Spec} R$. However, had $I \subseteq \mathfrak{p}$, then one of those primes would have to lie under the prime $\mathfrak{p}$ which would yield a contradiction with incomparability.

Thus, there has to be $a \in I$ such that $a \notin \mathfrak{p}$. This means that elements $g_2^{-1}(a), \dots, g_n^{-1}(a)$ belong to $\mathfrak{p}$. Similarly, we have $g_1^{-1}(a), \dots, g_n^{-1}(a) \in \mathfrak{q}$, thus $g_1^{-1}(a) + \dots + g_n^{-1}(a) \in \mathfrak{q}$, but this element is $G-$invariant. Therefore $g_1^{-1}(a) + \dots + g_n^{-1}(a) \in \mathfrak{p}$ which means that $a = g_1^{-1}(a) = (g_1^{-1}(a) + \dots + g_n^{-1}(a)) - (g_2^{-1}(a) + \dots + g_n^{-1}(a)) \in \mathfrak{p}$, a contradiction.

Since $R^G \subseteq R$ is an integral extension, which *a fortiori* by the going up theorem means that $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is a quotient map, it suffices to prove that

$\operatorname{Spec} R^G$ is naturally homeomorphic to $\operatorname{Spec} R \sqcup \operatorname{Spec} R/\sim$ where $\sim$ is the equivalence generated by $({}^a(\mathrm{id}_R, \dots, \mathrm{id}_R)(\mathfrak{r}), {}^a(g_1, \dots, g_n)(\mathfrak{r}))$ for all $\mathfrak{r}$ primes of R^n.

Suppose that $\mathfrak{p}, \mathfrak{q}$ are primes of R such that $\mathfrak{p} \cap R^G = \mathfrak{q} \cap R^G$, then $g_i(\mathfrak{q}) = \mathfrak{p}$ for some $1 \leq i \leq n$. Then let $\mathfrak{r} = (R, \dots, R, \mathfrak{p}, R, \dots, R)$ be a prime of R^n with $\mathfrak{p}$ on the $i-$th coordinate. Clearly, ${}^a(\mathrm{id}_R, \dots, \mathrm{id}_R)(\mathfrak{r}) = \mathfrak{p}$ and ${}^a(g_1, \dots, g_n)(\mathfrak{r}) = \mathfrak{q}$ as $(g_1(\mathfrak{q}), \dots, g_n(\mathfrak{q}))$ has $g_i(\mathfrak{q}) = \mathfrak{p}$ on the $i-$th coordinate. $\triangle$

2.3 Pushout of the form of a scheme

However, it is overly restrictive to limit ourselves to looking for pushouts of diagrams of affine schemes in the form of affine schemes. Consider this trivial example, let R be a ring and f, g be two non-nilpotent elements.

We have two morphisms of affine schemes $\operatorname{Spec} R_{fg} \hookrightarrow \operatorname{Spec} R_f$ and similarly $\operatorname{Spec} R_{fg} \hookrightarrow \operatorname{Spec} R_g$ which correspond to inclusions $\operatorname{Spec} R - V(fg) \subseteq \operatorname{Spec} R - V(f)$ and $\operatorname{Spec} R - V(fg) \subseteq \operatorname{Spec} R - V(g)$. The pushout of our two morphisms of affine schemes exists in the category of schemes and is given by the union of two corresponding open affine subsets of $\operatorname{Spec} R$ as $(\operatorname{Spec} R - V(f)) \cup (\operatorname{Spec} R - V(g))$. This open subset of $\operatorname{Spec} R$ is naturally equipped with a structure of a scheme, but needs not to be affine. This is the case for setting R to be $K[x, y]$ and f, g as x, y respectively, for instance.

We shall try to extend the idea in the proof of Theorem 42 to include some cases when $\operatorname{Spec} A \to \operatorname{Spec} \operatorname{Im} \varphi$ and $\operatorname{Spec} A \to \operatorname{Spec} \operatorname{Im} \psi$ do not have a pushout in the form of an affine scheme in the category of schemes. It is possible to do so under some additional assumptions.

At first, we will prove a generalisation of Theorem 40. After some work, the following theorem can be viewed as a stronger version of one of the implications of Théorème 7.1. on page 575 of Ferrand [2003] in a less general setting.

Theorem 50. *Let $Y' = \operatorname{Spec} A'$ and $Y = \operatorname{Spec} A$ be affine schemes, Z a scheme, and $f : Y \to Z$ morphisms of schemes. Suppose there exists a open covering $(\operatorname{Spec} B_j)_{j \in J}$ of Z by affine schemes such that $J \subseteq A$ and for each $j \in J$ we have $f^{-1}(\operatorname{Spec} B_j) = \operatorname{Spec} A_j$. Assume moreover that ${}^a\pi_I : Y \to Y'$ is a closed immersion. Then, the pushout of the corresponding diagram in the category ringed spaces is a scheme.*

Proof. At first, we examine the structure of the pushout ringed space $Y' \sqcup_Y Z$ with morphisms $i_{Y'} : Y' \to Y' \sqcup_Y Z$ and $i_Z : Z \to Y' \sqcup_Y Z$. We know that ${}^a\pi_I$ is a closed immersion, so we can identify Y with a closed set ${}^a\pi_I(Y)$ of Y'. This gives us that $i_{Y'}$ restricted to $Y' - Y$ is an open immersion, we shall identify $Y' - Y$ and its image under $i_{Y'}$. Moreover, i_Z is a closed immersion because so is ${}^a\pi_I : Y \to Y'$. Similarly, we shall identify Z and its image under i_Z. These arguments are analogous to those made by Schwede [2004] in his proof of 40.

We will cover the pushout $Y' \sqcup_Y Z$ with open affines. Clearly, we can do it for $Y' - Y \subseteq Y' \sqcup_Y Z$ and cover it by $\operatorname{Spec} A_i$ for each $i \in I$ where $Y' - Y = Y' - V(I)$ and I is an ideal of A.

Now, let us have $j \in J$. We have a commutative square:

$$\begin{array}{ccc} \operatorname{Spec} A_j & \xrightarrow{f} & \operatorname{Spec} B_j \\ \subseteq \big\downarrow & & \big\downarrow \subseteq \\ Y & \xrightarrow{f} & Z \end{array}$$

with open immersions as vertical arrows. Identify A with A'/I and let $j' \in A'$ be an element such that $\pi_I(j') = j$. Then, we have the following commutative diagram of rings:

$$\begin{array}{ccc} A'_{j'} & \xrightarrow{\pi_{Ij}} & A_j \\ \big\uparrow & & \big\uparrow \\ A' & \xrightarrow{\pi_I} & A \end{array}$$

with localisations as vertical arrows, as $A'_{j'}/I_{j'} \cong (A'/I)_j$, we know that π_{Ij} is a surjective homomorphism with kernel I_j. Take $X_j = (D(j') \cap (Y' - Y)) \cup \operatorname{Spec} B_j$. Clearly, $i_{Y'}^{-1}(X_j) = D(j') = \operatorname{Spec} A_{j'}$ and $i_Z^{-1}(X_j) = \operatorname{Spec} B_j$. Then, by Definition 39 of pushout of ringed spaces and Theorem 40, $X_j = \operatorname{Spec} P_j$ where P_j is the pullback of the following diagram:

$$\begin{array}{ccccc} A'_{j'} & & & & B_j \\ & \searrow^{\pi_{Ij}} & & \swarrow_{f^\flat_{\operatorname{Spec} B_j}} & \\ & & A_j & & \end{array}$$

To prove that $Y' \sqcup_Y Z$ is a scheme, it suffices to cover it by open affines $-$ we showed that X_j are open affine subsets $Y' \sqcup_Y Z$. By $\{X_j\}_{j \in J}$, it is possible to cover Z a closed subset of $Y' \sqcup_Y Z$. However, the complement of Z is identified with $Y' - Y$ has affine open cover inherited from Y'. Thus, we obtain that $Y' \sqcup_Y Z$ is a scheme. □

To apply the same reasoning as in the proof of Theorem 42, we will need to use another result due to Schwede [2004], which is an easy consequence of his Theorem 40.

Theorem 51 (Corollary 3.7. on page 7 in Schwede [2004])**.** *Suppose Z is a closed subscheme of both X and Y. Then $X \sqcup_Z Y$ is a scheme.*

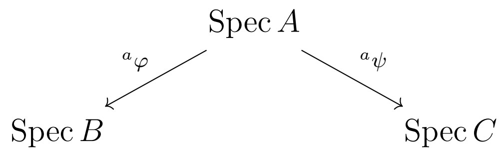
Remark. The claim of the theorem above trivially holds if we assume Z to be an open rather than closed subscheme of X and Y.

We can use the results above to give the aforementioned generalisation of Theorem 42 in a straightforward manner, then will use the theorem in some special cases.

Theorem 52. *Assume we have ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$. Such that the pushout of the diagram:*

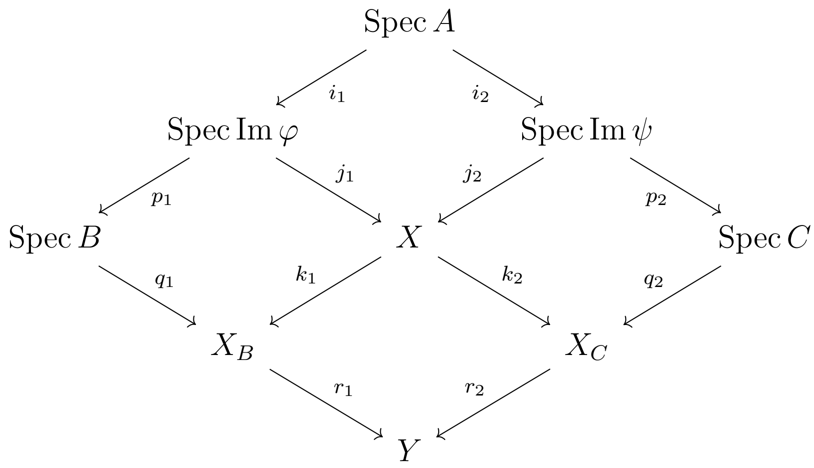
$$\begin{array}{ccccc} & & \operatorname{Spec} A & & \\ & \swarrow^{i_1} & & \searrow^{i_2} & \\ \operatorname{Spec} \operatorname{Im} \varphi & & & & \operatorname{Spec} \operatorname{Im} \psi \end{array}$$

where i_1 and i_2 are morphisms induced by respective inclusions is a scheme X together with morphisms $j_1 : \operatorname{Spec}\operatorname{Im}\varphi \to X$ and $j_2 : \operatorname{Spec}\operatorname{Im}\varphi \to X$ that satisfy the condition from the Theorem 50 on existence of a special affine open covering. Then the pushout of the diagram:



has a structure of a scheme.

Proof. We will proceed in similarly as in case of the Theorem 42. Let us draw the following diagram:



where we put X_B, X_C as the pushout of the respective diagrams, they are schemes by Theorem 50 and Y is also the pushout of the respective diagram. Moreover, Y is a scheme a by Theorem 51 as X is a closed subscheme of both X_B and X_C because p_1 and p_2 are closed immersions.

We deduce that Y is the pushout of ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ and ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} C$ in the same way as in case of the Theorem 42. □

Corollary. Suppose there are ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$. Provided that $i_1 : \operatorname{Spec} A \to \operatorname{Spec}\operatorname{Im}\varphi$ and $i_2 : \operatorname{Spec} A \to \operatorname{Spec}\operatorname{Im}\psi$ have a pushout which is naturally an open subset of $\operatorname{Spec}\operatorname{Im}\varphi \cap \operatorname{Im}\psi$, the pushout of ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ and ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} C$ is a scheme.

Proof. Assume that X together with morphisms of schemes $j_1 : \operatorname{Spec}\operatorname{Im}\varphi \to X$ and $j_2 : \operatorname{Spec}\operatorname{Im}\varphi \to X$ is the pushout of $i_1 : \operatorname{Spec} A \to \operatorname{Spec}\operatorname{Im}\varphi$ and $i_2 : \operatorname{Spec} A \to \operatorname{Spec}\operatorname{Im}\psi$. By the universal property, there is a morphism of schemes $X \to \operatorname{Spec}\operatorname{Im}\varphi \cap \operatorname{Im}\psi$ through which factor $\operatorname{Spec}\operatorname{Im}\varphi \to \operatorname{Im}\varphi \cap \operatorname{Im}\psi$ and $\operatorname{Spec}\operatorname{Im}\psi \to \operatorname{Im}\varphi \cap \operatorname{Im}\psi$.

If the morphism $X \to \operatorname{Spec}\operatorname{Im}\varphi \cap \operatorname{Im}\psi$ is an open immersion, then the open covering of X in question is simply the restriction of covering of $\operatorname{Spec}\operatorname{Im}\varphi \cap \operatorname{Im}\psi$ by principal open affine subsets. Clearly for any $f \in \operatorname{Im}\varphi \cap \operatorname{Im}\psi$, we have that the preimage of $D(f)$ under the morphism $\operatorname{Spec}\operatorname{Im}\varphi \to \operatorname{Im}\varphi \cap \operatorname{Im}\psi$ induced by the inclusion is again $D(f)$ as a principal open affine subset of $\operatorname{Spec}\operatorname{Im}\varphi$, the same works for $\operatorname{Spec}\operatorname{Im}\psi$ as well. □

Corollary. Suppose there are ring homomorphisms $B \xrightarrow{\varphi} A$ and $C \xrightarrow{\psi} A$. Provided that $i_1 : \operatorname{Spec} A \to \operatorname{Spec} \operatorname{Im} \varphi$ and $i_2 : \operatorname{Spec} A \to \operatorname{Spec} \operatorname{Im} \psi$ are open immersions, the pushout of ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ and ${}^a\varphi : \operatorname{Spec} A \to \operatorname{Spec} C$ is a scheme.

2.4 Another approach

In this section, we will try a possibly different approach than Ferrand [2003] and Schwede [2004] who try to find conditions under which the pushout of a diagram of schemes in the category of ringed spaces is a scheme. In the second half of the Section , showing that the spectra of two rings glue just right via a third affine schemes that maps to both of them proved very difficult as it is complicated to deal with general pushouts in topological spaces. Now we will show that it is possible to forego such discussions if we make some concessions, namely that we do not require that the pushout of the diagram of schemes in the category of schemes is not in the category of ringed spaces.

Proposition 53 (Proposition 3.4. on page 68 of Görtz and Wedhorn [2010])**.** *Let X be a scheme and $Y = \operatorname{Spec} R$ be an affine scheme then:*

$$\operatorname{Hom}(X, Y) \to \operatorname{Hom}(R, \Gamma(X, \mathcal{O}_X)), (f, f^\flat) \mapsto f^\flat_Y$$

is a bijection.

Proposition 54 (Proposition 3.5. *Gluing of morphisms* on pages 68 and 69 of Görtz and Wedhorn [2010])**.** *Let X and Y be a schemes, $(U_i)_{i \in I}$ an open covering of X, and $(U_i \to Y)_{i \in I}$ a family of morphisms of schemes, such that they coincide on intersections. Then there exists a unique morphism of schemes $X \to Y$ such that on U_i it is equal to $U_i \to Y$ for all $i \in I$.*

Proposition 55. *Suppose we have a commutative diagram of schemes:*

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ Yx & & & & Z \\ & \searrow^{\eta_1} & & \swarrow^{\eta_2} & \\ & & \operatorname{Spec} R & & \end{array}$$

that P with morphisms p_1, p_2 is the pullback of the induced diagram of rings:

$$\begin{array}{ccccc} & & \Gamma(X, \mathcal{O}_X) & & \\ & \nearrow & & \nwarrow & \\ \Gamma(Y, \mathcal{O}_Y) & & & & \Gamma(Z, \mathcal{O}_Z) \end{array}$$

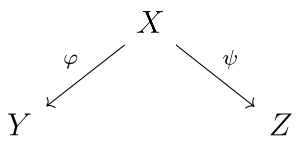
and that $\pi_1 : Y \to \operatorname{Spec} P$ and $\pi_2 : Z \to \operatorname{Spec} P$ are the unique morphisms of schemes induced by p_1, p_2 respectively. Then there is a unique homomorphism $\delta : \operatorname{Spec} P \to \operatorname{Spec} R$ such that $\eta_i = \delta \pi_i$ for $i = 1, 2$.

Proof. By the universal property of the pullback, there exists a unique ring homomorphism $\delta^\flat_{\operatorname{Spec} R} : R \to P$ such that $\eta_{i\,\operatorname{Spec} R}^\flat = p_i \delta^\flat_{\operatorname{Spec} R}$ for $i = 1, 2$. This homomorphism gives rise to a unique morphism of schemes $\delta : \operatorname{Spec} P \to \operatorname{Spec} R$. Proposition 53 gives us that $\eta_i = \delta \pi_i$ since $\eta_{i\,\operatorname{Spec} R}^\flat = p_i \delta^\flat_{\operatorname{Spec} R}$ and $p_i = \pi_{i\,\operatorname{Spec} P}^\flat$ for $i = 1, 2$. If there is another morphism $\delta' : \operatorname{Spec} P \to \operatorname{Spec} R$ such that $\eta_i = \delta' \pi_i$ for $i = 1, 2$, then $\eta_{i\,\operatorname{Spec} R}^\flat = p_i {\delta'}^\flat_{\operatorname{Spec} R}$ for $i = 1, 2$. We have that $\delta^\flat_{\operatorname{Spec} R} = {\delta'}^\flat_{\operatorname{Spec} R}$ by the uniqueness of $\delta^\flat_{\operatorname{Spec} R}$ from the universal property, moreover, $\delta' = \delta$ by Proposition 53. □

Theorem 56. *Let us have morphisms $X \xrightarrow{\varphi} Y$ and $X \xrightarrow{\psi} Z$ and a scheme W with morphisms $p_1 : Y \to W$ and $p_2 : Z \to W$ such that $p_1 \varphi = p_2 \psi$. We have a unique induced map α from the ringed space $Y \sqcup_X Z$ to W such that p_1 and p_2 factor through it. Provided that*

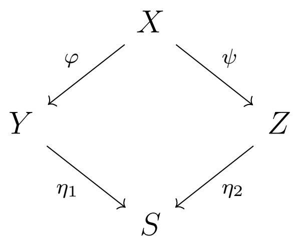
1. *α is onto,*
2. *$Y \sqcup_X Z$ has the coarsest possible topology that makes α continuous,*
3. *and for each open subscheme U of W, we have that $\alpha^\flat_U$ is an isomorphism,*

then W with morphisms p_1 and p_2 is the pushout of the diagram:

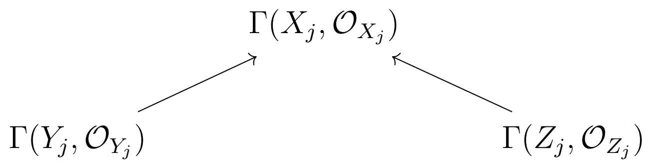


in the category of schemes.

Proof. Suppose S is a scheme with an affine open cover $(\operatorname{Spec} S_j)_{j \in J}$ and there is a commutative diagram:



Fix $j \in J$ and denote $Y_j = \eta_1^{-1}(\operatorname{Spec} S_j)$, $Z_j = \eta_2^{-1}(\operatorname{Spec} S_j)$, and $X_j = \varphi^{-1}(Y_j) = \psi^{-1}(Z_j)$ the open subschemes of Y, Z and X respectively. Let P_j with morphisms p_1^j, p_2^j is the pullback of the induced diagram of rings:

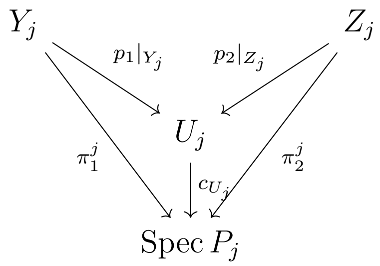


and that $\pi_1^j : Y_j \to \operatorname{Spec} P_j$ and $\pi_2^j : Z_j \to \operatorname{Spec} P_j$ are the unique morphisms of schemes induced by p_1^j, p_2^j respectively. By Proposition 55, there is a unique homomorphism $\delta^j : \operatorname{Spec} P_j \to \operatorname{Spec} R$ such that $\eta_i^j = \delta^j \pi_i^j$ for $i = 1, 2$ where

$\eta_1^j = \eta_1|_{Y_j}$ and $\eta_2^j = \eta_2|_{Z_j}$.

Our assumption (2.) and the definition of the ringed space $Y \sqcup_X Z$ (see Definition 39) ensure that there is U_j an open subscheme of W such that $Y_j = p_1^{-1}(U_j)$ and $Z_j = p_2^{-1}(U_j)$. Since Y_j and Z_j correspond to an open set $Y_j \sqcup_{X_j} Z_j$ of $Y \sqcup_X Z$, its topology is the coarsest such that α is continuous, thus $Y_j \sqcup_{X_j} Z_j$ must be a preimage of an open set of $\operatorname{Spec} P$. Moreover, by definition of pushout of ringed spaces and our assumption (3.), we can put $P_j = \Gamma(U_j, \mathcal{O}_W)$.

Proposition 53 gives us that the following diagram is commutative:



as the maps on corresponding global sections are identical. Altogether, this means that $\eta_1^j = \delta_j \circ c_{U_j} \circ p_1|_{Y_j}$ and $\eta_2^j = \delta_j \circ c_{U_j} \circ p_2|_{Z_j}$. Also, $\delta_j \circ c_{U_j}$ is the only morphism of schemes with such a property. This is due to the universal property of pullback $P_j = \Gamma(U_j, \mathcal{O}_W)$ and the fact that morphism of schemes from U_j to $\operatorname{Spec} S_j$ are uniquely determined by induced homomorphisms on global sections.

As preimages of open subschemes in the affine open cover $(\operatorname{Spec} S_j)_{j \in J}$ cover Y, Z, and X, corresponding $(U_j)_{j \in J}$ need to cover W by our assumption (1.) that $\alpha : Y \sqcup_X Z \to W$ is onto. We also have a system of maps $(\delta_j \circ c_{U_j} : U_j \to S)_{j \in J}$.

Pick different $j, j' \in J$, we shall show that $\delta_j \circ c_{U_j}$ and $\delta_{j'} \circ c_{U_{j'}}$ are equal on $U_j \cap U_{j'}$. Let $u \in U_j \cap U_{j'}$ be arbitrary and assume, without loss of generality, that there is $y \in Y$ such that $p_1(y) = u$. This is possible as the map $\alpha : Y \sqcup_X Z \to W$, induced by p_1 and p_2, is onto.

Clearly, $p_1|_{Y_j}(y) = p_1|_{Y_{j'}}(y)$ and $\eta_1^j(y) = \eta_1^{j'}(y)$. We have that $\eta_1^j = \delta_j \circ c_{U_j} \circ p_1|_{Y_j}$ and $\eta_1^{j'} = \delta_{j'} \circ c_{U_{j'}} \circ p_1|_{Y_{j'}}$. Therefore $(\delta_j \circ c_{U_j})(u) = (\delta_{j'} \circ c_{U_{j'}})(u)$.

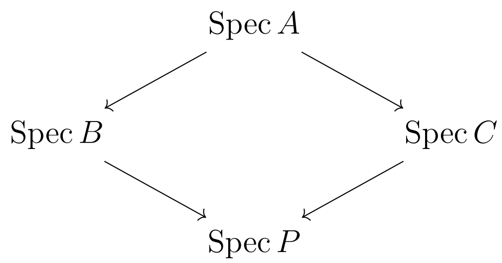
By Proposition 54, there is a unique morphism $\sigma : W \to S$ such that $\sigma|_{U_j} = \delta_j \circ c_{U_j}$ for all $j \in J$. This is the unique morphism such that $\eta_i = \sigma p_i$ for $i = 1, 2$ as it holds on all U_j which cover W and is uniquely determined on all of them.

Hence, W with morphisms p_1, p_2 is the pushout of $X \xrightarrow{\varphi} Y$ and $X \xrightarrow{\psi} Z$ in the category of schemes. □

Remark. The theorem above allows not to worry about some pathologies of the pushout of the diagram in the category of ringed spaces (namely that it has some topologically indistinguishable points which is not possible for a scheme) if we restrict ourselves to the category of schemes and find a scheme that looks almost like the pushout of the diagram in the category of ringed spaces.

Using Theorem 56, we will be able to give a very general and purely algebraic sufficient condition for the existence of pushouts of affine schemes in the category of schemes. That is because Theorem 56 spares us the need to glue the primes of B and C via A exactly right, however, it comes at a cost − the pushout we obtain is valid only in the category of schemes and not necessarily in the larger category of ringed spaces.

Theorem 57. *Let $B, C \subseteq A$ be rings and all the induced maps in the following diagram are surjective:*



where $P = B \cap C$. Provided that for any $I_1 \subseteq B$ and $I_2 \subseteq C$ such that $V_A(I_1) = V_A(I_2)$ there exists $I_3 \subseteq B \cap C$ with $V_A(I_1) = V_A(I_2) = V_A(I_3)$, then $\operatorname{Spec} P$ *is the pushout of the diagram above in the category of schemes.*

Proof. We use Theorem 56. Since induced maps $\operatorname{Spec} B \to \operatorname{Spec} P$ and $\operatorname{Spec} C \to \operatorname{Spec} P$ are surjective, so needs to be the induced map

$$\alpha : \operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C \to \operatorname{Spec} P.$$

Let $F \subseteq \operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C$ be a closed set, then, by definition, it needs to correspond to the sum of $V_B(I_1)$ in $\operatorname{Spec} B$ and $V_C(I_2)$ in $\operatorname{Spec} C$ amalgamated over $V_A(I_1) = V_A(I_2)$ in $\operatorname{Spec} A$ for $I_1 \subseteq B$ and $I_2 \subseteq C$. However, we assume that there is $I_3 \subseteq P$ such that $V_A(I_1) = V_A(I_2) = V_A(I_3)$. We know that $V_A(I_3)$ is the preimage of $V_B(I_1), V_C(I_2)$, and $V_P(I_3)$ under the respective induced map. Due to surjectivity of induced maps from $\operatorname{Spec} A$, $V_B(I_1)$ and $V_C(I_2)$ need to be preimages of $V_P(I_3)$ under the respective induced map. Thus F needs to be the preimage of $V_P(I_3)$. This gives us that any closed set of $\operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C$ is an inverse image of a closed set of $\operatorname{Spec} P$ under α.

The last assumption of Theorem 56 holds trivially by Proposition 41 and definition of $\operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C$ as a ringed space. □

Corollary. Suppose that A is a domain. Then it is possible to forego the assumption that induced maps $\operatorname{Spec} B \to \operatorname{Spec} P$ and $\operatorname{Spec} C \to \operatorname{Spec} P$.

Proof. Independently of surjectivity assumptions on the induced maps, we proved that any closed set of $\operatorname{Spec} B \sqcup_{\operatorname{Spec} A} \operatorname{Spec} C$ is an inverse image of a closed set of $\operatorname{Spec} P$ under α. *A fortiori,* α is a closed map. Since A is domain, so are B, C and P. Clearly, zero primes of B and C map to the zero prime of P by the induced maps, therefore the image α contains the zero prime and is closed. This means that α is surjective as $V_P(0) = \operatorname{Spec} P$. □

Remark. Recall that algebraic characterisation of surjectivity of maps of spectra induced by inclusions is given in Lemma 43.

It would be possible to use Theorem 57 and the gluing of pushouts as in Theorems 42 and 50 to obtain an analogous result, the pushout we would obtain would be in the category of schemes and not necessarily in the category of ringed spaces, of course.

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