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DIPLOMOVÁ PRÁCE

Pavel Čoupek

Vychylující teorie pro kvazikoherentní svazky

Katedra algebry

Vedoucí diplomové práce: doc. RNDr. Jan Šťovíček, Ph.D.
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Název práce: Vychylující teorie pro kvazikoherentní svazky

Autor: Pavel Čoupek

Katedra: Katedra algebry

Vedoucí diplomové práce: doc. RNDr. Jan Šťovíček, Ph.D., Katedra algebry

Abstrakt: V práci zavádíme definici 1-kovychylujícího objektu v Grothendieckově kategorii a studujeme vztah této definice k analogii standardní definice 1-kovychylujícího modulu. Zejména pak studujeme 1-kovychylující svazky na noetherovském schématu X: za požití klasifikace dědičných torzních párů kategorie kvazikoherentních svazků na X přiřadíme každé beztorzní dědičné třídě $\mathcal{F}$, která je generující, 1-kovychylující kvazikoherentní svazek, jehož 1-kovychylující třída je rovna $\mathcal{F}$. Obdržíme tak množinu po dvou neekvivalentních 1-kovychylujících kvazikoherentních svazků parametrizovaných podmnožinami X uzavřenými na specializace, které neobsahují množinu asociovaných bodů zvoleného generátoru kategorie kvazikoherentních svazků. V mnohých případech (např. pro separovaná schémata) lze tuto množinu zakázaných bodů volit jako množinu asociovaných bodů samotného schématu.

Klíčová slova: algebraická geometrie, homologická algebra, (ko)vychylující svazek

Title: Tilting theory for quasicoherent sheaves

Author: Pavel Čoupek

Department: Department of Algebra

Supervisor: doc. RNDr. Jan Šťovíček, Ph.D., Department of Algebra

Abstract: We introduce definition of 1-cotilting object in a Grothendieck category and investigate its relation to the analogue of the standard definition of 1-cotilting module. The 1-cotilting quasi-coherent sheaves on a Noetherian scheme are studied in particular: using the classification of hereditary torsion pairs in the category of quasi-coherent sheaves on a Noetherian scheme X, to each hereditary torsion-free class $\mathcal{F}$ that is generating we assign a 1-cotilting quasi-coherent sheaf whose 1-cotilting class is $\mathcal{F}$. This provides a family of pairwise non-equivalent 1-cotilting quasi-coherent sheaves which are parametrized by specialization closed subsets of X avoiding the set of associated points of a chosen generator of the category of quasi-coherent sheaves. In many cases (e.g. for separated schemes), this set of avoided points can be chosen as the set of associated points of the scheme.

Keywords: algebraic geometry, homological algebra, (co)tilting sheaf

Contents

List of symbols

$\mathsf{Mod}\text{–}\mathcal{O}_X$	category of $\mathcal{O}_X$-modules
Coh_X	category of coherent sheaves on X
QCoh_X	category of quasi-coherent sheaves on X
$\mathscr{F}, \mathscr{G}, \mathscr{H}, \mathscr{I}$	(typically) a quasi-coherent sheaf, or an $\mathcal{O}_X$-module
$\mathcal{O}_X$	the structure sheaf of a scheme X
res^U_V, $\restriction_V$	restriction morphism (from U) to V
$\bigoplus_{i\in I} \mathscr{F}_i$	direct sum of $\mathscr{F}_i$'s, $i \in I$
$\prod_{i\in I} \mathscr{F}_i$	direct product of $\mathscr{F}_i$'s, $i \in I$
$\mathscr{F}^{\oplus n}, \mathscr{F}^{\oplus I}$	direct sum of n copies / I copies of $\mathscr{F}$ (I possibly infinite set)
$\mathscr{F}^{\times n}, \mathscr{F}^{\times I}$	direct product of n copies / I copies of $\mathscr{F}$ (I possibly infinite set)
$\widetilde{M}$	quasi-coherent sheaf on $\operatorname{Spec} R$ associated to an R-module M
$\mathscr{F}_x$, $\mathcal{O}_{X,x}$	stalk of $\mathscr{F}$ / $\mathcal{O}_X$ at a point x
$\kappa(x)$	residue field at a point x
f_*	direct image functor induced by f
f^*	inverse image functor induced by f
$\operatorname{Supp} \mathscr{F}$	support of $\mathscr{F}$
$\operatorname{Ass} \mathscr{F}$	set of all associated points of $\mathscr{F}$
$\operatorname{Ann}(s)$	annihilator of element of a module (or section of a sheaf) s
$\overline{Y}$	closure of Y (in the topological sense)
$E(\mathscr{F})$	injective hull of the $\mathcal{O}_X$-module / quasi-coherent sheaf $\mathscr{F}$
$\operatorname{Add}(\mathcal{S})$	class of all direct summands of direct sums of objects from class $\mathcal{S}$
$\operatorname{Prod}(\mathcal{S})$	class of all direct summands of direct products of objects from class $\mathcal{S}$
$\operatorname{Gen}(\mathcal{S})$	class of all objects generated by class $\mathcal{S}$
$\operatorname{Cogen}(\mathcal{S})$	class of all objects cogenerated by class $\mathcal{S}$
$\varinjlim \mathcal{S}$	class of all direct limits of objects from class $\mathcal{S}$
${}^{\perp}\mathcal{S}$	left Ext^1-orthogonal class to $\mathcal{S}$
$\mathcal{S}^{\perp}$	right Ext^1-orthogonal class to $\mathcal{S}$

Introduction

Tilting theory originates in representation theory of finite-dimensional algebras. In 1979, S. Brenner and M. C. R. Butler introduced tilting module as a certain finite-dimensional module T of projective dimension at most 1 over a finite-dimensional algebra Λ in [BB80]. The purpose of such modules was that the functor $\mathrm{Hom}_\Lambda(T, -)$ provides an equivalence between certain subcategories of $\mathsf{mod}\text{–}\Lambda$ and $\mathsf{mod}\text{–}\Lambda'$ where $\Lambda' = \mathrm{End}_\Lambda(T)$. D. Happel later showed in [Hap87] that this behaviour can be viewed as a remnant of the equivalence between the derived categories $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b(\Lambda')$ induced by the derived functor $\mathbb{R}\mathrm{Hom}_\Lambda(T, -)$. In [Ric89], J. Rickard provided a necessary and sufficient condition for equivalence of the derived categories. The condition is the existence of the so-called tilting complex. From this point of view, tilting theory can be considered a continuation of Morita theory, and tilting modules a generalization of projective generators.

Since then, tilting modules and tilting theory have been substantially generalized. Tilting modules of arbitrarily larger projective dimension n (so-called n-tilting modules) were introduced and studied ([Miy86], [Hap88]) as well as non-finitely generated tilting modules over arbitrary rings were considered ([CT95a], [AHTT01], [HC01]). Dualization provided the notion of a cotilting module ([Col93], [CDT97], [CF00], [CTT97]), possibly non-finitely generated and of large (finite) injective dimension.

Tilting objects in Abelian and Grothendieck categories $\mathcal{A}$ were defined and studied in order to obtain a derived equivalence $\mathcal{D}^b(\mathcal{A}) \xrightarrow{\simeq} \mathcal{D}^b(R)$ with category of modules ([HRS96], [Col99], [CF07]). For example, the derived equivalence of the category of coherent sheaves on the projective n-space $\mathbb{P}^n$ and the category of finite-dimensional modules over a non-commutative finite-dimensional algebra obtained by A. A. Beĭlinson in [Beĭ78] can be understood in this context. An important justification for the notion of (big) cotilting modules is the fact that such modules occur on the module side of the equivalence as the image of an injective cogenerator ([HRS96], [CGM07], [Šťo14]).

In the context of this development, it seems justified to presume that suitable further generalizations of tilting and cotilting modules could lead to obtaining, or better understanding of, equivalences between more general derived categories.

One far-reaching generalization of modules over a (commutative) ring R was given by J. Dieudonné, A. Grothendieck and J.-P. Serre ([Ser55], [GD60]). The commutative ring R is replaced by a scheme X, an object of geometrical nature. The analogues of modules over R are then the so-called quasi-coherent sheaves on X.

In the present thesis, we take a step in this direction of generalization by studying 1-cotilting quasi-coherent sheaves on a scheme. For this purpose, a 1-cotilting object in a general Grothendieck category is defined and, in order to justify the given definition, its relation to the module-theoretic definition of a 1-cotilting module (cf. [HC01]) is studied.

In two consecutive papers [AHPŠT14] by L. Angeleri Hügel, D. Pospíšil, J. Šťovíček and J. Trlifaj and [ŠTH14] by J. Šťovíček, J. Trlifaj and D. Herbera, the authors classify all n-cotilting modules over a Noetherian commutative ring R up to equivalence of n-cotilting modules. In particular, 1-cotilting modules over R correspond, up to equivalence, to the so-called specialization closed subsets Y

of the spectrum of R with the additional requirement that such sets Y do not contain any of the associated primes of the regular module R. The classification relies on the correspondence between specialization closed subsets $Y \subseteq \operatorname{Spec} R$ and torsion classes $\mathcal{T}(Y)$ of hereditary torsion pairs in $\mathsf{Mod}\text{–}R$, which is a special case of a result of P. Gabriel in [Gab62], establishing the correspondence for categories of quasi-coherent sheaves on a Noetherian scheme X.

This general version of the correspondence is used in this thesis to construct a family of pairwise non-equivalent 1-cotilting sheaves on a Noetherian scheme. These are parametrized, analogically to the affine case, by specialization closed subsets $Y \subseteq X$ such that Y does not contain any of the associated points of a generator $\mathscr{G}$ (that can be chosen arbitrarily) of the category of quasi-coherent sheaves on X. To obtain an explicit description of $\mathcal{F}(Y)$, the torsion-free class associated to $\mathcal{T}(Y)$, by means of the specialization closed subset Y, we also present an alternative proof of the Gabriel's correspondence using the description of injective quasi-coherent sheaves from [Har66].

In Chapter 1, we remind the reader of most of the standard definitions and basic theorems needed for establishing the context of this thesis. The definition of Grothendieck category (as well as the definition of an Abelian category) is recalled and several its basic properties stated. We continue by summarizing some of the results on (1-)tilting and (1-)cotilting modules over a general ring and over a Noetherian commutative ring in particular, and we introduce the definitions of 1-tilting and 1-cotilting objects in a Grothendieck category. We close the chapter by establishing the category QCoh_X of quasi-coherent sheaves on a (locally) Noetherian scheme X, and the structure of injective objects in it.

Chapter 2 contains a description of hereditary torsion pairs in the category of quasi-coherent sheaves on a Noetherian scheme by means of specialization closed subsets. The reader is first reminded of (hereditary) torison pairs in an Abelian category. We recall the definitions of associated point and support of a quasi-coherent sheaf, and establish some of their properties needed for the classification. In particular, it is proved that an associated point x of a sheaf $\mathscr{F}$ on a Noetherian scheme can be detected by the existence of a coherent subsheaf $\mathscr{G} \subseteq \mathscr{F}$ such that x is its only associated point. Using this fact together with the structure of injective quasi-coherent sheaves, we prove the claimed correspondence.

At the beginning of Chapter 3, various definitions of the Ext functor are recalled and several complications that arise in Grothendieck categories in contrast to module categories are discussed. We continue this discussion by relating our definition of 1-cotilting object in a Grothendieck category to the module-theoretic axioms (that is, its analogues in a Grothendieck catgory). The last part of the chapter contains the description of a large family of 1-cotilting quasi-coherent sheaves on an arbitrary Noetherian scheme X. These correspond to some of the hereditary torsion pairs in the category of quasi-coherent sheaves, and hence to some of the specialization closed subset by results from Chapter 2.

Finally, an overview of various subjects connected to the thesis' topic is given in Appendix. These facts occasionaly come up in arguments of proofs throughout the first three chapters but are not central to the main topic. The subjects presented here include definitions and basic properties of sheaves and schemes, envelopes and covers in Abelian categories and the Yoneda's definition of the Ext functor.

1. Preliminaries

Before we begin, let us comment on some of the notational conventions used throughout this thesis.

If $\mathcal{C}$ is a category, the expression $A \in \mathcal{C}$ means that A is an object of the category $\mathcal{C}$. If $\mathcal{S}$ is a class of objects of $\mathcal{C}$, we assume that it is closed under isomorphic objects, except for the case when $\mathcal{S}$ is a set, since the closure under isomorphism would enlarge $\mathcal{S}$ to a proper class. Since all the subcategories we consider are always full, we do not make a strict distinction between a class $\mathcal{S}$ of objects of $\mathcal{C}$ and the full subcategory on $\mathcal{S}$.

When we talk about limits and colimits, we usually have the (co-)limit object in mind; the associated (co-)cone is then called the universal (co-)cone, the canonical morphisms from (to) the (co-)limit, and so on. One exception is made when talking about kernels and cokernels: if $f : A \to B$ is a morphism (in an Abelian category,) kernel of f means either the object $\operatorname{Ker} f$ itself or its inclusion into A. Dually, cokernel of f is either the cokernel morphism or its codomain.

Similarly, if A_i, B_i $i \in I$ is a collection of objects in a category, there are two situations that occur regularly:

(A) There is a collection of morphisms $f_i : A_i \longrightarrow B_i$ and they lift to a morphism

$$f' : \prod_{i \in I} A_i \longrightarrow \prod_{i \in I} B_i$$

by the universal property of products. That is, if $\pi_i^A : \prod_{i \in I} A_i \to A_i$, $\pi_i^B : \prod_{i \in I} B_i \to B_i$, $i \in I$ are the canonical projections, then f' is the unique map satisfying $\pi_i^B f' = f_i \pi_i^A$ for every $i \in I$.

(B) There is a collection of morphisms $f_i : C \to B_i$ $i \in I$ and they lift to a morphism

$$f' : C \longrightarrow \prod_{i \in I} B_i$$

by the universal property of product. That is, $f' \pi_i^A = f_i$ for every $i \in I$.

In both cases, we denote the resulting morphism f' by $\prod_{i \in I} f_i$. This should not lead to any confusion as in those cases the domain of the morphism $\prod_{i \in I} f_i$ is always explicitly mentioned.

The term "module category" always refers to the category $\mathsf{Mod}\text{–}R$ of right R-modules, where R is an associative ring with unit, not necessarily commutative.

If $\mathcal{A}$ is a category, $A \in \mathcal{A}$ an object and $f : B \to C$ a morphism in $\mathcal{A}$, we denote the image of f under the covariant (contravariant, resp.) $\operatorname{Hom}_{\mathcal{A}}$-functor $\operatorname{Hom}_{\mathcal{A}}(A, f)$ ($\operatorname{Hom}_{\mathcal{A}}(f, A)$, resp.) by $f \circ -$ ($- \circ f$, resp.). This notation is inspired by the action of such maps: $\operatorname{Hom}_{\mathcal{A}}(A, f)$ takes g to $f \circ g$ and $\operatorname{Hom}_{\mathcal{A}}(f, A)$ takes g to $g \circ f$. In particular, do not use the common notation f^* and f_* as the star-index notation is reserved for different functors (i.e. direct and inverse image functors) in the context of algebraic geometry.

As is usually the case in algebraic geometry, in order to keep the amount of notation at a reasonable level, objects and maps defined uniquely up to unique isomorphism are often identified. This occurs frequently e.g. in discussions of various localization morphisms. For example, if R is a commutative ring and $f, g \in R$ are two elements such that $\sqrt{(f)} = \sqrt{(g)}$, the localization morphisms $R \to R_f$ and $R \to R_g$ are usually identified.

1.1 Abelian and Grothendieck categories

We begin by recalling the notions of Abelian and Grothendieck category.

Definition 1.1. An additive category $\mathcal{A}$ is called *Abelian* if

(AB1) every morphism admits a kernel and a cokernel, and

(AB2) given a morphism $f : A \to B$ in $\mathcal{A}$, the induced morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism[1].

An Abelian category $\mathcal{A}$ is called *AB5-category* if additionally

(AB3) there exist arbitrary direct sums (i.e. coproducts) in $\mathcal{A}$, and

(AB5) direct limits in $\mathcal{A}$ are exact functors - that is, given a directed set $\mathcal{I}$ (considered as a small thin category), the colimit functor

$$\varinjlim : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$$

is an exact functor.

An AB5-category with a generator is called a *Grothendieck category.*

Among the important theorems of the general theory of Abelian categories, we mention the following result of B. Mitchell.

Theorem 1.2 (Mitchell's Embedding Theorem, [Mit65, Theorem 7.2]). *Suppose that $\mathcal{A}$ is a small Abelian category. Then there exists a (unitary, not necessarily commutative) ring R and a fully faithful exact (additive) functor*

$$F : \mathcal{A} \longrightarrow \mathsf{Mod}\text{–}R.$$

An important consequence of Mitchell's Theorem is that small diagrams can be chased element-by-element instead of using universal properties of kernels, cokernels, etc. This is very practical when one wishes to extend certain results of homological algebra from the concrete module category to an abstract Abelian one. This serves as a justification for the common diagram-chasing methods in abstract setting, such as the Five Lemma, Four Lemma, behavior of pushouts and pullbacks with respect to short exact sequences, etc. For further details and a precise formulation, the reader is referred to [Mit65, Metatheorem 7.3].

Convenient as the above theorem may be, much more important role for our purposes plays the well-known embedding theorem of Gabriel and Popescu.

Theorem 1.3 (Gabriel-Popescu Theorem, [GP64][2]). *Let $\mathcal{A}$ be a Grothendieck category and $G \in \mathcal{A}$ its generator. Let R be the ring of endomorphisms of G. Then the functor*

$$\operatorname{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \longrightarrow \mathsf{Mod}\text{–}R$$

is fully faithful, and admits a left adjoint which is exact.

[1] Recall that $\operatorname{Im} f = \operatorname{Ker}(\operatorname{Coker} f)$ and dually, $\operatorname{Coim} f = \operatorname{Coker}(\operatorname{Ker} f)$.

[2] Note that instead of N. Popescu, N. Popesco is listed among the authors of the paper; this is very likely a misspelling in the paper.

Thus, by the Gabriel–Popescu Theorem, every Grothendieck category can be treated as a reflective subcategory of the category $\mathsf{Mod}\text{–}R$ for a ring R, with the reflector being exact. Such subcategories are called *Giraud subcategories* (cf. [Ste75, X.1]). Among the important consequences, we mention the following.

Corollary 1.4 ([Ste75, X.4.3, X.4.4])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then*

(1) $\mathcal{A}$ is complete, and

(2) $\mathcal{A}$ has enough injective objects.

The statement (2) can be further strengthened. The following theorem by Gabriel gives a stronger condition.

Theorem 1.5 ([Gab62, Ch. II, Theorem 2])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathcal{A}$ has injective hulls.*

Proof (sketch). If $A \in \mathcal{A}$ is an object, by (2) we can treat A as a subobject of an injective object E. The axiom (AB5) implies that A has a pseudocomplement[3] A' in E - it is the union of all subobjects B of E with the property $A \cap B$. Similarly, one constructs a pseudocomplement A'' of A' containing A. Now it is enough to observe that A'' is a maximal essential extension of A in E and that it is complemented by A', so it splits. Thus, A'' is injective and $A \hookrightarrow A''$ is essential. □

One more thing that is crucial to us is the fact that every Grothendieck category $\mathcal{A}$ possesses an injective cogenerator. This was stated without proof by Grothendieck in [Gro57]. We present the proof here for reader's convenience.

Theorem 1.6. *Let $\mathcal{A}$ be a Grothendieck category. Then there exists an injective cogenerator W of $\mathcal{A}$.*

Proof. Consider the generator G of $\mathcal{A}$ and put $R = \operatorname{End}_{\mathcal{A}}(G)$. By the fact that the full embedding of $\mathcal{A}$ to $\mathsf{Mod}\text{–}R$ from Theorem 1.3 is a right adjoint, it preserves kernels, hence it takes monomorphisms to monomorphisms. As a consequence, the lattice $\operatorname{Sub}(G)$ of all subobjects of G (in $\mathcal{A}$) is small and thus, the lattice of all quotients of $\operatorname{Quo}(G)$ is small as well[4]. For each quotient of G, fix a representative $f_i : G \twoheadrightarrow G_i$, so that we have a set of representatives

$$f_i : G \longrightarrow\!\!\!\!\!\rightarrow G_i, \quad i \in I$$

(indexed by a set I).

For each $i \in I$, choose an embedding $g_i : G_i \hookrightarrow E_i$ where E_i is injective. Then put

$$W := \prod_{i \in I} E_i \ .$$

[3]For a lattice L with bottom element 0, by pseudocomplement of $x \in L$ it is meant an element $y \in L$ maximal with respect to the property $x \wedge y = 0$.

[4]$\operatorname{Quo}(G)$ is isomorphic or anti-isomorphic to $\operatorname{Sub}(G)$, depending on convention. The main point is that a quotient corresponds to the kernel of the quotient map, and conversely, every subobject prescribes the quotient given by cokernel of the inclusion map.

W is clearly injective. For $i \in I$, denote by $\iota_i : E_i \to W$ the inclusion coming from the biproduct structure

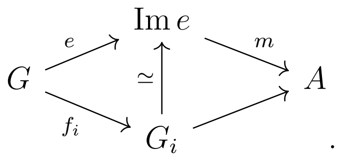
$$W = E_i \oplus \prod_{j \in I \setminus \{i\}} E_j \ .$$

Let us verify that W is an injective cogenerator. Since W is injective, it is enough to show that there is a nonzero morphism $A \to W$ for each nonzero object $A \in \mathcal{A}$.

Suppose that $A \neq 0$ is a nonzero object. Since G is a generator, there is a nonzero morphism $f : G \to A$. Consider its epi-mono factorization

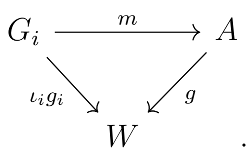
$$f = me, \quad m \text{ monic}, \ e \text{ epic}.$$

The epimorphism e belongs to a class of the lattice of quotients of G so there is an index $i \in I$ admitting a commutative diagram



This means that we may without loss of generality assume that $e = f_i$ (by replacing m by its composition with the vertical isomorphism in the above diagram). Note that $G_i \neq 0$ since $f \neq 0$.

Since m is monic, by injectivity of W there is a morphism g fitting into the commutative diagram



As $G_i \neq 0$, the monomorphism $\iota_i g_i$ is nonzero and thus, $g : A \to W$ is a nonzero morphism. Since A was chosen as an arbitrary nonzero object, this shows that W is an injective cogenerator. $\square$

The definition of Grothendieck category is, in contrast to the definition of Abelian category, not self-dual – that is, the opposite category of a Grothendieck category is not Grothedieck in general. One way to see this is to observe that inverse limit functors are typically not exact in such Grothendieck categories as $\mathsf{Mod}\text{–}R$.

Another way to see that the opposite category of a Grothendieck category is not Grothendieck in general is to make use of the following observation (which will be useful by itself later on).

Remark 1.7. Suppose that $A_i,\ i \in I$, is a collection of objects in a Grothendieck category $\mathcal{A}$. We claim that there is a canonical monomorphism

$$\bigoplus_{i \in I} A_i \hookrightarrow \prod_{i \in I} A_i \ .$$

For each finite subset $I' \subseteq I$, one has a canonical isomorphism

$$\bigoplus_{i\in I'} A_i \oplus \prod_{i\in I\setminus I'} A_i \xrightarrow{\simeq} \prod_{i\in I} A_i \ ,$$

in particular, there is a monomorphism

$$\bigoplus_{i\in I'} A_i \xhookrightarrow{i_{I'}} \prod_{i\in I} A_i$$

(obtained as the composition of the embedding of $\bigoplus_{i\in I'} A_i$ into the biproduct $\bigoplus_{i\in I'} A_i \oplus \prod_{i\in I\setminus I'} A_i$, and the canonical isomorphism mentioned above). Observe that $i_{I'} \restriction_{\bigoplus_{i\in I''} A_i} = i_{I''}$ for any two finite subsets $I'' \subseteq I' \subseteq I$. It follows that there is a monomorphism

$$\bigoplus_{i\in I} A_i = \varinjlim_{I'\subseteq I \text{finite}} \bigoplus_{i\in I'} A_i \xhookrightarrow{\varinjlim i_{I'}} \prod_{i\in I} A_i \ ,$$

since direct limits in a Grothendieck category are exact functors by (AB5) and thus, they preserve monomorphisms.

Again the dual property does not hold e.g. for the category of Abelian groups, since there is no epimorphism $\mathbb{Z}^{\times I} \to \mathbb{Z}^{\oplus I}$ if I is a countable (infinite) set[5]. Thus, the category $\mathsf{Ab}^{\mathrm{op}}$ is not a Grothendieck category.

Let us conclude this section by introducing some of the notation used throughout the rest of the thesis.

Notation 1.8. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories. For a class of objects $\mathcal{S} \subseteq \mathcal{A}$, we use the following notation:

(1) $\mathrm{Add}(\mathcal{S})$ denotes the class of all direct summands of arbitrary direct sums of objects from $\mathcal{S}$.

(2) $\mathrm{Prod}(\mathcal{S})$ denotes the class of all direct summands of arbitrary direct products of objects from $\mathcal{S}$.

(3) If $F : \mathcal{A} \to \mathcal{B}$ an additive functor, we denote by $\operatorname{Ker} F$ the class of objects (or the full subcategory on those objects) $A \in \mathcal{A}$ such that $F(A) = 0$.

(4) If $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is an additive bifunctor (meaning that it is additive separately in both arguments), we put

$$\operatorname{Ker} F(\mathcal{S}, -) = \bigcap_{S\in\mathcal{S}} \operatorname{Ker} F(S, -).$$

That is, $\operatorname{Ker} F(\mathcal{S}, -)$ is the class (full subcategory) of all objects B such that $F(S, B) = 0$ for all $S \in \mathcal{S}$.

(5) In the same situation as in (4), if $\mathcal{T}$ is a class of objects of $\mathcal{B}$, we denote the fact that $F(S, T) = 0$ for all $S \in \mathcal{S}$ and all $T \in \mathcal{T}$ by $F(\mathcal{S}, \mathcal{T}) = 0$.

(6) $\mathcal{S}^{\perp}$ denotes $\operatorname{Ker} \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{S}, -)$ and ${}^{\perp}\mathcal{S}$ denotes $\operatorname{Ker} \operatorname{Ext}^1_{\mathcal{A}}(-, \mathcal{S})$[6].

[5] This is because $\mathbb{Z}$ is a slender $\mathbb{Z}$-module. See e.g. [GT12, Chapter 4].

[6] If there are not enough injectives or projectives in $\mathcal{A}$, we use the Yoneda's definition of Ext (Appendix, Section D).

(7) $\mathrm{Gen}(\mathcal{S})$ denotes the class of all objects generated by $\mathcal{S}$, that is, the class of all epimorphic images of objects from $\mathrm{Add}(\mathcal{S})$. Dually, $\mathrm{Cogen}(\mathcal{S})$ is the class of all objects cogenerated by $\mathcal{S}$, i.e. the class of all subobjects of objects from $\mathrm{Prod}(\mathcal{S})$.

(8) $\varinjlim \mathcal{S}$ denotes the class of all direct limits of objects in $\mathcal{S}$. That is, $A \in \varinjlim \mathcal{S}$ if there is a direct system $\mathcal{D}$ consisting of objects from $\mathcal{S}$ (and morphisms between them in $\mathcal{A}$) such that $A = \varinjlim \mathcal{D}$.

(9) If $\mathcal{S} = \{S\}$, we use plain S instead of $\{S\}$ in the notation above . That is, ${}^{\perp}S$ denotes ${}^{\perp}\{S\}$, and so on.

1.2 Tilting and cotilting objects

For the purposes of this section, the reader may assume that the functor $\mathrm{Ext}^i_{\mathcal{A}}$ in a Grothendieck category $\mathcal{A}$ is defined as the right derived covariant $\mathrm{Hom}_{\mathcal{A}}$ functor; this makes always sense since there are enough injectives in $\mathcal{A}$ by Corollary 1.4 (2). A more detailed discussion of Ext and its behaviour in a Grothendieck category will be given in Chapter 3 (Section 3.1).

In this section, let R be a (unitary, not necessarily commutative) ring.

Definition 1.9. Let n be a nonnegative integer.

An R-module T is called *n-tilting* provided that the following conditions hold:

(T1) $\operatorname{projdim} T \leq n$.

(T2) $\mathrm{Ext}^i_R(T, T^{\oplus I}) = 0$ for every set I and every $i \geq 1$.

(T3) There is an exact sequence

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$$

with $T_0, T_1, \ldots, T_r \in \mathrm{Add}(T)$.

Dually, we call an R-module C *n-cotilting* if the following conditions hold:

(C1) $\operatorname{injdim} C \leq n$.

(C2) $\mathrm{Ext}^i_R(C^{\times I}, C) = 0$ for every set I and every $i \geq 1$.

(C3) There is an exact sequence

$$0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

where W is an injective cogenerator of $\mathsf{Mod}\text{–}R$ and $C_0, C_1, \ldots, C_r \in \mathrm{Prod}(C)$.

If a module T is n-tilting (n-cotilting, resp.), the exact sequence in the axiom (T3) ((C3), resp.) can always be chosen in a way that $r \leq n$. See [GT12, Lemma 13.10 (b) and Lemma 15.14 (b)] for proof.

Using this fact, note that 0-tilting modules are precisely projective generators and 0-cotilting modules are precisely injective generators for $\mathsf{Mod}\text{–}R$.

A starting point for the definition of 1-tilting and 1-cotilting object in a Grothendieck category is the following characterization of 1-tilting and 1-cotilting modules by Colpi and Trlifaj[7].

Theorem 1.10 ([CT95b, Proposition 1.3]). *Let R be a ring.*

(1) A right R-module T is 1*-tilting if and only if* $T^{\perp} = \mathrm{Gen}(T)$.

(2) A right R-module C is 1*-cotilting if and only if* ${}^{\perp}C = \mathrm{Cogen}(C)$.

Definition 1.11. (1) If T is a 1-tilting module, we call the class $T^{\perp} = \mathrm{Gen}(T)$ the 1-*tilting class associated to T*. Two 1-tilting modules T, T' are called *equivalent* if $T^{\perp} = (T')^{\perp}$.

(2) If C is a 1-cotilting module, we call the class ${}^{\perp}C = \mathrm{Cogen}(C)$ the 1-*cotilting class associated to C*. Two 1-cotilting modules C, C' are called *equivalent* if ${}^{\perp}C = {}^{\perp}C'$.

Remark 1.12. It can be proved that 1-tilting modules T, T' are equivalent if and only if $\mathrm{Add}(T) = \mathrm{Add}(T')$, that is, if each one of them is a direct summand in a direct sum of copies of the other. Dually, 1-cotilting modules C, C' are equivalent if and only if $\mathrm{Prod}(C) = \mathrm{Prod}(C')$, i.e. C is a direct summand of suitable direct power of C' and vice versa. We will not need this fact. The reader is referred to [GT12, Lemma 13.16, Remark 15.6] for proof.

The following definition of 1-tilting object in a Grothendieck category was given in [Col99], and recently used in [CF07], [AHK15].

Definition 1.13. Let $\mathcal{A}$ be a Grothendieck category. An object $T \in \mathcal{A}$ is called 1-*tilting* if $\mathrm{Gen}(T) = T^{\perp}$.

If this is the case, the class $\mathrm{Gen}(T) = T^{\perp}$ is called the 1-*tilting class associated to T*.

We now give a definition of 1-cotilting object in a Grothendieck category that is used throughout this thesis.

As stated in Corollary 1.4 (2), any Grothendieck category always has enough injectives. The dual fact, however, does not necessarily hold: a Grothendieck category does not always possess enough projectives (this is be demonstrated in the next section, Remark 1.29). As a result, any class of the form $\mathcal{S}^{\perp}$ is always cogenerating, but the dual statement does not need to hold, i.e. ${}^{\perp}\mathcal{S}$ is not necessarily generating. This asymmetry results in the seemingly-asymmetrical definition of the dual 1-cotilting object.

Definition 1.14. Let $\mathcal{A}$ be a Grothendieck category. An object $C \in \mathcal{A}$ is called 1-*cotilting* if $\mathrm{Cogen}(C) = {}^{\perp}C$ and this class is generating.

If C is a 1-cotilting object, the class $\mathrm{Cogen}(C) = {}^{\perp}C$ is called the 1-*cotilting class associated to C*.

To close this section, let us state the classification theorem for 1-cotilting classes over a commutative Noetherian ring proved by Angeleri Hügel, Pospíšil, Šťovíček and Trlifaj in [AHPŠT14, Theorem 2.11].

[7] The characterization was later generalized for n-tilting and n-cotilting modules by S. Bazzoni, see [Baz04, Theorem 3.11].

Theorem 1.15. *Let R be a commutative Noetherian ring. Then there are bijections between the sets of*

(1) 1*-tilting classes in* $\mathsf{Mod}\text{–}R$,

(2) 1*-cotilting classes in* $\mathsf{Mod}\text{–}R$,

(3) specialization closed subsets $Y \subseteq \operatorname{Spec} R$ *such that* $\operatorname{Ass} R \cap Y = \emptyset$,

(4) hereditary torsion pairs $(\mathcal{T}, \mathcal{F})$ *in* $\mathsf{Mod}\text{–}R$ *with* $R \in \mathcal{F}$.

The theorem will serve us as a guidance as we pass from the category of modules over a Noetherian commutative ring to the category of quasi-coherent sheaves over a Noetherian scheme.

1.3 $\mathcal{O}_X$-modules and quasi-coherent sheaves

The purpose of this section is to introduce the categories of $\mathcal{O}_X$-modules and of quasi-coherent sheaves on a scheme X as our main examples of Grothendieck categories. It also contains description of the main tools and results of the theory of quasi-coherent sheaves used throughout the rest of the thesis.

To make the presentation more to the point, we do not recall the definition of general sheaf (of commutative rings, or Abelian groups) on a topological space, or the notion of scheme at this moment. The reader is referred to Sections A and B of Appendix, where a brief overview of these topics is presented. Alternatively, we refer to the numerous algebraic geometry textbooks, e.g. [Har77], [GW10], or to the original source [GD60].

The overview presented in this section is far from comprehensive; the reason is, again, the effort to make this section as brief and compact as possible. Some of the proofs are only informally sketched and some are omitted completely. Despite this, we try to at least outline the arguments proving that the category of quasi-coherent sheaves on a scheme is a Grothendieck category, and describe differences from the module categories (most notably, not having enough projective objects).

Regarding schemes, we only briefly comment on some of the scheme-theoretic notation we use (cf. Notation B.5). If $x \in X$ is a point of a scheme, given an open affine neighbourhood $U \subseteq X$ of x, there is a prime ideal in the ring $\mathcal{O}_X(U)$ corresponding to the point; this prime ideal is denoted by $\mathfrak{p}_x$, or $\mathfrak{q}_x$ (in case we need to consider two such affine open neighbourhoods, and the corresponding prime ideals, at once). The unique maximal ideal of the stalk of the structure sheaf $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$. The field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is called *the residue field* at x and it is denoted by $\kappa(x)$. Recall the fact (of commutative algebra) that $\kappa(x)$ can be alternatively computed as the fraction field of the integral domain $\mathcal{O}_X(U)/\mathfrak{p}_x$.

If X is a scheme and $\operatorname{Spec} R = U \subseteq X$ is an affine open subset, recall that the collection of all *distinguished open sets*

$$D_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}, \ \ f \in R$$

forms a basis of the (Zariski) topology on U. Since all the sets D_f are affine open, and X has a cover by affine open sets, it follows that X has a basis consisting of affine open sets (cf. Remark B.4).

Let us start by recalling the definition of an $\mathcal{O}_X$-module.

Definition 1.16. Let X be a scheme. An *$\mathcal{O}_X$-module* consists of a sheaf of Abelian groups $\mathscr{F}$ endowed with an $\mathcal{O}_X(U)$-module action[8] on $\mathscr{F}(U)$ for each open subset $U \subseteq X$ such that the restrictions of the sheaf $\mathscr{F}$ $\mathscr{F}^U_V$ is an $\mathcal{O}_X(U)$-module homomorphism for every pair of open sets $V \subseteq U \subseteq X$. That is, given two open sets $U \subseteq V \subseteq X$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \\ \big\downarrow{\scriptstyle \mathrm{res}^V_U \times \mathscr{F}^V_U} & & \big\downarrow{\scriptstyle \mathscr{F}^V_U} \\ \mathcal{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \end{array}$$

(the horizontal arrows are the respective module action maps).

A *morphism of $\mathcal{O}_X$-modules* $\mathscr{F}$ and $\mathscr{G}$ is a morphism of sheaves of Abelian groups $f : \mathscr{F} \to \mathscr{G}$ such that for every open set $U \subseteq X$, the map $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.

Denote by $\mathsf{Mod}\text{–}\mathcal{O}_X$ the category of all $\mathcal{O}_X$-modules.

Remark 1.17. Suppose that $\mathscr{F}$ is an $\mathcal{O}_X$-module and $x \in X$ is a point. Given an open neighbourhoods U, V of x, a function $f \in \mathcal{O}_X(U)$ and a section of the sheaf $\mathscr{F}$ $s \in \mathscr{F}(U)$, define

$$f_x \cdot s_x := ((f \restriction_{U\cap V}) \cdot (s_{U\cap V}))_x$$

(recall that f_x, s_x denotes the germ of f, s, resp., in the stalk $\mathcal{O}_{X,x}$, $\mathscr{F}_x$ resp., cf. Definition A.9). This definition is correct in the following sense: if $s_x = s'_x$ and $f_x = f'_x$ for $f \in \mathcal{O}_X(U), f' \in \mathcal{O}_X(U')$, and $s \in \mathscr{F}(V), s' \in \mathscr{F}(V')$, then by definition of stalks, it is possible to find a neighbourhood of x $W \subseteq U \cap U' \cap V \cap V'$ small enough, so that

$$f \restriction_W = f' \restriction_W, \; s \restriction_W = s' \restriction_W .$$

Thus, using the commutative diagram from Definition 1.16 twice we have that

$$\begin{aligned} ((f \restriction_{U\cap V}) \cdot (s \restriction_{U\cap V})) \restriction_W &= ((f \restriction_W) \cdot (s \restriction_W)) \\ &= ((f' \restriction_W) \cdot (s' \restriction_W)) = ((f' \restriction_{U'\cap V'}) \cdot (s \restriction_{U'\cap V'})) \restriction_W, \end{aligned}$$

therefore we obtain (using the definition of stalks again)

$$\begin{aligned} ((f \restriction_{U\cap V}) \cdot (s \restriction_{U\cap V}))_x &= (((f \restriction_{U\cap V}) \cdot (s \restriction_{U\cap V})) \restriction_W)_x \\ &= (((f' \restriction_{U'\cap V'}) \cdot (s'_{U'\cap V'})) \restriction_W)_x = ((f \restriction_{U'\cap V'}) \cdot (s \restriction_{U'\cap V'}))_x. \end{aligned}$$

That is, $f_x \cdot s_x$ is independent of the choice of f, s, it depends on the germs only. A similar line of reasoning shows that the map $\cdot : \mathcal{O}_{X,x} \times \mathscr{F}_x \to \mathscr{F}_x$ defines a structure of $\mathcal{O}_{X,x}$-module on $\mathscr{F}_x$.

The category of $\mathcal{O}_X$-modules is always an Abelian category, which we state without proof. The second part of the statement is a useful criterion of exactness in $\mathsf{Mod}\text{–}\mathcal{O}_X$.

Proposition 1.18 ([GW10, (7.13)]). *Let X be a scheme. The category of $\mathcal{O}_X$-modules is an Abelian category. Moreover, a sequence of $\mathcal{O}_X$-modules and $\mathcal{O}_X$-module morphisms*

[8] For a commutative ring R, by an R-module action on an Abelian group M it is meant a map $\cdot : R \times M \to M$ making M into an R-module.

$$\mathscr{F}' \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{F}''$$

is exact if and only if the sequence

$$\mathscr{F}'_x \xrightarrow{\alpha_x} \mathscr{F}_x \xrightarrow{\beta_x} \mathscr{F}''_x$$

is exact sequence of $\mathcal{O}_{X,x}$-modules for all $x \in X$.

Remark 1.19. In [GW10, (7.13)], the proof of the assertion listed in Proposition 1.18 is done by an explicit construction of kernels, cokernels and biproducts. An important fact is that kernels can be computed "open set by open set". That is, given a morphism $f : \mathscr{F} \to \mathscr{G}$ of $\mathcal{O}_X$-modules, the presheaf of Abelian groups $\mathscr{K}$ defined by

$$\mathscr{K}(U) := \operatorname{Ker} f_U,\, U \subseteq X\text{open}$$

is actually a sheaf, and since each f_U is an $\mathcal{O}_X(U)$-module homomorphism, $\mathscr{K}(U)$ is naturally an $\mathcal{O}_X(U)$-module. Consequently, $\mathscr{K}$ is an $\mathcal{O}_X$-module, and together with the set-theoretical inclusions to $\mathscr{F}$, it is the kernel of f in $\mathsf{Mod}\text{–}\mathcal{O}_X$. As a consequence, monomorphisms in $\mathsf{Mod}\text{–}\mathcal{O}_X$ can be realized as (collections of) set-theoretical inclusions in the category $\mathsf{Mod}\text{–}\mathcal{O}_X$.

The fact that the category $\mathsf{Mod}\text{–}\mathcal{O}_X$ is a Grothendieck category was proved by Grothendieck in [Gro57]. This is, however, not enough for our purposes, since we are primarily interested in the full subcategory of $\mathcal{O}_X$-modules described by the following definition.

Definition 1.20. An $\mathcal{O}_X$-module is *quasi-coherent* if for every affine open subset $U \subseteq X$ and every $f \in \mathcal{O}_X(U)$, there is an isomorphism $\mathscr{F}(D_f) \xrightarrow{\simeq} \mathscr{F}(U)_f$ such that the following square

$$\begin{array}{ccc} \mathscr{F}(U) & = & \mathscr{F}(U) \\ \downarrow{\scriptstyle \mathrm{res}^U_{D_f}} & & \downarrow{\scriptstyle \mathrm{loc}_f} \\ \mathscr{F}(D_f) & \xrightarrow{\simeq} & \mathscr{F}(U)_f \end{array}$$

commutes (here loc_f denotes the localization morphism with respect to f). In other words, the restriction from U to D_f is the localization of $\mathscr{F}(U)$ with respect to f.

We will call quasi-coherent $\mathcal{O}_X$-modules quasi-coherent sheaves, as X will be usually clear from the context. Denote QCoh_X the full subcategory of $\mathsf{Mod}\text{–}\mathcal{O}_X$ consisting of all quasi-coherent sheaves.

There are several classes of objects in QCoh_X of interest. In order to define coherent sheaves in particular, recall that given a commutative ring R, an R-module M is said to be coherent if M is finitely generated and each finitely generated submodule of M is finitely presented.

Definition 1.21. Let X be a scheme and $\mathscr{F}$ a quasi-coherent sheaf.

(1) $\mathscr{F}$ is called *coherent* if for every affine open subset $U \subseteq X$, $\mathscr{F}(U)$ is a coherent $\mathcal{O}_X(U)$-module.

(2) $\mathscr{F}$ is *finite rank locally free* if there is an affine open cover $X = \bigcup_{i \in I} U_i$ such that $\mathscr{F}_{\restriction U_i}$ is isomorphic to $(\mathcal{O}_{X \restriction U_i})^{\oplus n_i}$ for every $i \in I$, where n_i are suitable integers[9].

Denote the full subcategory of QCoh_X consisting af all coherent sheaves by Coh_X.

Let us first comment on the heuristical reasons why is the category of quasi-coherent sheaves preferable to the category of all $\mathcal{O}_X$-modules.

A geometric motivation for the notion of quasi-coherent sheaf is that quasi-coherent sheaves are those $\mathcal{O}_X$-modules that restrict to smaller open sets in a similar way as does the structure sheaf of a scheme. This is apparent if the following construction of quasi-coherent sheaves over affine schemes is compared to the construction of the affine scheme itself (cf. Construction B.2).

Construction 1.22. Let R be a commutative ring and M an R-module. Consider the affine scheme $X = \operatorname{Spec} R$. Define a quasi-coherent sheaf $\widetilde{M}$ on X as follows:

1. When $U = D_f$ is a distinguished open set (if $U = D_1 = \operatorname{Spec} R$ in particular), put
$$\widetilde{M}(U) = M_f$$
(i.e. the localization of M with respect to $\{f^k \mid k \in \mathbb{N}\}$). Note that if $D_f = D_g$ for some $f, g \in R$, there is a canonical isomorphism $M_f \simeq M_g$ (i.e. the unique isomorphism making the diagram
$$\begin{array}{ccc} & M & \\ {\scriptstyle \mathrm{loc}_f}\swarrow & & \searrow{\scriptstyle \mathrm{loc}_g} \\ M_f & \xrightarrow{\ \simeq\ } & M_g \end{array}$$
commute, $\mathrm{loc}_f, \mathrm{loc}_g$ being the localization morphisms). That is, the module M_f is unique up to a canonical isomorphism.

2. If $D_f \subseteq D_g$ is a pair of distinguished open sets, we have $D_f = D_{fg}$, hence $M_f \simeq M_{fg}$ can be thought of as $(M_g)_f$, the localization of M_g whith respect to the multiplicative set $\{(f/1)^k \mid k \in \mathbb{N}\}$. Put
$$\left(\widetilde{M}(D_g) \xrightarrow{\mathrm{res}^{D_g}_{D_f}} \widetilde{M}(D_f) \right) := \left(M_g \xrightarrow{\mathrm{loc}_f} M_{gf} \right).$$
Note that this is again uniquely determined up to a canonical isomorphism.

3. Extend the collection to arbitrary open sets the only possible way so that the gluing axiom[10] is satisfied. That is, whenever $U \subseteq X$ is an open set,

[9]Coherent sheaves can also be defined using a chosen affine open cover only. That is, a quasi-coherent sheaf $\mathscr{F}$ is coherent if and only if there is an affine open cover $X = \bigcup_{i \in I} U_i$ such that each $\mathscr{F}(U_i)$ is a coherent $\mathcal{O}_X(U_i)$-module. This is because the assumptions of the Affine Communication Lemma (Lemma B.6) are satisfied. The definition of locally free sheaf, however, cannot be straightforwardly reformulated to the form analogous to the definition of coherent sheaves.

[10]Cf. Remark A.2.

consider all open sets $U_i,\ i \in I$, distinguished in X and contained in U, and all restriction among them (defined in the previous step). Then put

$$\widetilde{M}(U) := \lim_{i \in I} \widetilde{M}(U_i).$$

For every pair of open sets $V \subseteq U \subseteq X$, we obtain a unique morphism

$$\mathrm{res}^U_V : \widetilde{M}(U) \to \widetilde{M}(V)$$

coming from the universal property of limit.

A similar construction produces a morphism of quasi-coherent sheaves $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ out of an R-module homomorphism $f : M \to N$, making $\widetilde{(-)}$ into a functor

$$\widetilde{(-)} : \mathsf{Mod}\text{–}R \to \mathsf{QCoh}_X.$$

An algebraic motivation for working with quasi-coherent sheaves as opposed to all $\mathcal{O}_X$-modules is that quasi-coherent sheaves on a scheme directly generalize modules over a commutative ring, in the sense that QCoh_X is equivalent to $\mathsf{Mod}\text{–}R$ if $X = \operatorname{Spec} R$ is an affine scheme. This can be inferred from Construction 1.22: every step of the construction is necessary when trying to construct a quasi-coherent sheaf with the module of global sections being M. Thus, every quasi-coherent sheaf $\mathscr{F}$ on an affine scheme X is of the form $\widetilde{M}$ (for $M = \mathscr{F}(X)$), and similarly, every morphism f of quasi-coherent sheaves on X is of the form $\widetilde{f_X}$. The non-trivial part is to verify that the above construction always works, i.e. produces a quasi-coherent sheaf. But it is, indeed, always the case (cf. [GW10, (7.9)]). Let us state the conclusion of this discussion precisely.

Proposition 1.23 ([GW10, Corollary 7.17], [Har66, II.5.5])**.** *Let R be a commutative ring. Denote $X = \operatorname{Spec} R$. The functor $\widetilde{(-)} : \mathsf{Mod}\text{–}R \to \mathsf{QCoh}_X$ yields an equivalence of categories, with the inverse functor being the global sections functor*[11]

$$\Gamma(X, -) : \mathscr{F} \mapsto \Gamma(X, \mathscr{F}) = \mathscr{F}(X).$$

Moreover, the functor $\widetilde{(-)}$ takes coherent modules precisely to coherent sheaves, and finitely generated projective modules correspond precisely to finite rank locally free sheaves.

We now turn our attention to the fact that the category QCoh_X is Grothendieck, in particular Abelian. The latter assertion, together with the axiom (AB3), is a consequence of the following proposition.

Proposition 1.24 ([GW10, Corollary 7.19])**.** *Let X be a scheme.*

(1) If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of quasi-coherent sheaves on X, then the $\mathcal{O}_X$-modules $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *(computed in $\mathsf{Mod}\text{–}\mathcal{O}_X$) are quasi-coherent. That is, the kernels, cokernels and images in $\mathsf{Mod}\text{–}\mathcal{O}_X$ and in QCoh_X agree for morphisms from QCoh_X.*

(2) A direct sum of a family of quasi-coherent sheaves (computed, again, in $\mathsf{Mod}\text{–}\mathcal{O}_X$) is again quasi-coherent.

[11] Cf. Definition A.5.

(3) If $\mathscr{F}$ is a quasi-coherent sheaf and $\mathscr{F}_i$, $i \in I$ a collection of quasi-coherent subsheaves of $\mathscr{F}$, then the sum $\sum_{i\in I} \mathscr{F}_i$ is again quasi-coherent. If the set I is finite, then the intersection $\bigcap_{i\in I} \mathscr{F}_i$ is quasi-coherent.

In order to verify the axiom (AB5), it is enough to use Proposition 1.24 together with the following observation.

Lemma 1.25. *Consider a morphism of quasi-coherent sheaves $f : \mathscr{G} \to \mathscr{H}$. Then f is a monomorphism in $\mathsf{Mod}\text{–}\mathcal{O}_X$ if and only if f is a monomorphism in QCoh_X.*

Proof. Every monomorphism in $\mathsf{Mod}\text{–}\mathcal{O}_X$ is clearly a monomorphism in QCoh_X, since QCoh_X is a subcategory of $\mathsf{Mod}\text{–}\mathcal{O}_X$.

Conversely, if $f : \mathscr{F} \to \mathscr{G}$ is a monomorphism in QCoh_X, it is a monomorphism in $\mathsf{Mod}\text{–}\mathcal{O}_X$: since the kernel $\mathscr{K}$ of f in $\mathsf{Mod}\text{–}\mathcal{O}_X$ is quasi-coherent by Proposition 1.23 (1), it is easy to see that it is the kernel of f in QCoh_X, hence $\mathscr{K} = 0$. □

The subcategory QCoh_X of $\mathsf{Mod}\text{–}\mathcal{O}_X$ is thus closed under direct sums, kernels and cokernels of morphisms from QCoh_X. In particular, it is closed under direct limits[12]. Additionally, the monomorphisms in QCoh_X and $\mathsf{Mod}\text{–}\mathcal{O}_X$ agree by Lemma 1.25. Using the fact that direct limits are exact in $\mathsf{Mod}\text{–}\mathcal{O}_X$, we thus infer that direct limits are exact in QCoh_X as well. Thus, QCoh_X is an AB5-category.

What remains is to verify that QCoh_X admits a generator. This is more subtle than in the case of the category of $\mathcal{O}_X$-modules[13]. The following line of arguments is the approach of O. Gabber.

Given an infinite cardinal κ, let us call a quasi-coherent sheaf $\mathscr{F}$ *of type* κ if for each open set $U \subseteq X$, the $\mathcal{O}_X(U)$-module $\mathscr{F}(U)$ is $(< \kappa)$-generated. Gabber proved the following.

Lemma 1.26 (Gabber, [Con00, Lemma 2.1.7][14])**.** *Given an arbitrary scheme X, there exists an infinite cardinal κ such that every quasi-coherent sheaf $\mathscr{F}$ on X is the sum of its quasi-coherent subsheaves of type κ.*

On a given scheme X, there are clearly, up to isomorphism, set-many quasi-coherent sheaves of type κ. More precisely, there is a set S and a collection $\{\mathscr{F}^s \mid s \in S\}$ of quasi-coherent sheaves of type κ on X such that each quasi-coherent sheaf $\mathscr{F}$ of type κ is isomorphic to $\mathscr{F}^s$ for some s; as a consequence, the quasi-coherent sheaf $\bigoplus_{s\in S} \mathscr{F}^s$ is a generator for the category QCoh_X. Thus, QCoh_X admits a generator, hence it is a Grothendieck category.

The importance of Gabber's result is in the fact that it ensures the existence of a generator of QCoh_X for arbitrary scheme X. It is, however, not needed when suitable finiteness conditions are imposed on X. An example of this is the

[12] Or, more generally, all colimits.

[13] In order to produce a small family of generators for $\mathsf{Mod}\text{–}\mathcal{O}_X$, Grothendieck uses an $\mathcal{O}_X$-module construction called *extension by zero*. The problem is that this construction typically produces $\mathcal{O}_X$-modules that are not quasi-coherent.

[14] The assertion is stated without proof in [Con00]. Apparently, the proof itself was written down only later, in a generalized version, by E. Enochs and S. Estrada as Corollary 3.5 in [EE05].

following well-known exercise from Hartshorne's book [Har77] on extensions of coherent sheaves.

Theorem 1.27 ([Har77, Exercise II.5.15], [Sta16, Tag 01PF[15]])**.** *Let X be a Noetherian scheme, $U \subseteq X$ be an open subset and $\mathscr{G}$ be a coherent sheaf on U.*

(1) There is a coherent sheaf $\mathscr{G}'$ on X extending $\mathscr{G}$, i.e. such that $\mathscr{G}'_{\restriction U} = \mathscr{G}$.

(2) If $\mathscr{F}$ is a quasi-coherent sheaf on X such that $\mathscr{G} \subseteq \mathscr{F}_{\restriction U}$, then there is a coherent subsheaf $\mathscr{G}' \subseteq \mathscr{F}$ on X such that $\mathscr{G}'_{\restriction U} = \mathscr{F}'_{\restriction U}$.

A consequence of Theorem 1.27 is that κ can be taken as $\aleph_0$ in Gabber's lemma when X is Noetherian.

Corollary 1.28. *Let X be a Noetherian scheme. Then every quasi-coherent sheaf $\mathscr{F}$ on X is a directed union of its coherent subsheaves.*

Proof. It is enough to show that given any open set $U \subseteq X$ and any section $s \in \mathscr{F}(U)$, there is a coherent subsheaf $\mathscr{G}$ of $\mathscr{F}$ containing s. Consider an affine open cover[16] of U, $U = \underbrace{\bigcup_{i=1}^{k}} U_i$. For each i, put $s_i := s \restriction_{U_i}$. By Theorem 1.27 (2), the quasi-coherent sheaf $\langle s_i \rangle$ can be extended to a coherent subsheaf $\mathscr{G}^i$ on X. In particular, $s_i \in \mathscr{G}^i(U_i)$.

Now consider the coherent subsheaf

$$\mathscr{G} = \sum_{i=1}^{k} \mathscr{G}^i \subseteq \mathscr{F}.$$

$\mathscr{G}$ has the property that $s_i \in \mathscr{G}(U_i)$ for every i. It follows that $s \in \mathscr{G}(U)$, since s_i's are gluable collection of sections[17]. This finishes the proof. □

Example 1. A basic example of a non-affine scheme is the projective line $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, where k is an algebraically closed field[18].

The following is a rough desription, without any proof, of the category $\mathsf{Coh}_{\mathbb{P}^1_k}$. A very thorough discussion of this category is given in [CK09, Chapter 5].

The category $\mathsf{Coh}_{\mathbb{P}^1_k}$ is a Krull-Shmidt category, hence every coherent sheaf $\mathscr{F}$ decomposes as a direct sum of indecomposable coherent sheaves.

There is a family of indecomposable, pairwise non-isomorphic, locally free sheaves $\mathcal{O}(n)$, $n \in \mathbb{Z}$, such that $\mathcal{O}(0)$ is the structure sheaf $\mathcal{O}_X$. We call these sheaves *locally free sheaves of rank* 1, or *line bundles*[19]. For each $n, m \in \mathbb{Z}$ we have

$$\operatorname{Hom}_{\mathbb{P}^1_k}(\mathcal{O}(m), \mathcal{O}(n)) \simeq (k[x_0, x_1])_{n-m} ,$$

[15]As of May 2nd 2016, the Lemma under the Tag 01PF was listed on the Stacks Project site [Sta16] as Lemma 27.22.1.

[16]Note that the affine open cover is taken finite. This is because if X is a Noetherian scheme, any subspace of X is quasi-compact, see Remark B.8.

[17]Cf. Remark A.2.

[18]An explicit construction of $\mathbb{P}^1_k$ can be found in Appendix, Example B.9. Proj is a (functorial) construction that makes $\mathbb{Z}_{\geq 0}$-graded k-algebras into projective k-schemes, in some ways analogous to the Spec functor. For details, see [GW10, (13.2)].

[19]The terminology stems from the fact that a locally free sheaf of rank 1 can be obtained as the sheaf of sections of a 1-dimensional vector bundle over X. This was proved by Serre in [Ser55].

where $(k[x_0, x_1])_{n-m}$ denotes the $(n-m)$-graded part of $k[x_0, x_1]$ (graded by degree). In particular, there are no nonzero morphisms $\mathcal{O}(m) \to \mathcal{O}(n)$ if $m > n$.

For each closed point $x \in X$ and every integer $r > 0$, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}(-rd(x)) \longrightarrow \mathcal{O}(0) \longrightarrow \mathcal{O}_{x,r} \longrightarrow 0 \,,$$

where $d(x)$ denotes an integer associated to the point x in a suitable way[20]. This prescribes an indecomposable coherent sheaf $\mathcal{O}_{x,r}$ of length r. These are called *torsion indecomposable sheaves.*

The above is a comprehensive list of indecomposable coherent sheaves of $\mathbb{P}^1_k$.

Remark 1.29. In contrast to the case of the category $\mathsf{Mod}\text{–}R$, direct products in a general Grothendieck category do not need to be exact. More precisely, given an infinite set I, the functor

$$\prod_{i \in I} : \mathcal{A}^I \longrightarrow \mathcal{A}$$

(which takes collection of objects indexed by I to their product) preserves kernels (it is the right adjoint to the diagonal embedding functor) but it does not preserve cokernels in general.

The phenomenon is not rare among the categories we are interested in, i.e. $\mathcal{A} = \mathsf{QCoh}_X$ for a (non-affine) scheme X. Even in the case when $X = \mathbb{P}^1_k$, the exactness of direct product fails. This is demonstrated by H. Krause in [Kra05, Example 4.9][21].

Another problem is that the category QCoh_X does not have enough projectives in general. This can actually be viewed as a consequence of the non-exactness of direct product functors: thus, the category $\mathbb{P}^1_k$ does not have enough projectives.

To demonstrate this, suppose that QCoh_X has enough projectives. We prove that the direct product functors are exact in QCoh_X [22]. Since there is a generator for QCoh_X, this in particular means that there is a projective generator $\mathscr{P}$ (obtained as a projective sheaf admitting an epimorphism onto a generator). Put $R = \operatorname{End}_X(\mathscr{P})$.

The functor

$$\operatorname{Hom}_X(\mathscr{P}, -) : \mathsf{QCoh}_X \longrightarrow \mathsf{Mod}\text{–}R$$

is fully faithful (since $\mathscr{P}$ is a generator, e.g. by Theorem 1.3), and additionally, it is exact (since $\mathscr{P}$ is projective). A general fact is that a fully faithful exact functor reflects exactness, i.e. if

$$(\xi) \qquad 0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is a sequence of quasi-coherent sheaves such that the sequence

[20] The point x corresponds to a homogeneous prime ideal $\mathfrak{p}$ of $k[x_0, x_1]$ and $d(x)$ is chosen as the degree of a homogeneous polynomial P such that $(P) = \mathfrak{p}$. More details are in [CK09, Chapter 5].

[21] The example is attributed to B. Keller.

[22] The author learned of this fact through an online discussion from L. Positselski. Although the proof was worked out independently, it is certainly known. The proof is presented here since the author of this thesis was not able to find any reference for it.

$$0 \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}') \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}) \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}'') \longrightarrow 0$$

is exact, then (ξ) is itself exact.

Now, suppose that

$$(\xi_i) \qquad 0 \longrightarrow \mathscr{F}_i' \longrightarrow \mathscr{F}_i \longrightarrow \mathscr{F}_i'' \longrightarrow 0\,, \qquad i \in I$$

is a collection of short exact sequences in QCoh_X. Applying the product functor $\prod_{i\in I}$ yields a complex

$$(\textstyle\prod_i \xi_i) \qquad 0 \longrightarrow \prod_{i\in I} \mathscr{F}_i' \longrightarrow \prod_{i\in I} \mathscr{F}_i \longrightarrow \prod_{i\in I} \mathscr{F}_i'' \longrightarrow 0$$

which is taken by the functor $\mathrm{Hom}_X(\mathscr{P}, -)$ to the complex

$$0 \longrightarrow \mathrm{Hom}_X\Big(\mathscr{P}, \prod_{i\in I} \mathscr{F}_i'\Big) \longrightarrow \mathrm{Hom}_X\Big(\mathscr{P}, \prod_{i\in I} \mathscr{F}_i\Big) \longrightarrow \mathrm{Hom}_X\Big(\mathscr{P}, \prod_{i\in I} \mathscr{F}_i''\Big) \longrightarrow 0$$

and this complex is, in turn, isomorphic to

$$0 \longrightarrow \prod_{i\in I} \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i') \longrightarrow \prod_{i\in I} \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i) \longrightarrow \prod_{i\in I} \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i'') \longrightarrow 0$$

by the fact the universal property of products, i.e. by the fact that $\mathrm{Hom}_X(\mathscr{P}, -)$ preserves direct products. But this resulting complex is just a direct product of the exact sequences

$$0 \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i') \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i) \longrightarrow \mathrm{Hom}_X(\mathscr{P}, \mathscr{F}_i'') \longrightarrow 0\,,$$

and thus, is exact since direct products are exact in $\mathsf{Mod}\text{–}R$. By the fact that $\mathrm{Hom}_X(\mathscr{P}, -)$ reflects exactness it thus follows that the original product complex ($\prod_i \xi_i$) is exact. That is, X has the property that direct products are exact in QCoh_X. The above mentioned example thus shows that $\mathsf{QCoh}_{\mathbb{P}^1_k}$ does not have enough projectives.

Remark 1.30 ([GW10, (7.8)])**.** Let us close this section by a remark on functors between the categories of $\mathcal{O}_X$-modules induced by morphisms of schemes.

Consider a morphism of schemes

$$\pi : X \longrightarrow Y.$$

Then the pushforward functor $\pi_* : \mathsf{Ab}_X \to \mathsf{Ab}_Y$ (as described in Definition A.6 in Appendix) takes an $\mathcal{O}_X$-module to an $\mathcal{O}_Y$-module; more precisely, there is a natural way how to make the sheaf $\pi_*\mathscr{F}$, given the fact that $\mathscr{F}$ is an $\mathcal{O}_X$-module, into an $\mathcal{O}_Y$-module. This way, a functor

$$\pi_* : \mathsf{Mod}\text{–}\mathcal{O}_X \to \mathsf{Mod}\text{–}\mathcal{O}_Y$$

is obtained. The functor π_* is called the *direct image functor* induced by π.

The analogous statement for the left adjoint $\pi^{-1} : \mathsf{Ab}_Y \to \mathsf{Ab}_X$ does not hold. However, the direct image functor still admits an adjoint, denoted by

$$\pi^* : \mathsf{Mod}\text{–}\mathcal{O}_Y \to \mathsf{Mod}\text{–}\mathcal{O}_X\,.$$

The functor π^* is called the *direct image functor* induced by π.

Inverse image functors enjoy the property that quasi-coherence is preserved by them, i.e. they take quasi-coherent sheaves to quasi-coherent sheaves (see [GW10, Remark 7.23] for proof). This is not true for the direct image functor, but it is true in several important cases. In particular, the fact that π_* preserves quasi-coherence when π is quasi-compact and quasi-separated morphism[23] (see [GW10, Corollary 10.27] for proof).

1.4 Injective $\mathcal{O}_X$-modules and quasi-coherent sheaves

This section is devoted to the investigation of the structure of injectives on a locally Noetherian scheme. The goal is to demonstrate that the structure of injective quasi-coherent sheaves on a locally Noetherian scheme resembles the structure of injective modules over a Noetherian ring. The main reference used in this section is [Har66, II.7] which treats primarily the structure of injective $\mathcal{O}_X$-modules.

The following fact will be crucial to the consequent arguments.

Proposition 1.31 ([Har66, proof of II.7.18]). *Let X be a locally Noetherian scheme. Then the subcategory QCoh_X of $\mathsf{Mod}\text{–}\mathcal{O}_X$ is closed under injective hulls. That is, whenever $\mathscr{F}$ is a quasi-coherent sheaf and $\mathscr{E}$ is its injective hull as an $\mathcal{O}_X$-module, then $\mathscr{E}$ is quasi-coherent.*

We now proceed to the description of the structure of injectives in QCoh_X if X is locally Noetherian. We describe a family $\mathscr{J}(x)$, $x \in X$, of injective quasi-coherent sheaves that plays similar role as the injective hulls of the indecomposable R-modules in the affine (Noetherian) case. In order to do that, we need first to describe a natural embedding $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ for every point $x \in X$.

Let X be a locally Noetherian scheme. For $x \in X$, denote the set of all generizations of x by Y. That is, Y consists of all points y such that $x \in \overline{\{y\}}$. Clearly we have that

$$Y = \bigcap \{U \subseteq X \mid U \text{ is open and } x \in U\},$$

and using the fact that the set of all affine open sets forms a basis of the topology on X, we obtain

$$Y = \bigcap \{U \subseteq X \mid U \text{ is affine open and } x \in U\}.$$

Fix an affine open neighbourhood U of x. By the above considerations, we have $Y \subseteq U$. Interpreting the (topological) closure operator in $\operatorname{Spec} A$, where $A = \mathcal{O}_X(U)$, we immediately see that

$$Y = \{y \in U \mid \mathfrak{p}_y \subseteq \mathfrak{p}_x\}.$$

The stalk $\mathcal{O}_{X,x}$ can be computed using U as the localization[24]

$$\mathcal{O}_X(U) \xrightarrow{\text{loc}} \mathcal{O}_X(U)_{\mathfrak{p}_x} = \mathcal{O}_{X,x},$$

[23]see [GW10, Definition 1.10 and Definition 10.22] for the definitions of quasi-compact morphism, quasi-separated morphism, resp.

[24]Cf. Construction B.2. The reason is that the structure sheaf $\mathcal{O}_X$ behaves over an affine open set as an affine scheme, and for affine scheme the assertion holds. Similarly, given a quasi-coherent sheaf $\mathscr{F}$ and affine open set U, the stalk $\mathscr{F}_x$ may be computed as $\mathscr{F}(U)_{\mathfrak{p}_x}$.

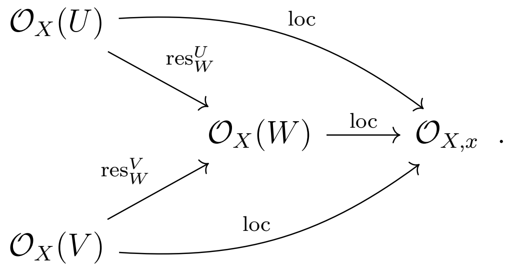
which induces an embedding

$$\operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{j} \operatorname{Spec} O_X(U) = U.$$

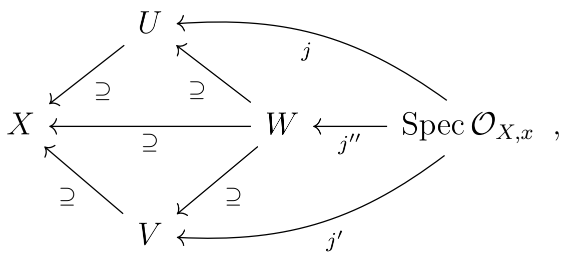
Composition of j with the open embedding of schemes $U \subseteq X$ yields an embedding of schemes

$$\operatorname{Spec} \mathcal{O}_{X,x} \overset{i}{\hookrightarrow} X$$

with $i(\operatorname{Spec} \mathcal{O}_{X,x}) = Y$. This embedding is independent of the choice of U. If V is another affine open neighbourhood of x and j, j' denote the respective embeddings of $\operatorname{Spec} \mathcal{O}_{X,x}$ into U and V, we can choose an affine open neighbourhood $x \in W \subseteq U \cap V$ which is distinguished in both U and V, and compute the embedding j'' of $\operatorname{Spec} \mathcal{O}_{X,x}$ into W. We have a commutative diagram (with all arrows being suitable localizations) of ring homomorphisms:



Applying the Spec functor and filling in the respective open embeddings of U, V, W into X yields a commutative diagram



whose common composite (i.e. from $\operatorname{Spec} \mathcal{O}_{X,x}$ to X) is i.

Define

$$\mathscr{J}(x) = i_* \left(\widetilde{E_x} \right),$$

where E_x is the injective hull of $\kappa(x)$, the residue field at x (in the category $\mathsf{Mod}\text{–}\mathcal{O}_{X,x}$). Note that $\operatorname{Spec} \mathcal{O}_{X,x}$ is a Noetherian (affine) scheme and thus, any morphism of schemes with domain $\operatorname{Spec} \mathcal{O}_{X,x}$ (i, in particular) is quasi-compact and quasi-separated (cf. Remark 1.30, or [GW10, Remark 10.2 and Definition 10.22]). Thus, $\mathscr{J}(x)$ is a quasi-coherent sheaf by Remark 1.30. In fact, $\mathscr{J}(x)$ is an indecomposable injective quasi-coherent sheaf on X, which is a part of the following classification theorem.

Theorem 1.32 ([Har66, II.7.17]). *Let X be a locally Noetherian scheme and $\mathscr{F}$ a quasi-coherent sheaf. Then the following are equivalent:*

(1) $\mathscr{F}$ is an injective $\mathcal{O}_X$-module.

(2) $\mathscr{F}$ is an injective quasi-coherent sheaf (i.e. it is an injective object in QCoh_X).

(3) For every $x \in X$, $\mathscr{F}_x$ is an injective $\mathcal{O}_{X,x}$-module.

(4) $\mathscr{F}$ is a direct sum of sheaves of the form $\mathscr{J}(x)$ for various $x \in X$.

Proof. The equivalence of (1), (3) and (4) is proved in [Har66, II.7.17], let us only comment on the equivalence of (1) and (2)[25].

The implication (1) ⇒ (2) follows directly from Lemma 1.25.

Conversely, suppose that $\mathscr{F}$ is injective as a quasi-coherent sheaf. By Proposition 1.31 we may consider an embedding $\mathscr{F} \subseteq \mathscr{G}$, where $\mathscr{G}$ is quasi-coherent sheaf which is injective as an $\mathcal{O}_X$-module. Using the injectivity of $\mathscr{F}$ (in QCoh_X), it follows that $\mathscr{F}$ is a direct summand of $\mathscr{G}$, i.e. the inclusion $\mathscr{F} \subseteq \mathscr{G}$ splits (in QCoh_X, thus also in $\mathsf{Mod}\text{–}\mathcal{O}_X$). Thus, $\mathscr{F}$ is a direct summand of injective $\mathcal{O}_X$-module, hence it is an injective $\mathcal{O}_X$-module as well. □

We close this chapter by the following consequence of the classification theorem.

Corollary 1.33. *Let X be a locally Noetherian scheme. The class of injective quasi-coherent sheaves on X is closed under taking direct limits.*

Proof. Suppose a sheaf $\mathscr{F}$ is given by

$$\mathscr{F} = \varinjlim_i \mathscr{E}_i$$

with all $\mathscr{E}_i$'s injective quasi-coherent sheaves. Consider an arbitrary point $x \in X$. As the stalk functor $(-)_x$ is a left adjoint, it preserves all colimits, hence

$$\mathscr{F}_x = \varinjlim_i (\mathscr{E}_i)_x .$$

By Theorem 1.32, all the $\mathcal{O}_{X,x}$-modules $(\mathscr{E}_i)_x$ are injective. As $\mathcal{O}_{X,x}$ is a Noetherian ring, it follows that the direct limit $\mathscr{F}_x$ is injective as well. Using Theorem 1.32 again, we infer that $\mathscr{F}$ is an injective quasi-coherent sheaf. □

[25] In [Har66, II.7, Example on p. 135], Hartshorne warns that the structure of injectives in QCoh_X, as well as which of the injective quasi-coherent sheaves are injective as $\mathcal{O}_X$-modules, is unclear even when X is locally Noetherian. The example goes on to describe a locally Noetherian scheme such that the category QCoh_X is not locally Noetherian. The proof of the equivalence "(1)⇔(2)" given here shows that this caution is not necessary when dealing with locally Noetherian schemes. The used argument seems to be folklore – cf. [TT90, B.4].

2. Hereditary torsion pairs in QCoh_X

In this chapter, we describe hereditary torsion pairs in the category of quasi-coherent sheaves on a Noetherian scheme X. The result is, in the form of description of the hereditary torsion classes[1], originally proved by Gabriel in his dissertation thesis ([Gab62, VI.4.b]).

Let us start by recalling the definitions of a torsion pair and the useful tools for dealing with quasi-coherent sheaves, namely, the associated point of a sheaf and the support of a sheaf.

2.1 Torsion pairs in an Abelian category

Definition 2.1. Let $\mathcal{A}$ be an Abelian category. A *torsion pair in* $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ such that

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, i.e. for all $T \in \mathcal{T}$ and all $F \in \mathcal{F}$, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$, and

(2) for every $A \in \mathcal{A}$, there is an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. We call the object T the $(\mathcal{T}, \mathcal{F})$-*torsion part* of A, and the object F the $(\mathcal{T}, \mathcal{F})$-*torsion-free part* of A.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *hereditary* if $\mathcal{T}$ is closed under subobjects.

Note that the definition of a torsion pair is self-dual. That is, if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in an Abelian category $\mathcal{A}$, then $(\mathcal{F}^{\mathrm{op}}, \mathcal{T}^{\mathrm{op}})$ is a torsion pair in $\mathcal{A}^{\mathrm{op}}$.

In order to make the thesis reasonably self-contained, we present proofs even for some of the standard results regarding torsion pairs. Let us start the discussion by the following observation.

Lemma 2.2. *Suppose $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$ are two torsion pairs in an Abelian category $\mathcal{A}$ such that $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{F} \subseteq \mathcal{F}'$. Then $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}', \mathcal{F}')$.*

Proof. Consider $G \in \mathcal{T}'$. Then there is a short exact sequence

$$0 \longrightarrow T \xrightarrow{\alpha} G \xrightarrow{\beta} F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since $F \in \mathcal{F}'$, $\beta = 0$ and thus, α is necessarily an isomorphism. It follows that $G \in \mathcal{T}$. Using the symmetric argument one proves that any $H \in \mathcal{F}'$ is in fact a member of $\mathcal{F}$. □

The notion of torsion pair in an Abelian category was introduced by S. E. Dickson in [Dic66]. The following discussion appeared, in some form, in the same paper as well.

[1] In Gabriel's thesis, these are called localizing subcategories. The reason is that hereditary torsion classes have a deep connection to the localization theory of Abelian categories. See [Gab62] or [Ste75] for further information.

Remark 2.3 ([Dic66]).

1. If $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{A}$, then it follows that $\mathcal{F} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, -)$ and $\mathcal{T} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$. To see the first equality, first observe that Definition 2.1 (1) gives the inclusion '$\subseteq$'. Now clearly $\mathcal{T}' := \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$ and $\mathcal{F}' := \mathcal{F}$ form a torsion pair (to check the property (2) for some $A \in \mathcal{A}$, use the same $T \in \mathcal{T} \subseteq \mathcal{T}'$ and $F \in \mathcal{F} = \mathcal{F}'$ as for the torsion pair $(\mathcal{T}, \mathcal{F})$). By Lemma 2.2, the torsion pairs $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$ agree, so $\mathcal{T} = \mathcal{T}' = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$. The other equality is dual.

2. From the above equalities, it is clear that $\mathcal{T}$ is closed under all colimits that exist in $\mathcal{A}$ (that is, if D is a diagram consisting of obects of $\mathcal{T}$ such that $\operatorname{colim} D$ exists in $\mathcal{A}$, then $\operatorname{colim} D \in \mathcal{T}$), and dually, $\mathcal{F}$ is closed under all limits that exist in $\mathcal{A}$. Also both $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions.

3. Suppose that $\mathcal{A}$ is an Abelian category where unions of arbitrary collections of subobjects (i.e. colimit of the associated monomorphisms) exist - this is fulfilled in particular when $\mathcal{A}$ is a Grothendieck category (e.g. QCoh_X) or a Noetherian category (e.g. Coh_X). Assume that $(\mathcal{T}, \mathcal{F})$ is a pair of subcategories of $\mathcal{A}$ such that $\mathcal{F} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, -)$ and $\mathcal{T}$ is closed under colimits (that exist in $\mathcal{A}$, as in 2.) and extensions. We claim that $(\mathcal{T}, \mathcal{F})$ is a torsion pair. Indeed, consider $A \in \mathcal{A}$, and take $T \overset{\alpha}{\hookrightarrow} A$ the maximum subobject of A with $T \in \mathcal{T}$ (T is given as the union of all subobjects $T' \hookrightarrow A$ with $T' \in \mathcal{T}$; by the fact that $\mathcal{T}$ is closed under colimits, from the assumption on $\mathcal{A}$ we have that $T \in \mathcal{T}$). We have the short exact sequence

$$0 \longrightarrow T \xrightarrow{\alpha} A \xrightarrow{\beta} F \longrightarrow 0$$

(where $\beta = \operatorname{Coker}\alpha$). It is enough to show that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, F) = 0$ so that $F \in \mathcal{F}$. Consider $T' \in \mathcal{T}$ and a morphism $\gamma : T' \to F$. Taking a pullback P of β and γ yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & F & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle\tilde{\gamma}} & & \uparrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & T & \longrightarrow & P & \xrightarrow{\tilde{\beta}} & T' & \longrightarrow & 0
\end{array}$$

with exact rows. In particular, $P \in \mathcal{T}$ as it is an extension of T' and T. Consider the epi-mono factorization of $\tilde{\gamma}$, $\tilde{\gamma} = me$, where $P \overset{e}{\to} P'$ is epic, hence $P' \in \mathcal{T}$, and $P' \overset{m}{\to} A$ is monic, i.e. it is a subobject of A with $P' \in \mathcal{T}$. It follows that we have a (unique) factorization

$$\begin{array}{ccc}
T & \xrightarrow{\alpha} & A \\
{\scriptstyle m'}\nwarrow & & \nearrow{\scriptstyle m} \\
 & P' &
\end{array},$$

i.e. $m = \alpha m'$. Thus, we have

$$\gamma\tilde{\beta} = \beta\tilde{\gamma} = \beta me = \beta\alpha m'e = 0m'e = 0,$$

and since $\tilde{\beta}$ is epic, it follows that $\gamma = 0$ and we are done.

4. Dually to 3., if $\mathcal{A}$ is a category where "counions" (i.e. limits) of arbitrary collections of quotients exist and if $\mathcal{F}$ is closed under extensions and under all limits that exist in $\mathcal{A}$, by setting $\mathcal{T} = \operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$, we obtain a torsion pair $(\mathcal{T}, \mathcal{F})$.

The following assertion is standard as well, cf. [Ste75, VI.3.2].

Lemma 2.4. *Let $\mathcal{A}$ be an Abelian category wich has injective hulls and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then the following are equivalent:*

(1) $(\mathcal{T}, \mathcal{F})$ is hereditary, i.e. $\mathcal{T}$ is closed under subobjects.

(2) $\mathcal{F}$ is closed under injective hulls.

Proof. (1) $\Rightarrow$ (2). Consider $F \in \mathcal{F}$, its injective hull $E(F)$ and a short exact sequence

$$0 \longrightarrow T' \longrightarrow E(F) \longrightarrow F' \longrightarrow 0$$

with $T' \in \mathcal{T}$ and $F' \in \mathcal{F}$. Then we have the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T' & \longrightarrow & E(F) & \xrightarrow{\beta} & F' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle\alpha} & & & & \\
0 & \longrightarrow & T' \cap F & \longrightarrow & F & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathcal{T}$ is hereditary (and the torsion-free class $\mathcal{F}$ is closed under subobjects automatically), $T' \cap F \in \mathcal{T} \cap \mathcal{F}$. Thus, $T' \cap F = 0$ (as the identity morphism on $T' \cap F$ must be the zero morphism). From essentiality of the monomorphism α it follows that $T' = 0$, hence β is an isomorphism and $E(F) \in \mathcal{F}$.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{F}$ is closed under taking injective hulls. Consider $T \in \mathcal{T}$ and its subobject $T' \overset{i}{\hookrightarrow} T$. We have a short exact sequence

$$0 \longrightarrow T'' \xrightarrow{\alpha} T' \xrightarrow{\beta} F'' \longrightarrow 0$$

with $T'' \in \mathcal{T}$ and $F'' \in \mathcal{F}$. Denote $F'' \overset{j}{\hookrightarrow} E(F'')$ the embedding of F'' into its injective hull. By injectivity of $E(F'')$, there exists a morphism $\gamma : T \to E(F'')$ such that the following square

$$\begin{array}{ccc}
T' & \xrightarrow{\beta} & F'' \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\
T & \xrightarrow{\gamma} & E(F'')
\end{array}$$

is commutative. However, $T \in \mathcal{T}$ and $E(F'') \in \mathcal{F}$, hence $\gamma = 0$. Thus, we have that $j\beta = 0$, hence $\beta = 0$ (since j is a monomorphism). It follows that α is an isomorphism and thus, $T' \in \mathcal{T}$. $\square$

The following two propositions are concerned with the relations of torsion pairs in the categories QCoh_X and Coh_X, in analogy to the Mod–R and mod–R case. For module categories, the assertion can be found e.g. in [GT12, Proposition 8.36].

Proposition 2.5. *Let X be a Noetherian scheme.*

(1) Consider a torsion pair $(\mathcal{T}, \mathcal{F})$ in QCoh_X. Then $(\mathcal{T} \cap \mathsf{Coh}_X, \mathcal{F} \cap \mathsf{Coh}_X)$ is a torsion pair in Coh_X.

(2) Consider a torsion pair $(\mathcal{T}, \mathcal{F})$ in Coh_X. Then $(\varinjlim \mathcal{T}, \varinjlim \mathcal{F})$ is a torsion pair in QCoh_X.

Proof.

(1) Consider $\mathscr{T} \in \mathsf{Coh}_X$ such that $\operatorname{Hom}_X(\mathscr{T}, \mathcal{F} \cap \mathsf{Coh}_X) = 0$. We claim that $\operatorname{Hom}_X(\mathscr{T}, \mathcal{F}) = 0$. To see this, choose a sheaf $\mathscr{F} \in \mathcal{F}$ arbitrarily and consider a morphism $f : \mathscr{T} \to \mathscr{F}$. As $\mathscr{T} \in \mathsf{Coh}_X$ and X is Noetherian, the subsheaf $\operatorname{Im} f \subseteq \mathscr{F}$ is coherent as well. Then $\operatorname{Im} f \in \mathcal{F} \cap \mathsf{Coh}_X$, which shows that the corestriction $f : \mathscr{T} \to \operatorname{Im} f$ is zero, hence $f = 0$.

Similarly, if $\mathscr{F} \in \mathsf{Coh}_X$ is a sheaf such that $\operatorname{Hom}_X(\mathcal{T} \cap \mathsf{Coh}_X, \mathscr{F}) = 0$, then $\operatorname{Hom}_X(\mathcal{T}, \mathscr{F}) = 0$. Consider an arbitrary morphism $g : \mathscr{T} \to \mathscr{F}$ with $\mathscr{T} \in \mathcal{T}$. Then $\operatorname{Im} g \subseteq \mathscr{F}$ is coherent, since $\mathscr{F}$ is (and X is noetherian). Thus, $\operatorname{Im} g \in \mathcal{T} \cap \mathsf{Coh}_X$, hence the inclusion $\subseteq : \operatorname{Im} g \hookrightarrow \mathscr{F}$ is necessarily zero, i.e. $g = 0$.

The above considerations show that

$$\begin{aligned} \mathcal{T} \cap \mathsf{Coh}_X &= \operatorname{Ker} \operatorname{Hom}_{\mathsf{Coh}_X}(-, \mathcal{F} \cap \mathsf{Coh}_X) , \\ \mathcal{F} \cap \mathsf{Coh}_X &= \operatorname{Ker} \operatorname{Hom}_{\mathsf{Coh}_X}(\mathcal{T} \cap \mathsf{Coh}_X, -) , \end{aligned}$$

hence $(\mathcal{T} \cap \mathsf{Coh}_X, \mathcal{F} \cap \mathsf{Coh}_X)$ is a torsion pair in Coh_X.

(2) Suppose $\mathscr{T} \in \mathcal{T}$ is a coherent sheaf and $\mathscr{F}_i \in \mathcal{F}$, $i \in I$, is a directed system. Since the canonical morphism

$$\varinjlim_i \operatorname{Hom}_X(\mathscr{T}, \mathscr{F}_i) \to \operatorname{Hom}_X(\mathscr{T}, \varinjlim_i \mathscr{F}_i)$$

is an isomorphism, we have that

$$0 = \varinjlim_i \operatorname{Hom}_X(\mathscr{T}, \mathscr{F}_i) \simeq \operatorname{Hom}_X(\mathscr{T}, \varinjlim_i \mathscr{F}_i).$$

This shows that $\operatorname{Hom}_X(\mathcal{T}, \varinjlim \mathcal{F}) = 0$.

Suppose now that $\mathscr{T} \in \varinjlim \mathcal{T}$, $\mathscr{F} \in \varinjlim \mathcal{F}$. We can express $\mathscr{T}$ as $\varinjlim_i \mathscr{T}_i$, for some collection (more precisely, directed system) $\{\mathscr{T}_i\}_{i \in I}$ of members of $\mathcal{T}$. As every cocone $\{\mathscr{T}_i \to \mathscr{F}\}_{i \in I}$ is trivial (i.e. all the maps are zero), it follows that $\operatorname{Hom}_X(\mathscr{T}, \mathscr{F}) = 0$. Thus, $\operatorname{Hom}_X(\varinjlim \mathcal{T}, \varinjlim \mathcal{F}) = 0$.

Consider an arbitrary quasi-coherent sheaf $\mathscr{G}$. By Corollary 1.28, $\mathscr{G}$ is a directed union of its coherent subsheaves. Thus, we can write

$$\mathscr{G} = \varinjlim_{i \in I} \mathscr{G}_i ,$$

where $\mathscr{G}_i \subseteq \mathscr{G}$ are coherent and all the morphisms in the directed system $\eta_{ij} : \mathscr{G}_i \to \mathscr{G}_j$ are inclusions. Denote $\tau(\mathscr{G}_i)$ the $(\mathcal{T}, \mathcal{F})-$torsion part of $\mathscr{G}$. That is, we have exact sequences

$$0 \longrightarrow \tau(\mathscr{G}_i) \xrightarrow{\nu_i} \mathscr{G}_i \longrightarrow \mathscr{F}_i \longrightarrow 0, \quad i \in I$$

with $\mathscr{F}_i = \operatorname{Coker} \nu_i \in \mathcal{F}$. The inclusions

$$\eta_{ij} : \mathscr{G}_i \to \mathscr{G}_j$$

induce inclusions

$$\eta'_{ij} : \tau(\mathscr{G}_i) \to \tau(\mathscr{G}_j),$$

and together they induce morphisms on the cokernels

$$\eta''_{ij} : \mathscr{F}_i \to \mathscr{F}_j.$$

In other words, we have a short exact sequence of directed systems

$$0 \longrightarrow \{\tau(\mathscr{G}_i), \eta'_{ij}\} \longrightarrow \{\mathscr{G}_i, \eta_{ij}\} \longrightarrow \{\mathscr{F}_i, \eta''_{ij}\} \longrightarrow 0$$

As direct limits are exact in QCoh_X we have a short exact sequence

$$0 \longrightarrow \varinjlim_i \tau(\mathscr{G}_i) \longrightarrow \mathscr{G} \longrightarrow \varinjlim_i \mathscr{F}_i \longrightarrow 0$$

with $\varinjlim_i \tau(\mathscr{G}_i) \in \varinjlim \mathcal{T}$ and $\varinjlim_i \mathscr{F}_i \in \varinjlim \mathcal{F}$.

This shows $(\varinjlim \mathcal{T}, \varinjlim \mathcal{F})$ is a torsion pair in QCoh_X.

□

Proposition 2.6. *Let X be a Noetherian scheme. Then any hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ is of finite type, i.e.*

$$\mathcal{T} = \varinjlim (\mathcal{T} \cap \mathsf{Coh}_X) \quad \textit{and} \quad \mathcal{F} = \varinjlim (\mathcal{F} \cap \mathsf{Coh}_X).$$

Proof. By Proposition 2.5, $(\varinjlim (\mathcal{T} \cap \mathsf{Coh}_X), \varinjlim (\mathcal{F} \cap \mathsf{Coh}_X))$ is a torsion pair in QCoh_X. If $\mathscr{T} \in \mathcal{T}$ is an arbitrary member of the torison class $\mathcal{T}$, then $\mathscr{T}$ is a direct union of its coherent subsheaves (by Corollary 1.28). Since the torsion pair is hereditary, all these subsheaves are members of $\mathcal{T}$ as well. This shows that $\mathcal{T} \subseteq \varinjlim (\mathcal{T} \cap \mathsf{Coh}_X)$. Similarly, we have that $\mathcal{F} \subseteq \varinjlim (\mathcal{F} \cap \mathsf{Coh}_X)$ ($\mathcal{F}$ is closed under subsheaves since it is a torsion-free class). Application of Lemma 2.2 yields the result. □

The last lemma of this section connects the discussion of hereditary torsion pairs with the structure of injective quasi-coherent sheaves obtained at the end of Chapter 1.

Lemma 2.7. *Let X be a locally Noetherian scheme. A hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ in QCoh_X is determined by the set of indecomposable injective quasi-coherent sheaves contained in $\mathcal{F}$.*

Proof. Any hereditary torsion pair is determined by the set of torsion-free injectives, since the class of torsion-free objects is closed under taking injective hulls. Remark 1.7 shows that the torsion-free class in a Grothendieck category is closed under direct sums since it is closed under products and subobjects. In particular, this holds for QCoh_X. By Theorem 1.32, every injective quasi-coherent sheaf decomposes into a direct sum of indecomposable injectives. Thus, the set of indecomposable injective quasi-coherent sheaves in $\mathcal{F}$ determines the set of all injective quasi-coherent sheaves in $\mathcal{F}$. Consequently, the indecomposable injectives in $\mathcal{F}$ determines the torsion-free class $\mathcal{F}$, since $\mathcal{F}$ consists precisely of all subobjects of injectives in $\mathcal{F}$. □

2.2 Support and associated points of a quasi-coherent sheaf

We now proceed to the topic of associated points and points in support of quasi-coherent sheaves. In what follows in this section, let X be a Noetherian scheme.

Definition 2.8. Let $\mathscr{F}$ be a quasi-coherent sheaf on X.

Define the *support of the sheaf* $\mathscr{F}$ by

$$\operatorname{Supp} \mathscr{F} = \{x \in X \mid \mathscr{F}_x \neq 0\} .$$

We say that a point $x \in X$ is an *associated point of* $\mathscr{F}$ provided that there is a monomorphism of $\mathcal{O}_{X,x}$-modules

$$\kappa(x) \hookrightarrow \mathscr{F}_x ,$$

i.e. there is an (affine) open set $U \subseteq X$ and a section $s \in \mathscr{F}(U)$ such that $\operatorname{Ann}_{\mathcal{O}_{X,x}}(s_x) = \mathfrak{m}_x$. Denote the set of all associated points of $\mathscr{F}$ by $\operatorname{Ass} \mathscr{F}$.

Recall that given a commutative ring R, its prime ideal $\mathfrak{p}$ and an R-module M, we say that $\mathfrak{p}$ is an *associated prime of* M if $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M$. That is, there is an injection of R-modules $R/\mathfrak{p} \hookrightarrow M$ (taking $1+\mathfrak{p}$ to x). Denote the set of all primes associated to M by $\operatorname{Ass} M$. Similarly, define *support of* M, denoted by $\operatorname{Supp} M$, as the set of all primes $\mathfrak{p}$ such that $M_\mathfrak{p} \neq 0$.

The following lemma describes the basic well-known properties of associated primes. The proof can be found e.g. in [Eis95, Sections 3.1 and 3.2].

Lemma 2.9. *Let R be a commutative Noetherian ring, M an R-module and $\mathfrak{p} \subseteq R$ a prime ideal.*

(1) $\mathfrak{p} \in \operatorname{Ass} M$ *if and only if* $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass} M_\mathfrak{p}$.

(2) $\operatorname{Ass} M \subseteq \operatorname{Supp} M$.

(3) $\operatorname{Ass} M = \emptyset$ *if and only if* $M = 0$.

(4) Given any collection M_i, $i \in I$, of R-modules, we have

$$\operatorname{Ass} \bigoplus_{i \in I} M_i = \bigcup_{i \in I} \operatorname{Ass} M_i .$$

(5) *Given a short exact sequence of R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\,,$$

we have that $\operatorname{Ass} B \subseteq \operatorname{Ass} A \cup \operatorname{Ass} C$.

In addition, we prove the following lemma on associated primes of direct limits for later use.

Lemma 2.10. *Let R be a commutative Noetherian ring and M an R-module. If M is a direct limit of a directed system of modules $(M_i \mid i \in I)$, then*

$$\operatorname{Ass} M \subseteq \bigcup_{i \in I} \operatorname{Ass} M_i.$$

Proof. Denote $\nu_i : M_i \to M$ the canonical homomorphisms (coming from the description $M = \varinjlim_i M_i$). Consider $\mathfrak{p} \in \operatorname{Ass} M$. Then we have an injective homomorphism $R/\mathfrak{p} \overset{\iota}{\hookrightarrow} M$. Since R is Noetherian, $\mathfrak{p}$ is a finitely generated module, therefore the module $R/\mathfrak{p}$ is finitely presented. Thus, it follows that ι has a factorization

$$\iota = \nu_i \iota_i,$$

where $i \in I$ is some index and $\iota_i : R/\mathfrak{p} \to M_i$ is a suitable homomorphism (see e.g. [GT12, Lemma 2.8]). Such ι_i is necessarily injective since ι is. It follows that $\mathfrak{p} \in \operatorname{Ass} M_i$. □

Now we use these algebraic facts to prove its algebro-geometric counterparts.

Corollary 2.11. *Let X be a Noetherian scheme, $\mathscr{F}$ a quasi-coherent sheaf on X and $x \in X$ a point.*

(1) *$x \in \operatorname{Ass} \mathscr{F}$ if and only if there exists an affine open neighbourhood U of x such that $\mathfrak{p}_x \in \operatorname{Ass} \mathscr{F}(U)$.*

(1') *$x \in \operatorname{Ass} \mathscr{F}$ if and only if for every affine open neighbourhood U of x, $\mathfrak{p}_x \in \operatorname{Ass} \mathscr{F}(U)$.*

(2) $\operatorname{Ass} \mathscr{F} \subseteq \operatorname{Supp} \mathscr{F}$.

(3) $\operatorname{Ass} \mathscr{F} = \emptyset$ *if and only if* $\mathscr{F} = 0$.

(4) *Given any collection $\mathscr{F}_i$, $i \in I$, of quasi-coherent sheaves on X, we have*

$$\operatorname{Ass} \bigoplus_{i \in I} \mathscr{F}_i = \bigcup_{i \in I} \operatorname{Ass} \mathscr{F}_i.$$

(5) *Given a short exact sequence of quasi-coherent sheaves on X*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0,$$

we have that $\operatorname{Ass} \mathscr{G} \subseteq \operatorname{Ass} \mathscr{F} \cup \operatorname{Ass} \mathscr{H}$.

Proof. If U is an affine open neighbourhood of x, the stalk $\mathscr{F}_x$ may be computed as $(\mathscr{F}(U))_{\mathfrak{p}_x}$ and thus, $x \in \operatorname{Ass}\mathscr{F}$ iff $\mathfrak{m}_x = (\mathfrak{p}_x)_{\mathfrak{p}_x} \in \operatorname{Ass}(\mathscr{F}(U)_{\mathfrak{p}_x})$. Application of Lemma 2.9 (1) thus proves (1) and (1'). The statement (2) is clear from the definition. Statements (3)–(5) follow directly from its algebraic counterparts using the facts that for any $x \in X$, the stalk functor $(-)_x$ is exact (to prove (4)) and preserves direct sums (to prove (5)). □

The presented definition of associated point is "stalk-local" and by Corollary 2.11 (1), (1') it can be considered "affine-local" in a strong sense as well. It is therefore not surprising that if a quasi-coherent sheaf on a scheme is induced from a quasi-coherent sheaf on an affine open subcheme, the set of associated points does not change. Let us make this precise.

Let X be a Noetherian scheme and $U \subseteq X$ an affine open subset. Denote $i : U \hookrightarrow X$ the open immersion. Suppose that $\mathscr{F}$ is a quasi-coherent sheaf on U. By Remark 1.30, the $\mathcal{O}_X$-module $\mathscr{G} = i_*\mathscr{F}$ is again quasi-coherent (as the immersion i is quasi-compact and quasi-separated[2]). That is, $\mathscr{G}$ is a quasi-coherent $\mathcal{O}_X$-module such that for every pair of open sets $W' \subseteq W$,

$$\mathscr{G}(W) \xrightarrow{\operatorname{res}^W_{W'}} \mathscr{G}(W') \quad \text{equals} \quad \mathscr{G}(W \cap U) \xrightarrow{\operatorname{res}^{W\cap U}_{W'\cap U}} \mathscr{G}(W' \cap U)\,.$$

In particular, taking $W' = W \cap U$, we have that

$$\mathscr{G}(W) \xrightarrow{\operatorname{res}^W_{W\cap U}} \mathscr{G}(W \cap U) \quad \text{equals} \quad \mathscr{G}(W \cap U) \xrightarrow{1_{\mathscr{G}(W\cap U)}} \mathscr{G}(W \cap U).$$

Proposition 2.12. *In the situation as above:*

(1) Consider $x \in X \setminus U$. Then $x \notin \operatorname{Ass}\mathscr{G}$.

(2) Suppose $x \in U$. Then $x \in \operatorname{Ass}\mathscr{G}$ if and only if $x \in \operatorname{Ass}\mathscr{F}$.

Proof.

(1) Fix an affine open neighbourhood V of x and without loss of generality, suppose that $V \cap U \neq \emptyset$ (otherwise $\mathscr{G}(V) = 0$, hence $x \notin \operatorname{Ass}\mathscr{G}$). Assume for contradiction that for $s \in \mathscr{G}(V)$, $\operatorname{Ann}_{\mathcal{O}_X(V)}(s) = \mathfrak{p}_x$ (where $\mathfrak{p}_x$ denotes the prime ideal corresponding to x in $\mathcal{O}_X(V)$).

Consider an arbitrary point $y \in V \cap U$. Then there is $f \in \mathcal{O}_X(V)$ such that for $V_y = D_f$ we have $y \in V_y \subseteq V \cap U$. Since $\mathscr{G}$ is quasi-coherent, both the maps

$$\mathscr{G}(V) \xrightarrow{\operatorname{res}^V_{V_y}} \mathscr{G}(V_y)\ \text{, resp. } \mathcal{O}_X(V) \xrightarrow{\operatorname{res}^V_{V_y}} \mathcal{O}_X(V_y)$$

are localisations of $\mathscr{G}(V)$, resp. $\mathcal{O}_X(V)$, with respect to f.

The fact that $x \notin V_y = D_f$ means that $f \in \mathfrak{p}_x$. That is, $f \cdot s = 0$, since $\mathfrak{p}_x = \operatorname{Ann}_{\mathcal{O}_X(V)}(s)$. In particular, we have $(f \restriction_{V_y}) \cdot (s \restriction_{V_y}) = 0$. However, $(f \restriction_{V_y})$ is invertible in $\mathcal{O}_X(V_y)$ and hence, $s \restriction_{V_y} = 0$.

Therefore, there is an open cover $\{V_y \mid y \in V \cap U\}$ of $V \cap U$ such that all the restrictions $s \restriction_{V_y}$ are zero. By gluing axioms, we have that $s \restriction_{V\cap U} = 0$.

[2] Similarly to Section 1.4, this follows from the fact U is a Noetherian scheme from the definitions ([GW10, Definition 1.10 and Definition 10.22]).

However, the restriction map $\mathrm{res}^V_{V\cap U}$ (of the sheaf $\mathscr{G}$) is the identity $1_{\mathscr{G}(V\cap U)}$. Thus, $0 = s \restriction_{U\cap V} = s$, hence $\mathrm{Ann}_{\mathcal{O}_X(V)}(s) = \mathcal{O}_X(V) \neq \mathfrak{p}_x$. This is a contradiction with the choice of s.

(2) This part easily follows from Corollary 2.11 (1), (1'). U is affine open, hence we have $x \in \mathrm{Ass}\,\mathscr{G}$ if and only if $\mathfrak{p}_x \in \mathrm{Ass}\,\mathscr{G}(U)$ and $x \in \mathrm{Ass}\,F$ if and only if $\mathfrak{q}_x \in \mathrm{Ass}\,\mathscr{F}(U)$ (here $\mathfrak{q}_x$ denotes the prime ideal of $\mathcal{O}_{X\restriction U}(U) = \mathcal{O}_X(U)$). But $\mathscr{F}(U) = \mathscr{G}(U)$, and so the claim follows.

□

Given a commutative Noetherian ring R and an R-module M, an associated prime $\mathfrak{p}$ of M can always be "isolated" in a submodule $N \subseteq M$. That is, there is a submodule N with $\mathrm{Ass}\,N = \{\mathfrak{p}\}$. This is obvious, one simply needs to take N to be an isomorphic copy of $R/\mathfrak{p}$ that is embedded into M. It will be useful to generalize this property for Noetherian schemes. For this, we use the following lemma, which is stated without proof.

Lemma 2.13 ([Sta16, Tag 01YE[3]]). *Let X be a Noetherian scheme. Let $i : Z \longrightarrow X$ be the inclusion of an integral, closed subscheme, let ξ be its generic point. Consider a coherent sheaf $\mathscr{F}$ on X such that $\mathscr{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exists an integer $r \geq 0$, a sheaf of ideals $\mathscr{I} \subseteq \mathcal{O}_Z$ and an injective map of coherent sheaves*

$$i_*(\mathscr{I}^{\oplus r}) \longrightarrow \mathscr{F}$$

which is an isomorphism in a neighbourhood[4] of ξ.

Let us now proceed to the proof of the claimed assertion.

Lemma 2.14. *Let X be a Noetherian scheme, $x \in X$ and $\mathscr{F}$ a quasi-coherent sheaf with $x \in \mathrm{Ass}\,\mathscr{F}$. Then there is a coherent subsheaf $\mathscr{G} \subseteq \mathscr{F}$ such that $\mathrm{Ass}\,\mathscr{G} = \{x\}$. Moreover, there is an affine cover $V_1, \ldots, V_k$ of X such that any section $s \in \mathscr{G}(V_i)$ is annihilated by $\mathfrak{p}_x$, i.e. $\mathfrak{p}_x \subseteq \mathrm{Ann}(s)$.*

Proof. By the assumptions, there is an affine neighbourhood $U \subseteq X$ of x and a section $g \in \mathscr{F}(U)$ with $\mathrm{Ann}_{\mathcal{O}_X(U)}(g) = \mathfrak{p}_x$. By Theorem 1.27 (2), there is a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{F}$ (on X) extending the coherent sheaf $\widetilde{\langle g\rangle}$ (on U). Thus,

$$\mathscr{F}'_x \simeq \left(\widetilde{\langle g\rangle}\right)_x \simeq (\mathcal{O}_X(U)/\mathfrak{p}_x)_{\mathfrak{p}_x} \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

in particular, $\mathscr{F}'_x$ is nonzero and is annihilated by $\mathfrak{m}_x$.

Let Z be the integral closed subscheme of X with generic point x (i.e. whose set of points is $\overline{\{x\}}$)[5], and denote by i the embedding of schemes $Z \hookrightarrow X$. By Lemma 2.13, there is a monomorphism

$$h : i_*(\mathscr{I}^{\oplus r}) \longrightarrow \mathscr{F}',$$

[3]As of May 2nd 2016, the Lemma under the Tag 01YE was listed on the Stacks Project site [Sta16] as Lemma 29.12.2.

[4]A morphism of quasi-coherent sheaves $f : \mathscr{F} \to \mathscr{G}$ is an *isomorphism in an open set* U if $f\restriction_U : \mathscr{F}_{\restriction U} \to \mathscr{G}_{\restriction U}$ is an isomorphism of quasi-coherent sheaves on U.

[5]This exists and is unique, cf. [GW10, Proposition 3.50].

which is an isomorphism at a neighbourhood of x, where $\mathscr{I}$ is a sheaf of ideals on Z. Clearly h induces the isomorphism of stalks

$$h_x : (i_*(\mathscr{I}^{\oplus r}))_x \xrightarrow{\sim} \mathscr{F}'_x.$$

In particular, $\mathscr{G}_0 = \operatorname{Im} h$ is a (nonzero) coherent subsheaf of $\mathscr{F}$ with $x \in \operatorname{Ass} \mathscr{G}_0$ and $\operatorname{Supp} \mathscr{G}_0 = \overline{\{x\}}$. What remains is to find a coherent subsheaf of $\mathscr{G}_0$ whose only associated point in $\overline{\{x\}}$ is x.

For $y \in \overline{\{x\}}$, choose an affine neighbourhood V_y (then $x \in V_y$ since $y \in \overline{\{x\}}$). Since $\overline{\{x\}}$ is quasi-compact[6], there is a collection of finitely many points $y_1, \ldots, y_k$ such that

$$\overline{\{x\}} \subseteq \bigcup_{j=1}^{k} V_{y_j}.$$

Inductively, when $\mathscr{G}_j$ is defined, choose a quasi-coherent sheaf $\mathscr{G}_{j+1} \subseteq \mathscr{G}_j$ such that $\mathscr{G}_{j+1}(V_{y_{j+1}}) \simeq \mathcal{O}_X(V_{y_{j+1}})/\mathfrak{p}_x$, which is possible by Theorem 1.27 (2) similarly as above. We obtain a chain of coherent subsheaves of $\mathscr{F}$

$$\mathscr{G} := \mathscr{G}_k \subseteq \mathscr{G}_{k-1} \subseteq \mathscr{G}_{k-2} \subseteq \cdots \subseteq \mathscr{G}_1 \subseteq \mathscr{G}_0.$$

By the construction, it is obvious that $\operatorname{Ass} \mathscr{G}_j \cap V_{y_j} = \{x\}$ and thus, we clearly have

$$\operatorname{Ass} \mathscr{G} \cap \bigcup_{j=1}^{k} V_{y_j} = \{x\}.$$

That is, $\operatorname{Ass} \mathscr{G} = \{x\}$.

The additional condition follows easily for the affine open cover $V_i := V_{y_i}$ supplemented by an arbitrary affine open cover $\{U_j\}_j$ of $X \setminus \overline{\{x\}}$ (since $\operatorname{Supp} \mathscr{G} = \overline{\{x\}}$, it is easy to see that $\mathscr{G}(U_j) = 0$, so these extra open sets do not spoil the condition). ☐

Remark 2.15. In general, there is, in contrast to the affine case, no single coherent sheaf $\mathscr{G}^x$ such that quasi-coherent sheaves $\mathscr{F}$ with $x \in \operatorname{Ass} \mathscr{F}$ are characterized by the existence of a monomorphism $\mathscr{G}^x \hookrightarrow \mathscr{G}$ (in the affine case, the module $R/\mathfrak{p}$ plays such a role for associated prime $\mathfrak{p}$).

To see this, let ξ be the generic point of a projective line $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. Then it easily follows from Lemma 2.14 that any quasi-coherent sheaf $\mathscr{F}$ with $\xi \in \operatorname{Ass} \mathscr{G}$ contains some line bundle $\mathscr{F}$. In fact, any line bundle $\mathscr{L}$ satisfies $\operatorname{Ass} \mathscr{L} = \{\xi\}$ and it is not difficult to observe that such a testing sheaf $\mathscr{G}^\xi$ needs to be a line bundle[7].

However, there is no single line bundle that would embed into arbitrary line bundle, since $\operatorname{Hom}_X(\mathcal{O}(m), \mathcal{O}(n)) = 0$ if $n < m$.

Lemma 2.14 has the following consequence regarding the associated points of injective hulls.

[6] Cf. Remark B.8.

[7] This is because of the structure of coherent sheaves over $\mathbb{P}^1_k$ as described in Example 1 together with the fact that the torsion sheaf $\mathcal{O}_{x,r}$ is supported only at the closed point x, see [CK09, Chapter 5].

Corollary 2.16. *Let X be a Noetherian scheme and $\mathscr{F} \in \mathsf{QCoh}_X$. Denote $E(\mathscr{F})$ the injective hull of $\mathscr{F}$. Then* $\operatorname{Ass}\mathscr{F} = \operatorname{Ass} E(\mathscr{F})$.

Proof. Obviously, $\operatorname{Ass}\mathscr{F} \subseteq \operatorname{Ass} E(\mathscr{F})$ as $\mathscr{F}$ is a subsheaf of $E(\mathscr{F})$.

Suppose for contradiction that there is a point $x \in \operatorname{Ass} E(\mathscr{F}) \setminus \operatorname{Ass}\mathscr{F}$. By Lemma 2.14, there is a nonzero coherent subsheaf $\mathscr{G} \subseteq E(\mathscr{F})$ with $\operatorname{Ass}\mathscr{G} = \{x\}$. However, $\operatorname{Ass}(\mathscr{F} \cap \mathscr{G}) \subseteq (\operatorname{Ass}\mathscr{F}) \cap \{x\} = \emptyset$. Thus, $\mathscr{F} \cap \mathscr{G} = 0$, which contradicts the essentiality of the inclusion $\mathscr{F} \subseteq E(\mathscr{F})$. □

Let us now compute the associated points of the indecomposable injectives $\mathscr{J}(x)$ from Section 1.4.

Lemma 2.17. *For a point $x \in X$,* $\operatorname{Ass}\mathscr{J}(x) = \{x\}$.

Proof. First let us show that there are no associated points of $\mathscr{J}(x)$ outside $i(\operatorname{Spec}\mathcal{O}_{X,x})$. By the discussion in Section 1.4, we have

$$i(\operatorname{Spec}\mathcal{O}_{X,x}) = \bigcap\{U \subseteq X \mid x \in U,\, U \text{ is affine open}\}.$$

If $y \notin i(\operatorname{Spec}\mathcal{O}_{X,x})$ is an arbitrary point, then there is an affine neighbourhood U of x such that $y \notin U$. Denote the inclusion $U \to X$ by j and the inclusion $\operatorname{Spec}\mathcal{O}_{X,x} \to U$ by i'. Then clearly $i_* = j_* \circ i'_*$. However, by Proposition 2.12, $y \notin \operatorname{Ass} j_*\left(i'_*\left(\widetilde{E_x}\right)\right)$, so $y \notin \operatorname{Ass} i_*\left(\widetilde{E_x}\right) = \operatorname{Ass}\mathscr{J}(x)$.

Thus, $\operatorname{Ass}\mathscr{J}(x) \subseteq i(\operatorname{Spec}\mathcal{O}_{X,x})$. However, since the fact whether a point $z \in i(\operatorname{Spec}\mathcal{O}_{X,x})$ is associated to $\mathscr{J}(x)$ or not depends on the stalks $\mathscr{J}(x)_z$, $\mathcal{O}_{X,z} (= \mathcal{O}_{(\operatorname{Spec}\mathcal{O}_{X,x}),z})$ (the latter understood as a ring) only, we see that $\operatorname{Ass}\mathscr{J}(x) = i\left(\operatorname{Ass}\widetilde{E_x}\right) = \{x\}$. □

Remark 2.18. Alternatively, one can prove Lemma 2.17 without computations at stalks, using only the fact that $\operatorname{Supp}\mathscr{J}(x) \subseteq \overline{\{x\}}$. The argument is as follows.

First, observe that every point x is an associated point of a quasi-coherent sheaf. This can be done e.g. by considering the sheaf $i_*(\widetilde{\mathcal{O}_X(U)/\mathfrak{p}_x})$, where $U \subseteq X$ is an affine open neighbourhood of x and $i : U \hookrightarrow X$ the corresponding open immersion, and applying Proposition 2.12. From Lemma 2.14 it follows that for every point x, there is a quasi-coherent sheaf $\mathscr{G}$ with $\operatorname{Ass}\mathscr{G} = \{x\}$, and Corollary 2.16 implies that these $\mathscr{G}$'s can be taken injective. However, since for injective $\mathscr{G}$ we have

$$\mathscr{G} \simeq \bigoplus_{j \in J} \mathscr{J}(x_j),$$

by Corollary 2.11 (4) it follows that for every $x \in X$, there is a point $y \in X$ with $\operatorname{Ass}\mathscr{J}(y) = \{x\}$.

Now we use a version of Noetherian induction. Suppose that x_0 is a counterexample to the claim $\operatorname{Ass}\mathscr{J}(x) = \{x\}$. Then by the above considerations, there is a point x_1 such that $\mathscr{J}(x_1) = \{x_0\}$. Since $\operatorname{Supp}\mathscr{J}(x_1) \subseteq \overline{\{x_1\}}$ by the computation in the above proof of Lemma 2.17, it follows that x_1 is a generization of x_0 which is proper in the sense that $x_1 \neq x_0$. In particular, $\operatorname{Ass}\mathscr{J}(x_1) \neq \{x_1\}$, hence the argument can be repeated to obtain a proper generization x_2 of x_1 such that $\operatorname{Ass}\mathscr{J}(x_2) \neq \{x_2\}$. Continuing in this manner, we obtain a sequence of points

$$x_0,\ x_1,\ x_2,\ \cdots,\ x_k,\ x_{k+1},\ \cdots$$

such that each x_{k+1} is a proper generization of x_k. This is a contradiction to the Noetherian hypothesis, since by choosing affine open neighbourhood U of x_0, in the Noetherian ring $\mathcal{O}_X(U)$ there is an infinite descending chain of primes

$$\mathfrak{p}_{x_0} \supsetneq \mathfrak{p}_{x_1} \supsetneq \mathfrak{p}_{x_2} \supsetneq \cdots \supsetneq \mathfrak{p}_{x_k} \supsetneq \mathfrak{p}_{x_{k+1}} \supsetneq \cdots,$$

which is not possible.

2.3 Parametrization of hereditary torsion pairs in QCoh_X

We are now ready to proceed to the main part of the chapter. We classify the hereditary torsion pairs in QCoh_X for X Noetherian in terms of associated points and supports of quasi-coherent sheaves. To recognize the relevant subsets of X for this goal, the following definition is needed.

Definition 2.19. Let X be a topological space. A subset $Y \subseteq X$ is *specialization closed* if $\overline{\{y\}} \subseteq Y$ for every $y \in Y$.

Alternatively, a set is specialization closed if it is a union of closed subsets. We start the classification by showing that to every specialization closed subset Y, a hereditary torsion pair $(\mathcal{T}(Y), \mathcal{F}(Y))$ can be assigned.

Proposition 2.20. *Let $Y \subseteq X$ be a specialization closed subset. Define*

$$\begin{aligned}\mathcal{T}(Y) =& \{\mathscr{T} \in \mathsf{QCoh}_X \mid \operatorname{Supp} \mathscr{T} \subseteq Y\},\\ \mathcal{F}(Y) =& \{\mathscr{F} \in \mathsf{QCoh}_X \mid \operatorname{Ass} \mathscr{F} \cap Y = \emptyset\}.\end{aligned}$$

Then the pair $(\mathcal{T}(Y), \mathcal{F}(Y))$ is a hereditary torsion pair in QCoh_X.

Proof. It is easy to observe that the class $\mathcal{T}(Y)$ is closed under arbitrary direct sums, since, by Lemma 2.11 (4), for any collection $\{\mathscr{G}_i \mid i \in I\}$ of quasi-coherent sheaves we have that

$$\operatorname{Supp} \bigoplus_{i \in I} \mathscr{G}_i = \bigcup_{i \in I} \operatorname{Supp} \mathscr{G}_i.$$

Similarly, whenever

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

is a short exact sequence of quasi-coherent sheaves, by Lemma 2.11 (5) we have that

$$\operatorname{Supp} \mathscr{G} \subseteq \operatorname{Supp} \mathscr{G}' \cup \operatorname{Supp} \mathscr{G}''.$$

This shows that $\mathcal{T}(Y)$ is closed under (quasi-coherent) subsheaves, homomorphic images and extensions. Thus, $\mathcal{T}(Y)$ is a hereditary torsion class in QCoh_X.

Next we show that $\operatorname{Hom}_X(\mathcal{T}(Y), \mathcal{F}(Y)) = 0$. Consider a morphism of quasi-coherent sheaves $f : \mathscr{T} \longrightarrow \mathscr{F}$ with $\mathscr{T} \in \mathcal{T}(Y)$, $\mathscr{F} \in \mathcal{F}(Y)$. Then $\operatorname{Im} f \in \mathcal{T}(Y)$, as established above. In particular, $\operatorname{Ass} \operatorname{Im} f \subseteq \operatorname{Supp} \operatorname{Im} f \subseteq Y$. On the other hand, $\operatorname{Ass} \operatorname{Im} f \subseteq \operatorname{Ass} \mathscr{F} \subseteq X \setminus Y$, hence $\operatorname{Ass} \operatorname{Im} f = \emptyset$. It follows that $\operatorname{Im} f = 0$ by Lemma 2.11 (3). Thus, $f = 0$.

What remains is to show that if $\mathscr{G}$ is a quasi-coherent sheaf on X such that $\operatorname{Hom}_X(\mathscr{T}, \mathscr{G}) = 0$ for all $\mathscr{T} \in \mathcal{T}(Y)$, then $\mathscr{G} \in \mathcal{F}(Y)$. Equivalently, whenever $\mathscr{G}$ is a quasi-coherent sheaf with $\operatorname{Ass} \mathscr{G} \cap Y \neq \emptyset$, there is a quasi-coherent sheaf $\mathscr{T}$ with $\operatorname{Supp} \mathscr{T} \subseteq Y$, and a nonzero morphism $\mathscr{T} \longrightarrow \mathscr{G}$.

Suppose $x \in \operatorname{Ass} \mathscr{G} \cap Y$. By Lemma 2.14, there is a coherent subsheaf $\mathscr{G}' \subseteq \mathscr{G}$ with $\operatorname{Supp} \mathscr{G}' \subseteq \overline{\{x\}}$ and $\operatorname{Ass} \mathscr{G}' = \{x\}$. In particular, $\mathscr{G}' \in \mathcal{T}(Y)$ and $\mathscr{G}'$ is nonzero.

Thus, there is a nonzero morphism

$$h : \mathscr{G}' \overset{\subseteq}{\hookrightarrow} \mathscr{G}$$

with $\mathscr{G}' \in \mathcal{T}(Y)$, which proves the claim. □

Finally, we prove the classification of hereditary torsion pairs in QCoh_X for a Noetherian scheme X.

Theorem 2.21. *There is a bijective correspondence between hereditary torsion pairs in* QCoh_X *and specialization closed subsets of* X*, given by the mutually inverse bijections*

$$(\mathcal{T}, \mathcal{F}) \mapsto \operatorname{Supp} \mathcal{T} (= \{x \in X \mid \exists \mathscr{T} \in \mathcal{T} : \mathscr{T}_x \neq 0\})$$

and

$$Y \mapsto (\mathcal{T}(Y), \mathcal{F}(Y)).$$

Proof. Clearly for any hereditary torsion pair $(\mathcal{T}, \mathcal{F})$, the set $\operatorname{Supp} \mathcal{T}$ is specialization closed (if $\mathscr{G}$ is a quasi-coherent sheaf with $\mathscr{G}_x \neq 0$ and $y \in \overline{\{x\}}$, then $\mathscr{G}_x = \mathscr{G}_y \otimes_{\mathcal{O}_{X,y}} \mathcal{O}_{X,x}$, in particular, $\mathscr{G}_y \neq 0$). Together with Proposition 2.20, this shows that both the maps in the statement are well-defined.

Suppose that $Y \subseteq X$ is a specialization closed subset. Clearly $\operatorname{Supp} \mathcal{T}(Y) \subseteq Y$ from the definitions. On the other hand, suppose that $x \in Y$ is an arbitrary point. Consider a closed subscheme $i : Z \hookrightarrow X$ whose underlying topological space is $\overline{\{x\}}$. Then $\mathscr{G} = i_*(\mathcal{O}_Z)$ is a quasi-coherent sheaf on X (by [GW10, Corollary 10.27]) with $\operatorname{Supp} \mathscr{G} = \overline{\{x\}} \subseteq Y$. Thus, $\mathscr{G} \in \mathcal{T}(Y)$ and $x \in \operatorname{Supp} \mathcal{T}(Y)$. This proves that $\operatorname{Supp} \mathcal{T}(Y) = Y$.

What remains is to prove that given a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$, we have $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}(\operatorname{Supp} \mathcal{T}), \mathcal{F}(\operatorname{Supp} \mathcal{T}))$. Clearly we have $\mathcal{T} \subseteq \mathcal{T}(\operatorname{Supp} \mathcal{T})$, or, equivalently, $\mathcal{F} \supseteq \mathcal{F}(\operatorname{Supp} \mathcal{T})$. It remains to prove the other inclusion, that is, to show that $\mathcal{F} \subseteq \mathcal{F}(\operatorname{Supp} \mathcal{T})$. Suppose not, then we have

$$\mathcal{F} \supsetneq \mathcal{F}(\operatorname{Supp} \mathcal{T}).$$

As both $\mathcal{F}, \mathcal{F}(\operatorname{Supp} \mathcal{T})$ are hereditary torsion-free classes in QCoh_X, it follows from Lemma 2.7 that there exists an indecomposable injective quasi-coherent sheaf $\mathscr{J}(x) \in \mathcal{F} \setminus \mathcal{F}(\operatorname{Supp} \mathcal{T})$, in particular, $\operatorname{Ass} \mathscr{J}(x) \cap \operatorname{Supp} \mathcal{T} \neq \emptyset$. Since $\operatorname{Ass} \mathscr{J}(x) = \{x\}$ by Lemma 2.17, it follows that $x \in \operatorname{Supp} \mathcal{T}$, so one can choose a quasi-coherent sheaf $\mathscr{T} \in \mathcal{T}$ with $\mathscr{T}_x \neq 0$. Our aim is to prove that $\operatorname{Hom}_X(\mathscr{T}, \mathscr{J}(x)) \neq 0$, to get a contradiction with $\mathscr{J}(x) \in \mathcal{F}$. To this end, it is enough to prove that $\operatorname{Hom}_{\operatorname{Spec} \mathcal{O}_{X,x}}(i^*(\mathscr{T}), \widetilde{E_x}) \neq 0$, since

$$\operatorname{Hom}_X(\mathscr{T}, \mathscr{J}(x)) = \operatorname{Hom}_X\left(\mathscr{T}, i_*\left(\widetilde{E_x}\right)\right) \simeq \operatorname{Hom}_{\operatorname{Spec} \mathcal{O}_{X,x}}\left(i^*(\mathscr{T}), \widetilde{E_x}\right).$$

Thus, what remains is to prove that $i^*(\mathscr{T}) \neq 0$, since $\widetilde{E_x}$ is an injective cogenerator for $\mathsf{QCoh}_{\operatorname{Spec} \mathcal{O}_{X,x}}$. This follows from [GW10, p. 180], since

$$i^*(\mathscr{T})_x \simeq \mathcal{O}_{(\operatorname{Spec} \mathcal{O}_{X,x}),x} \otimes_{\mathcal{O}_{X,x}} \mathscr{T}_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathscr{T}_x = \mathscr{T}_x \neq 0.$$

□

Consequently, we are able to prove that all hereditary torsion-free classes in QCoh_X are closed under direct limits.

Lemma 2.22. *Let X be a Noetherian scheme and $Y \subseteq X$ a specialization closed subset. The class $\mathcal{F}(Y)$ is closed under direct limits.*

Proof.
Consider $\mathcal{S}$ to be the class of all coherent $\mathcal{O}_X$-modules $\mathscr{G}$ such that $\operatorname{Ass}\mathscr{G} = \{y\}$ for some $y \in Y$. Then it follows from Lemma 2.14 that

$$\mathcal{F}(Y) = \operatorname{Ker}\operatorname{Hom}_X(\mathcal{S}, -) = \bigcap_{\mathscr{G}\in\mathcal{S}} \operatorname{Ker}\operatorname{Hom}_X(\mathscr{G}, -).$$

Indeed, $\mathcal{F}(Y) \subseteq \operatorname{Ker}\operatorname{Hom}_X(\mathcal{S}, -)$ by the fact that $\mathcal{S} \subseteq \mathcal{T}(Y)$. On the other hand, if $\mathscr{F} \notin \mathcal{F}(Y)$, then there is a point $y \in Y \cap \operatorname{Ass}\mathscr{F}$. Then by Lemma 2.14, there is a $\mathscr{G} \in \mathcal{S}$ such that $\mathscr{G} \subseteq \mathscr{F}$, in particular, $\operatorname{Hom}_X(\mathscr{G}, \mathscr{F}) \neq 0$.

However, since X is Noetherian, every coherent sheaf on X is finitely presented object of QCoh_X, i.e. given a direct limit

$$\mathscr{F} = \varinjlim_i \mathscr{F}_i$$

of quasi-coherent sheaves and a coherent sheaf $\mathscr{G}$, the canonical homomorphism

$$\varinjlim_i \operatorname{Hom}_X(\mathscr{G}, \mathscr{F}_i) \longrightarrow \operatorname{Hom}_X(\mathscr{G}, \varinjlim_i \mathscr{F}_i)$$

is an isomorphism, it follows that the class $\operatorname{Ker}\operatorname{Hom}_X(\mathcal{S}, -)$ is closed under direct limits. □

3. 1-cotilting sheaves over a Noetherian scheme

3.1 Ext in Grothendieck categories

With the goal of studying the Ext-orthogonality relation of quasi-coherent sheaves on a scheme X in mind, we begin this chapter by several remarks on the Ext functor in Grothendieck categories.

Let $\mathcal{A}$ be a Grothendieck category. As in any Abelian category, the notion of the Yoneda Ext groups $\mathrm{YExt}^i_{\mathcal{A}}(A, B),\ A, B \in \mathcal{A}$ makes sense and is functorial both in A and in B.

Since $\mathcal{A}$ has enough injectives, we may also consider, for $A \in \mathcal{A}$, the right derived functor $\mathrm{Ext}^i_{\mathcal{A}}(A, -) := \mathbb{R}^i\mathrm{Hom}_{\mathcal{A}}(A, -)$.

Remark 3.1. Given quasi-coherent sheaves $\mathscr{F}, \mathscr{G}$ on a scheme X, we may consider either

$$\mathrm{YExt}^i_{\mathsf{Mod}\text{–}\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \simeq \mathbb{R}^i_{\mathsf{Mod}\text{–}\mathcal{O}_X}\mathrm{Hom}_{\mathsf{Mod}\text{–}\mathcal{O}_X}(\mathscr{F}, -)(\mathscr{G}), \quad \text{or}$$
$$\mathrm{YExt}^i_{\mathsf{QCoh}_X}(\mathscr{F}, \mathscr{G}) \simeq \mathbb{R}^i_{\mathsf{QCoh}_X}\mathrm{Hom}_{\mathsf{QCoh}_X}(\mathscr{F}, -)(\mathscr{G}).$$

We take the second one as our definition of Ext.

Note, however, that if X is locally Noetherian, $\mathbb{R}^i\mathrm{Hom}_{\mathsf{QCoh}_X}(\mathscr{F}, -)$ is just the restriction of the functor $\mathbb{R}^i\mathrm{Hom}_{\mathsf{Mod}\text{–}\mathcal{O}_X}(\mathscr{F}, -)$ to the category of quasi-coherent sheaves. This is because $\mathrm{Hom}_{\mathsf{QCoh}_X}$ is just a restriction of $\mathrm{Hom}_{\mathsf{Mod}\text{–}\mathcal{O}_X}$ to QCoh_X, and the injective objects in QCoh_X are just injective $\mathcal{O}_X$-modules which happen to be quasi-coherent; thus, for quasi-coherent sheaves $\mathscr{F}, \mathscr{G}$, both $\mathbb{R}^i\mathrm{Hom}_{\mathsf{Mod}\text{–}\mathcal{O}_X}(\mathscr{F}, -)(\mathscr{G})$ and $\mathbb{R}^i\mathrm{Hom}_{\mathsf{Mod}\text{–}\mathcal{O}_X}(\mathscr{F}, -)(\mathscr{G})$ may be computed using the same injective coresolution of $\mathscr{G}$.

Since we consider primarily Noetherian schemes, we do not need to distinguish between the above possible definitions.

For the rest of this section, let us assume that $\mathcal{A}$ is a Grothendieck category.

The fact that Ext exists as a right derived covariant Hom functor implies that we have the following long exact sequence for Ext. This is a standard fact and the proof (using the Horseshoe Lemma) is very well-known. We refer the reader e.g. to [EJ11, Theorem 8.2.5 (1)] for proof.

Proposition 3.2. *Consider a short exact sequence*

$$(\xi) \qquad 0 \longrightarrow B' \xrightarrow{\ \iota\ } B \xrightarrow{\ \pi\ } B'' \longrightarrow 0$$

in $\mathcal{A}$, and an object $A \in \mathcal{A}$. Then there is a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, B') \xrightarrow{\iota\circ-} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\pi\circ-} \mathrm{Hom}_{\mathcal{A}}(A, B'') \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, B') \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, B'') \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(A, B') \longrightarrow \cdots$$

Since a Grothendieck category does not have enough projectives in general, we cannot use a dual argument to prove the existence of the second (i.e. "contravariant") well-known long exact sequence for Ext. Despite that, it is well-known that the result still holds.

Proposition 3.3 ([Har77, III.6.4])**.** *Consider a short exact sequence*

$$(\xi) \qquad 0 \longrightarrow A' \xrightarrow{\iota} A \xrightarrow{\pi} A'' \longrightarrow 0$$

in $\mathcal{A}$, and an object $B \in \mathcal{A}$. Then there is a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A'',B) \xrightarrow{-\circ\pi} \mathrm{Hom}_{\mathcal{A}}(A,B) \xrightarrow{-\circ\iota} \mathrm{Hom}_{\mathcal{A}}(A',B) \\ \hookrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A'',B) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A',B) \\ \hookrightarrow \mathrm{Ext}^2_{\mathcal{A}}(A'',B) \longrightarrow \cdots \end{array}$$

Proof. Applying the bifunctor $\mathrm{Hom}_{\mathcal{A}}(-,-)$ on an injective coresolution of B

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^1 \longrightarrow \cdots$$

and the short exact sequence (ξ) give rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A'',I^0) & \xrightarrow{-\circ\pi} & \mathrm{Hom}_{\mathcal{A}}(A,I^0) & \xrightarrow{-\circ\iota} & \mathrm{Hom}_{\mathcal{A}}(A',I^0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A'',I^1) & \xrightarrow{-\circ\pi} & \mathrm{Hom}_{\mathcal{A}}(A,I^1) & \xrightarrow{-\circ\iota} & \mathrm{Hom}_{\mathcal{A}}(A',I^1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A'',I^2) & \xrightarrow{-\circ\pi} & \mathrm{Hom}_{\mathcal{A}}(A,I^2) & \xrightarrow{-\circ\iota} & \mathrm{Hom}_{\mathcal{A}}(A',I^2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where the rows are exact (they are induced by $\mathrm{Hom}_{\mathcal{A}}(-,I^j)$, which is an exact functor since I^j is injective).

That is, we have a short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A'',I^\bullet) \xrightarrow{-\circ\pi} \mathrm{Hom}_{\mathcal{A}}(A,I^\bullet) \xrightarrow{-\circ\iota} \mathrm{Hom}_{\mathcal{A}}(A',I^\bullet) \longrightarrow 0$$

Now taking the long exact sequence in cohomology of chain complexes gives the exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Ext}^0_{\mathcal{A}}(A'', B) & \longrightarrow & \mathrm{Ext}^0_{\mathcal{A}}(A, B) & \longrightarrow & \mathrm{Ext}^0_{\mathcal{A}}(A', B) & \\ \hookrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(A'', B) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(A, B) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(A', B) & \\ \hookrightarrow & \mathrm{Ext}^2_{\mathcal{A}}(A'', B) & \longrightarrow & \cdots & & & \end{array}$$

and since $\mathrm{Hom}_{\mathcal{A}}(A'', -)$ is naturally isomorphic to $\mathrm{Ext}^0_{\mathcal{A}}(A'', -)$ (and the same for $A,\ A''$), the result follows.

□

As stated in Remark 1.29, another problematic feature of general Grothendieck categories is that the direct product functors are not exact, in general. This means, in particular, that we cannot expect to have the isomorphism

$$\mathrm{Ext}^j_{\mathcal{A}}\Big(A, \prod_{i\in I} B_i\Big) \simeq \prod_{i\in I} \mathrm{Ext}^j_{\mathcal{A}}(A, B_i)\ ,$$

which is known to exist in the case of modules (or, at least, we cannot obtain it by the same methods as in $\mathsf{Mod}\text{–}R$). The obstacle, in contrast to the categories of modules, is that if one takes an injective coresolution of B_i for every $i \in I$, their product does not need to be exact. Thus, one cannot use the resulting complex as an injective coresolution of $\prod_{i\in I} B_i$.

We are, however, able to prove the following weaker statement, sufficient for our purposes.

Proposition 3.4. *For any object $A \in \mathcal{A}$ and any collection of objects $B_i \in \mathcal{A}$, $i \in I$, we have that*

$$\mathrm{Ext}^1_{\mathcal{A}}\Big(A, \prod_{i\in I} B_i\Big) = 0 \textit{ if and only if } \forall i \in I:\ \mathrm{Ext}^1_{\mathcal{A}}\Big(A, B_i\Big) = 0\ .$$

In order to prove this, we need the following lemma. Although it is a standard lemma when working in the category $\mathsf{Mod}\text{–}R$ for a ring R, it is especially useful in our context, i.e. for the category QCoh_X (or a Grothendieck category with products that are not exact, in general), as is demonstrated by the consequent proof of Proposition 3.4.

Lemma 3.5. *For a pair of object A, B of $\mathcal{A}$, the following conditions are equivalent:*

(1) $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$.

(2) Every extension of A by B splits.

(3) Whenever there is a short exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow A \longrightarrow 0\ ,$$

any morphism $f : K \to B$ can be extended to C, so that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \tilde{f}} & & & & & \\ & & B & & & & & & . \end{array}$$

Proof. The equivalence of (1) and (2) is just a consequence of the fact that the Yoneda Ext $\mathrm{YExt}^1_{\mathcal{A}}(A, B)$ is isomorpic to $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$.

Suppose (1) and consider a short exact sequence

$$0 \longrightarrow K \xrightarrow{\ \iota\ } C \longrightarrow A \longrightarrow 0\ .$$

Applying $\mathrm{Hom}_{\mathcal{A}}(-, B)$ produces an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(C, B) \xrightarrow{-\circ\iota} \mathrm{Hom}_{\mathcal{A}}(K, B) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0\ ,$$

hence the above map $- \circ \iota : \tilde{f} \mapsto \tilde{f} \circ \iota$ is surjective. This is clearly equivalent to (3).

Conversely, suppose that (3) holds. Given an extension

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0\ ,$$

we obtain a commutative diagram (by putting $f = 1_B$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0\ , \\ & & \| & \swarrow{\scriptstyle r} & & & & & \\ & & B & & & & & & \end{array}$$

so r is clearly the splitting map of the extension. Thus, (2) holds. □

Proof of Proposition 3.4. The left-to-right implication is clear: If there is $j \in I$ such that $\mathrm{Ext}^1_{\mathcal{A}}(A, B_j) \neq 0$, then we have

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{A}}\Big(A, \prod_{i\in I} B_i\Big) &= \mathrm{Ext}^1_{\mathcal{A}}\Big(A, \Big(\prod_{i\in I\setminus\{j\}} B_i\Big) \oplus B_j\Big) \\ &= \mathrm{Ext}^1_{\mathcal{A}}\Big(A, \prod_{i\in I\setminus\{j\}} B_i\Big) \oplus \mathrm{Ext}^1_{\mathcal{A}}\Big(A, B_j\Big) \neq 0, \end{aligned}$$

hence $\mathrm{Ext}^1_{\mathcal{A}}\big(A, \prod_{i\in I} B_i\big) \neq 0$.

Conversely, assume that for every i, $\mathrm{Ext}^1_{\mathcal{A}}(A, B_i) = 0$. We will verify the extension property (3) from Lemma 3.5 for A and $\prod_{i\in I} B_i$.

Consider a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow K \xrightarrow{\ \iota\ } C \longrightarrow A \longrightarrow 0$$

and a morphism $f : K \to \prod_{i\in I} B_i$.

Fix $i \in I$ and put $f_i := \pi_i f$, where π_i denotes the canonical projection $\prod_{j\in I} B_j \to B_i$. Since $\operatorname{Ext}^i_{\mathcal{A}}(A, B_i) = 0$, from Lemma 3.5 (3) it follows that there is a morphism g_i fitting into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\iota} & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_i} & \swarrow{\scriptstyle g_i} & & & & & \\ & & B_i & & & & & & . \end{array}$$

By the universal property of product, the collection $g_i : C \to B_i$ can be lifted to a morphism $g : C \to \prod_{i\in I} B_i$, i.e. so that $\pi_i g = g_i$ for every i.

It follows that we have the desired commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\iota} & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & & & & & \\ & & \prod_{i\in I} B_i & & & & & & \end{array}$$

by checking that $\pi_i g \iota = g_i \iota = f_i = \pi_i f$ for every $i \in I$ and using (the uniqueness part of) the universal property of product once again. $\square$

3.2 1-cotilting in Grothendieck categories

The goal of this section is to compare the definition of 1-cotilting object in a Grothendieck category (Definition 1.14) with the definition of 1-cotilting modules, more precisely, the direct analogues of axioms (C1)–(C3) in a Grothendieck category.

We start our discussion by showing that a 1-cotilting object of a Grothendieck category $\mathcal{A}$ always satisfies the axioms (C1)–(C3).

Proposition 3.6. *Let C be a 1-cotilting object in $\mathcal{A}$. Then* $\operatorname{injdim} C \leq 1$.

Proof. Let F be an object of $\mathcal{A}$. We show that $\operatorname{Ext}^2_{\mathcal{A}}(F, C) = 0$ by showing that every 2-fold extension of F by C represents the trivial class of $\operatorname{YExt}^2_{\mathcal{A}}(F, C)$.

Consider a 2-fold extension

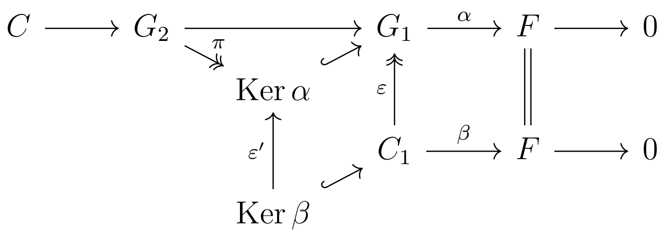
$$(\xi) \qquad 0 \longrightarrow C \longrightarrow G_2 \longrightarrow G_1 \xrightarrow{\alpha} F \longrightarrow 0 .$$

Since ${}^{\perp}C = \operatorname{Cogen}(C)$ is generating (and closed under direct sums, e.g. by Remark 1.7), there is an object $C_1 \in \operatorname{Cogen}(C)$ and an epimorphism $\varepsilon : C_1 \to G_2$. Denote $\beta = \alpha\varepsilon$, so that we have a commutative diagram

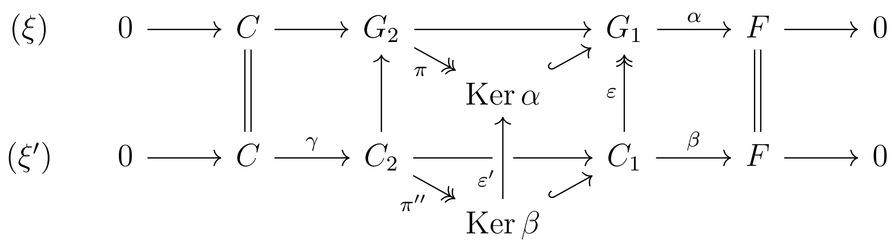
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C & \longrightarrow & G_2 & \longrightarrow & G_1 & \xrightarrow{\alpha} & F & \longrightarrow & 0 \\ & & & & & & {\scriptstyle\varepsilon}\uparrow & & \| & & \\ & & & & & & C_1 & \xrightarrow{\beta} & F & \longrightarrow & 0 \end{array}$$

with exact rows (note that we used the fact that ε is an epimorphism to ensure that β is an epimorphism).

The map ε restricts to kernels of α and β, hence we have a commutative diagram:



Note that $\operatorname{Ker}\beta \in \operatorname{Cogen}(C)$ as it is a subobject of $C_1 \in \operatorname{Cogen}(C)$. Let C_2 be the pullback of π and ε, fitting into the diagram (with the canonical pullback morphisms) with exact rows as follows:



That is, the 2-extensions (ξ) and (ξ') are equivalent in the sense of Yoneda Ext. Now it is enough to show that (ξ') is trivial by showing that γ is split monic. Thus, we consider the short exact sequence

$$0 \longrightarrow C \xrightarrow{\gamma} C_2 \xrightarrow{\pi''} \operatorname{Ker}\beta \longrightarrow 0$$

and since $\operatorname{Ker}\beta \in \operatorname{Cogen}(C) = {}^{\perp}C$, we have that $\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{Ker}\beta, C) = 0$ and so the above short exact sequence splits. In particular, γ is split monic and we are done. □

We prove the axiom (C2) next. This is rather straightforward.

Proposition 3.7. *Let $C \in \mathcal{A}$ be 1-cotilting. Then $\operatorname{Ext}^1_{\mathcal{A}}(C^{\times I}, C) = 0$ for any set I.*

Proof. This is immediate since $\operatorname{Prod}(C) \subseteq \operatorname{Cogen}(C) = {}^{\perp}C$. □

Remark 3.8. Proposition 3.4 says that the condition (C2) is equivalent to the seemingly stronger condition

$$\operatorname{Ext}^1_{\mathcal{A}}(\operatorname{Prod}(C), \operatorname{Prod}(C)) = 0,$$

as is the case for the category $\mathsf{Mod}\text{–}R$ for a ring R.

More generally, for any class $\mathcal{S}$ of objects in $\mathcal{A}$ we have

$$\operatorname{Ker}\operatorname{Ext}^1_{\mathcal{A}}(-, \mathcal{S}) = \operatorname{Ext}^1_{\mathcal{A}}(-, \operatorname{Prod}(\mathcal{S})).$$

Finally, we prove the axiom (C3). The proof requires the following lemma, which is an application of Proposition 1.8 of [CDT97] by R. Colpi, G. D'Este and A. Tonolo in abstract setting.

Lemma 3.9. *Let G be an object of $\mathcal{A}$ satisfying ${}^{\perp}G = \mathrm{Cogen}(G)$. Assume that $K \in \mathrm{Cogen}(G)$. Then there is a short exact sequence*

$$0 \longrightarrow K \longrightarrow G^{\times X} \longrightarrow L \longrightarrow 0\,,$$

where X is a set and $L \in \mathrm{Cogen}(G)$.

Proof. Consider the set $X = \mathrm{Hom}_{\mathcal{A}}(K, G)$. Consider the power $G^{\times X}$ together with the canonical projections $\pi_\chi : G^{\times X} \to G$, $\chi \in X$. Define Δ as the diagonal map $\Delta : K \to G^{\times X}$, that is, Δ is given as the unique map satisfying

$$\pi_\chi \Delta = \chi, \;\; \chi \in X\ .$$

By the fact that $K \in \mathrm{Cogen}(G)$, we immediately see that Δ is a monomorphism (if $\alpha, \beta : F \to K$ are two different morphisms, then, as G cogenerates K, there is a morphism $\chi : K \to G$ with $\chi\alpha \neq \chi\beta$; it follows that $\Delta\alpha \neq \Delta\beta$, since $\pi_\chi \Delta\alpha = \chi\alpha \neq \chi\beta = \pi_\chi \Delta\beta$).

Thus, we obtain a short exact sequence

$$0 \longrightarrow K \xrightarrow{\;\Delta\;} G^{\times X} \longrightarrow L \longrightarrow 0$$

(with $L = \mathrm{Coker}\,\Delta$), and it remains to check that $L \in \mathrm{Cogen}(G) = {}^{\perp}G$. Applying $\mathrm{Hom}_{\mathcal{A}}(-, G)$, we obtain a long exact sequence

$$\cdots\ \mathrm{Hom}_{\mathcal{A}}(G^{\times X}, G) \xrightarrow{-\circ\Delta} \mathrm{Hom}_{\mathcal{A}}(K, G) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(L, G) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(G^{\times X}, G)\ \cdots$$

The map $(- \circ \Delta) = \mathrm{Hom}_{\mathcal{A}}(\Delta, G)$ is clearly surjective from construction (if $\chi \in \mathrm{Hom}_{\mathcal{A}}(K, G)$, then $(- \circ \Delta)(\pi_\chi) = \pi_\chi \circ \Delta = \chi$) and $\mathrm{Ext}^1_{\mathcal{A}}(G^{\times X}, G) = 0$. Thus, from exactness it follows that $\mathrm{Ext}^1_{\mathcal{A}}(L, G) = 0$, which concludes the proof. $\square$

The following argument is very similar to the one used by Colpi and Trlifaj in [CT95b] where it is used for the category Mod–R. The purpose of the slight modifications given here is to overcome the fact that we don't have projective generators and that covariant Ext does not preserve direct products, in general.

Proposition 3.10. *Let C be a 1-cotilting object of $\mathcal{A}$. Given an injective object W (an injective cogenerator for $\mathcal{A}$ in particular), there is a short exact sequence*

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

with $C_0, C_1 \in \mathrm{Prod}(C)$.

Proof. Since ${}^{\perp}C = \mathrm{Cogen}(C)$ is generating, one can consider an epimorphism $e : F \twoheadrightarrow W$, where $F \in \mathrm{Cogen}(C)$. By definition of $\mathrm{Cogen}(C)$, there is a monomorphism $\iota : F \hookrightarrow C^{\times I}$ for some set I. By injectivity of W, e' extends along ι to a morphism $e : C^{\times I}_Y \twoheadrightarrow W$, i.e. $e\iota = e'$ (clearly e is an epimorphism as well).

Thus, we have a short exact sequence

$$0 \longrightarrow K \xrightarrow{i} C^{\times I} \xrightarrow{e} W \longrightarrow 0$$

(where $i = \operatorname{Ker} e$), and, obviously, $K \in \operatorname{Cogen}(C)$. By Lemma 3.9 there is a short exact sequence

$$0 \longrightarrow K \xrightarrow{j} C^{\times J} \longrightarrow L \longrightarrow 0$$

for some set J and some $L \in \operatorname{Cogen}(C)$.

Consider the pushout P of i and j, which gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{i} & C^{\times I} & \xrightarrow{e} & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j} & & \downarrow & & \| & & \\
0 & \longrightarrow & C^{\times J} & \longrightarrow & P & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns. Observe that since $L, C^{\times I} \in \operatorname{Cogen}(C)$ and $\operatorname{Cogen}(C) = {}^{\perp}C$ is closed under extensions, we have $P \in \operatorname{Cogen}(C_Y)$ by the exactness of the second column. The claim is that the exact sequence

$$(\xi) \qquad 0 \longrightarrow C^{\times J} \longrightarrow P \longrightarrow W \longrightarrow 0$$

(second row of the above diagram) is the desired exact sequence from the statement of the proposition. To verify this, it remains to show that $P \in \operatorname{Prod}(C)$.

Applying Lemma 3.9 again for $P \in \operatorname{Cogen}(C)$, there is a short exact sequence

$$0 \longrightarrow P \longrightarrow C^{\times L} \longrightarrow M \longrightarrow 0$$

with $M \in \operatorname{Cogen} C = {}^{\perp}C$. We compute that $\operatorname{Ext}^1_{\mathcal{A}}(M, P) = 0$, hence the above exact sequence splits and thus, $P \in \operatorname{Prod}(C)$. Indeed, applying $\operatorname{Hom}_{\mathcal{A}}(M, -)$ to the exact sequence (ξ) yields a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(M, C^{\times J}) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(M, P) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(M, W) \longrightarrow \cdots$$

where $\operatorname{Ext}^1_{\mathcal{A}}(M, W) = 0$ (W is injective) and $\operatorname{Ext}^1_{\mathcal{A}}(M, C^{\times J}) = 0$ (using Proposition 3.4 and the fact that $M \in {}^{\perp}C$). Thus, from exactness it follows that $\operatorname{Ext}^1_{\mathcal{A}}(M, P) = 0$, which completes the proof.

□

Note that the proof relied on the fact that the cotilting class ${}^{\perp}C$ is generating. We have thus proved that a 1-cotilting object always satisfies the axioms (C1)–

(C3).

We now turn our attention to the converse implication. In order to do that, a preparatory lemma is needed. Note that the proof of the lemma is basically a "local version"[1] of the Horseshoe Lemma.

Lemma 3.11. *Consider an object $G \in \mathcal{A}$ such that* $\mathrm{Cogen}(G) \subseteq {}^{\perp}G$. *Then the class* $\mathrm{Cogen}(G)$ *is closed under subobjects, direct products and extensions. Thus, the class* $\mathrm{Cogen}(G)$ *is a torsion-free class in a torsion pair in* $\mathcal{A}$.

Proof. The fact that $\mathrm{Cogen}(G)$ is closed under subobjects and direct products is obvious (for direct products, note that this follows from the fact that direct product functors are left exact). We only need to show that $\mathrm{Cogen}(G)$ is closed under extensions.

Consider a short exact sequence

$$0 \longrightarrow F' \xrightarrow{\ j\ } F \xrightarrow{\ p\ } F'' \longrightarrow 0$$

in $\mathcal{A}$, where $F', F'' \in \mathrm{Cogen}(G)$. That is, we have monomorphisms i', i''

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{\ j\ } & F & \xrightarrow{\ p\ } & F'' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle i'} & & & & \Big\downarrow{\scriptstyle i''} & & \\ & & G^{\times I} & & & & G^{\times J} & & \end{array},$$

where I and J are some sets. Without loss of generality, assume that I and J are disjoint. Consider $C^{\times(I\cup J)} \simeq C^{\times I} \oplus C^{\times J}$. This biproduct structure on $C^{\times(I\cup J)}$ comes with the canonical biproduct morphisms $\iota_I, \iota_J, \pi_I, \pi_J$, fitting into a split exact sequence as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{\ j\ } & F & \xrightarrow{\ p\ } & F'' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle i'} & & & & \Big\downarrow{\scriptstyle i''} & & \\ 0 & \longrightarrow & C^{\times I} & \underset{\pi_I}{\overset{\iota_I}{\rightleftarrows}} & C^{\times(I\cup J)} & \underset{\iota_J}{\overset{\pi_J}{\rightleftarrows}} & C^{\times J} & \longrightarrow & 0 \end{array}.$$

Put $\beta := i''p$. By the fact that $F'' \in \mathrm{Cogen}(G) \subseteq {}^{\perp}G$, we have $\mathrm{Ext}^1_{\mathcal{A}}(F'', G) = 0$ and thus, by Lemma 3.5, there is an extension α of i' along j, i.e. so that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{\ j\ } & F & \xrightarrow{\ p\ } & F'' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle i'} & \swarrow{\scriptstyle \alpha} & & \searrow{\scriptstyle \beta} & \Big\downarrow{\scriptstyle i''} & & \\ 0 & \longrightarrow & C^{\times I} & \underset{\pi_I}{\overset{\iota_I}{\rightleftarrows}} & C^{\times(I\cup J)} & \underset{\iota_J}{\overset{\pi_J}{\rightleftarrows}} & C^{\times J} & \longrightarrow & 0 \end{array}$$

---

[1]This is not meant geometrically. The intended meaning is that one uses the fact that particular Ext^1 group vanishes, instead of working with projectives or injectives.

Put $i := \iota_I\alpha + \iota_J\beta$. Then

$$\begin{aligned} ij &= \iota_I\alpha j + \iota_J\beta j = \iota_I i' + \iota_J i'' pj = \iota_I i', \\ \pi_J i &= \pi_J\iota_I\alpha + \pi_J\iota_J\beta = 0 + \beta = i'' p. \end{aligned}$$

That is, the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \xrightarrow{j} & F & \xrightarrow{p} & F'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i''} & & \\ 0 & \longrightarrow & C^{\times I} & \xrightarrow{\iota_I} & C^{\times(I\cup J)} & \xrightarrow{\pi_J} & C^{\times J} & \longrightarrow & 0 \end{array}$$

By the Four Lemma, it follows that i is again a monomorphism. This shows that $F \in \mathrm{Cogen}(G)$.

Since $\mathrm{Cogen}(G)$ is closed under subobjects, direct products and extensions, it follows by Remark 2.3 4. that $\mathrm{Cogen}(G)$ is a torsion-free class in $\mathcal{A}$. □

Proposition 3.12. *Suppose that C is an object of $\mathcal{A}$ satisfying the axioms (C1)–(C3). Then ${}^{\perp}C = \mathrm{Cogen}(C)$.*

Proof. By (C1) we have $\mathrm{Prod}(C) \subseteq {}^{\perp}C$. Thus, in order to prove the inclusion $\mathrm{Cogen}(C) \subseteq {}^{\perp}C$ it is enough to show that ${}^{\perp}C$ is closed under subobjects. Let us therefore consider an object $F \in \mathcal{A}$ and its subobject $G \overset{i}{\hookrightarrow} F$. Applying the functor $\mathrm{Hom}_{\mathcal{A}}(-, C)$ to the exact sequence

$$0 \longrightarrow G \xrightarrow{i} F \longrightarrow C \longrightarrow 0$$

(where $C = \mathrm{Coker}\, i$) yields an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(F, C) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(G, C) \longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(H, C) \longrightarrow \cdots,$$

where $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ by assumption on F and $\mathrm{Ext}^2_{\mathcal{A}}(H, C) = 0$ by the fact that $\mathrm{injdim}\, C \leq 1$. It follows that $\mathrm{Ext}^1_{\mathcal{A}}(G, C) = 0$, i.e. $G \in {}^{\perp}C$.

Let us now prove the converse inclusion. By Lemma 3.11, the class $\mathrm{Cogen}(C)$ is a torsion-free class of a torsion pair in $\mathcal{A}$.

Consider any object $A \in {}^{\perp}C$. There is a short exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $F \in \mathrm{Cogen}(C)$ and $T \in \mathrm{Ker}\,\mathrm{Hom}_{\mathcal{A}}(-, \mathrm{Cogen}(C))$. In order to prove that $A \in \mathrm{Cogen}(C)$ we only need to show that $T = 0$.

To this end, consider the short exact sequence

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

ensured by the axiom (C3). That is, W is an injective cogenerator for $\mathcal{A}$ and $C_0, C_1 \in \mathrm{Prod}(C)$. Applying $\mathrm{Hom}_{\mathcal{A}}(T, -)$ to it, we obtain an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(T, C_0) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(T, W) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(T, C_1) \longrightarrow \cdots$$

Since $\mathrm{Prod}(C) \subseteq \mathrm{Cogen}(C)$, we have $\mathrm{Hom}_{\mathcal{A}}(T, C_0) = 0$, and by the fact that ${}^{\perp}C$ is closed under subobjects, we have $T \in {}^{\perp}C$. Thus, we also have $\mathrm{Ext}^1_{\mathcal{A}}(T, C_0) = 0$ (by Remark 3.8). It follows that $\mathrm{Hom}_{\mathcal{A}}(T, W) = 0$. But since W is a cogenerator, this can happen only if $T = 0$. This concludes the proof.

□

Let us summarize the results of this section (in a less general but more compact form).

Theorem 3.13. *Let $\mathcal{A}$ be a Grothendieck category and $C \in \mathcal{A}$ an object in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) C is 1*-cotilting.*

(2) The class ${}^{\perp}C$ is generating, and C satisfies

- *(C1)* $\mathrm{injdim}\, C \leq n$.
- *(C2)* $\mathrm{Ext}^i_R(C^{\times\kappa}, C) = 0$ *for every cardinal κ and every $i \geq 1$.*
- *(C3) There is an exact sequence*

$$0 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow W \longrightarrow 0$$

where W is an injective cogenerator for $\mathcal{A}$ and $C_0, C_1 \in \mathrm{Prod}(C)$.

3.3 Construction of 1-cotilting sheaves on a Noetherian scheme

Let X be a fixed Noetherian scheme. The goal of this section is to construct a 1-cotilting sheaf $\mathscr{C}_Y$ on X, such that the cotilting class $\mathrm{Cogen}(\mathscr{C}_Y) = {}^{\perp}\mathscr{C}_Y$ is equal to $\mathcal{F}(Y)$. Our approach is analogical to the one used in [ŠTH14] for modules.

Suppose that $Y \subseteq X$ is a specialization closed subset. Denote $\mathcal{I}(Y)$ the class of all injective quasi-coherent sheaves $\mathscr{E}$ with $\mathrm{Ass}\,\mathscr{E} \cap Y = \emptyset$. That is, $\mathcal{I}(Y)$ consists of all the injectives contained in $\mathcal{F}(Y)$. From this description it follows that $\mathcal{I}(Y)$ is closed under direct limits: $\mathcal{F}(Y)$ is closed under direct limits by Lemma 2.22, and injectives are closed under direct limits by Corollary 1.33. Similarly, $\mathcal{I}(Y)$ is closed under taking direct sums (also note that $\mathcal{I}(Y)$ is closed under direct products, which will be useful later on). Moreover, from the structure theorem for injective quasi-coherent sheaves it follows that there is a set $\mathcal{S} \subseteq \mathcal{I}(Y)$ such that $\mathcal{I}(Y) = \varinjlim \mathcal{S}$ - it is enough to take the set of all finite direct sums of the sheaves $\mathscr{J}(x)$ for various (possibly repeating) points $x \in X \setminus Y$. Hence, by [EB06, Theorem 3.2], $\mathcal{I}(Y)$ is a covering class[2].

The following proposition is a key to our construction. The argument of the proof is based on the proof of implication (i)⇒(iii) of [ŠTH14, Lemma 2.10].

Proposition 3.14. *Suppose that there is a generator $\mathscr{G}$ of QCoh_X such that $\mathrm{Ass}\,\mathscr{G} \cap Y = \emptyset$. Then every $\mathcal{I}(Y)$-precover of an injective quasi-coherent sheaf is an epimorphism.*

[2] See Definition C.2 of Appendix for definition of covers and precovers.

Proof. Consider an $\mathcal{I}(Y)$-precover of an injective quasi-coherent sheaf $f : \mathscr{I} \to \mathscr{E}$. Since $\mathscr{G}$ is a generator for QCoh_X, there is an epimorphism $g : \mathscr{G}^{\oplus I} \to \mathscr{E}$. Denote $i : \mathscr{G}^{\oplus I} \hookrightarrow E(\mathscr{G}^{\oplus I})$ the injective envelope of $\mathscr{G}^{\oplus I}$. By the injectivity of $\mathscr{E}$, there is a morphism $h : E(\mathscr{G}^{\oplus I}) \to \mathscr{E}$ such that $h \circ i = g$. Since g is an epimorphism, so is h. From the assumption it follows that $\mathscr{G}^{\oplus I} \in \mathcal{F}(Y)$, hence $E(\mathscr{G}^{\oplus I}) \in \mathcal{I}(Y)$. Thus, there is a morphism $k : E(\mathscr{G}^{\oplus I}) \to \mathscr{I}$ such that $f \circ k = h$. Thus, f is an epimorphism, since so is h. □

Recall that our definition of 1-cotilting sheaf implies that ${}^{\perp}\mathscr{C}_Y$ is generating. Since the class $\mathcal{F}(Y)$ is closed under direct sums (by Remark 1.7), this means that we restrict our attention to specialization closed subsets Y such that the class $\mathcal{F}(Y)$ contains a generator. That is, in the cases we are interested in, there indeed is a generator $\mathscr{G}$ of QCoh_X such that $\operatorname{Ass}\mathscr{G} \cap Y = \emptyset$, and so the Proposition 3.14 applies.

Before proceeding to the construction of $\mathscr{C}_Y$, let us briefly discuss several circumstances that allow us to control (by a reasonable choice of the generator for QCoh_X) the set of points that Y needs to avoid. In both cases we mention, the resulting chosen generator has the set of associated points equal to (or is a subset of) $\operatorname{Ass}\mathcal{O}_X$, so the condition on Y reduces to $Y \cap \operatorname{Ass}\mathcal{O}_X = \emptyset$.

Definition 3.15. (1) We say that a scheme X *has an ample family of line bundles* if there are global sections f_i of line bundles $\mathscr{L}_i$, $i \in I$, such that the sets $D(f_i) = \{x \in X \mid f(x) \neq 0\}$, $i \in I$ form an affine open cover of X.

(2) We say that a scheme X *has the resolution property* if every coherent sheaf is an epimorphic image of a vector bundle (i. e. a locally free sheaf of finite rank).

A Noetherian scheme X which has an ample family of line bundles has the resolution property, which was proved by S. Kleiman and M. Borelli in [Bor67], and independently by L. Illusie in [Ill71]. The above properties are satisfied for a large class of Noetherian schemes, e.g. for quasi-projective schemes on affine schemes. See [TT90, section 2.1] for more detailed discussion.

Let X be a Noetherian scheme. For any vector bundle $\mathscr{F}$ of nonzero rank we have $\operatorname{Ass}\mathscr{F} = \operatorname{Ass}\mathcal{O}_X$, as the associated points depend on the stalks only, and any stalk $\mathscr{F}_x$ of $\mathscr{F}$ is isomorphic to (nonzero) direct sum of the stalk of the structure sheaf $\mathcal{O}_{X,x}$.

Assume that X has the resolution property. Since any quasi-coherent sheaf is a direct union of its coherent subsheaves, it follows that the quasi-coherent sheaf $\mathscr{G}$ obtained as a direct sum of representatives of all vector bundles (up to isomorphism) is a generator for QCoh_X, and $\operatorname{Ass}\mathscr{G} = \operatorname{Ass}\mathcal{O}_X$.

The second case we mention is the case when the category QCoh_X has enough flats.

Definition 3.16. Let X be a scheme. A quasi-coherent sheaf $\mathscr{F}$ on X is called *flat* if for every $x \in X$, $\mathscr{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Let $\mathscr{F}$ be a flat quasi-coherent sheaf on a scheme X. For each $x \in X$, $\mathscr{F}_x$ is a flat $\mathcal{O}_{X,x}$-module, hence by Govorov-Lazard Theorem [Rot08, Theorem 5.40], $\mathscr{F}_x$ is a direct limit of a direct system of finite-rank free $\mathcal{O}_{X,x}$-modules, say $(F_i \mid i \in I)$.

By Lemmas 2.10 and 2.9 (1) we have

$$\operatorname{Ass}_{\mathcal{O}_{X,x}} \mathscr{F}_x \subseteq \bigcup_{i \in I} \operatorname{Ass}_{\mathcal{O}_{X,x}} F_i = \operatorname{Ass}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}.$$

It follows that $\operatorname{Ass} \mathscr{F} \subseteq \operatorname{Ass} \mathcal{O}_X$.

A result of D. Murfet states the following.

Proposition 3.17 ([Mur07, Corollary 3.21]). *Let X be a quasi-compact separated scheme. Then for every quasi-coherent sheaf on $\mathscr{G}$ on X, there exists a flat quasi-coherent sheaf $\mathscr{F}$ on X and an epimorphism $\mathscr{F} \to \mathscr{G}$.*

Suppose that X is a Noetherian separated scheme. Let us fix a set $\mathcal{S} = \{\mathscr{G}^s \mid s \in S\}$ of representatives of all isomorphism classes of coherent sheaves (note that this is possible since Coh_X is skeletally small). For each $\mathscr{G} \in \mathcal{S}$, choose a flat quasi-coherent sheaf $\mathscr{F}^s$ admitting an epimorphism $\mathscr{F}^s \to \mathscr{G}^s$. Put $\mathscr{F} := \bigoplus_{s \in \mathcal{S}} \mathscr{F}^s$. Then we have

$$\operatorname{Gen}(\mathscr{F}) \supseteq \operatorname{Gen}(\mathcal{S}) = \mathsf{QCoh}_X,$$

i.e. $\mathscr{F}$ is a flat generator, and thus $\operatorname{Ass} \mathscr{F} \subseteq \operatorname{Ass} \mathcal{O}_X$.

Let us summarize the above discussion.

Proposition 3.18. *Let X be a Noetherian scheme which either has the resolution property, or is separated. Consider a specialization closed subset $Y \subseteq X$ satisfying $\operatorname{Ass} \mathcal{O}_X \cap Y = \emptyset$. Then any $\mathcal{I}(Y)$-precover of an injective quasi-coherent sheaf is an epimorphism.*

Finally, we proceed to the construction itself. In what follows throughout the rest of the chapter, let us fix a specialization closed subset $Y \subseteq X$ that does not contain any associated point of a fixed generator of the category QCoh_X.

Construction 3.19. Let X be a Noetherian scheme with the resolution property and $Y \subseteq X$ a specialization closed subset satisfying $\operatorname{Ass} \mathcal{O}_X \cap Y = \emptyset$. For any $y \in Y$, we have an exact sequence

$$0 \longrightarrow \mathscr{K}(y) \xrightarrow{\alpha_y} \mathscr{I}(y) \xrightarrow{\beta_y} \mathscr{J}(y) \longrightarrow 0,$$

where $\alpha_y : \mathscr{I}(y) \longrightarrow \mathscr{J}(y)$ is a $\mathcal{I}(Y)$-cover.

Define quasi-coherent sheaves

$$\mathscr{K}_Y := \prod_{y \in Y} \mathscr{K}(y), \quad \mathscr{J}_Y := \prod_{x \in X \setminus Y} \mathscr{J}(x),$$

and finally, put

$$\mathscr{C}_Y := \mathscr{K}_Y \times \mathscr{J}_Y.$$

Our aim is to prove that ${}^{\perp}\mathscr{C}_Y = \operatorname{Cogen}(\mathscr{C}_Y) = \mathcal{F}(Y)$. We do this by showing the equalities $\operatorname{Cogen}(\mathscr{C}_Y) = \mathcal{F}(Y)$, ${}^{\perp}\mathscr{C}_Y = \mathcal{F}(Y)$ separately. Then we will be done, since the specialization closed subset Y was chosen in a way that ensures that the class $\mathcal{F}(Y)$ is generating.

Remark 3.20. If $Y = \emptyset$, the resulting quasi-coherent sheaf is easily seen to be an injective cogenerator for QCoh_X, and

$$^{\perp}\mathscr{C}_Y = \mathrm{Cogen}(\mathscr{C}_Y) = \mathcal{F}(Y) = \mathsf{QCoh}_X.$$

From now on, let us additionally assume that $Y \neq \emptyset$.

We start with the less difficult equality $\mathrm{Cogen}(\mathscr{C}_Y) = \mathcal{F}(Y)$.

Proposition 3.21. *Under the above assumptions, $\mathcal{F}(Y) = \mathrm{Cogen}\,(\mathscr{C}_Y)$.*

Proof. For any $y \in Y$, we clearly have $\mathscr{K}(y) \in \mathcal{F}(Y)$ (as it is a subobject of $\mathscr{I}(y) \in \mathcal{F}(Y)$). Also, for any $x \in X \setminus Y$, we have $\mathscr{J}(x) \in \mathcal{F}(Y)$ directly from the definition of $\mathcal{F}(Y)$. This shows that $\mathscr{C}_Y \in \mathcal{F}(Y)$ and $\mathrm{Cogen}\,(\mathscr{C}_Y) \subseteq \mathcal{F}(Y)$, since $\mathcal{F}(Y)$ is closed under direct products and subobjects.

Conversely, consider any $\mathscr{F} \in \mathcal{F}(Y)$ and its injective hull

$$\mathscr{F} \hookrightarrow E(\mathscr{F}) \simeq \textstyle\bigoplus_{i\in I} \mathscr{J}(x_i),$$

where $x_i,\ i \in I$ is a suitable collection of points in X ($E(\mathscr{F})$ can be written in this form by structure theorem for injective quasi-coherent sheaves, Theorem 1.32). Since the torsion pair $(\mathcal{T}(Y), \mathcal{F}(Y))$ is hereditary, $\mathcal{F}(Y)$ is closed under injective hulls and it follows that $\bigoplus_{i\in I} \mathscr{J}(x_i) \in \mathcal{F}(Y)$. From this it is clear that all the points x_i lie outside Y.

Thus, we can write

$$E(\mathscr{F}) \simeq \bigoplus_{x\in S} \mathscr{J}(x)^{\oplus I_x},$$

where $S \subseteq X \setminus Y$ is a subset of X disjoint with Y and $I_x,\ x \in S$ are suitable index sets. Denote $J = \bigcup_{x\in S} I_x$. We have a monomorphism obtained as a composition of the following inclusions and isomorphisms (the obvious ones):

$$\mathscr{F} \subseteq E(\mathscr{F}) \simeq \bigoplus_{x\in S} \mathscr{J}(x)^{\oplus I_x} \subseteq \left(\bigoplus_{x\in S} \mathscr{J}(x)\right)^{\oplus J} \subseteq$$

$$\subseteq \left(\prod_{x\in S} \mathscr{J}(x)\right)^{\times J} \subseteq \left(\prod_{y\in Y} \mathscr{K}(y) \times \prod_{x\in X\setminus Y} \mathscr{J}(x)\right)^{\times J} = \mathscr{C}_Y^{\times J}.$$

This shows that $\mathscr{F} \in \mathrm{Cogen}(\mathscr{C}_Y)$.

□

Next we prove the equality $^{\perp}\mathscr{C}_Y = \mathcal{F}(Y)$. Both inclusions of this equality are non-trivial. The proof is split to a series of lemmas, studying the behaviour (both vanishing and non-vanishing) of $\mathrm{Ext}_X(-, \mathscr{C}_Y)$. The proof is conducted in analogy to [ŠTH14, Chapter 4].

Lemma 3.22. *Consider $y \in Y$. Then $\mathrm{Ext}^1_X(\mathscr{E}, \mathscr{K}(y)) = 0$ for any $\mathscr{E} \in \mathcal{I}(Y)$.*

Proof. Applying the functor $\mathrm{Hom}_X(\mathscr{E}, -)$ to the exact sequence

$$0 \longrightarrow \mathscr{K}(y) \xrightarrow{\alpha_y} \mathscr{I}(y) \xrightarrow{\beta_y} \mathscr{J}(y) \longrightarrow 0$$

as described in Construction 3.19 yields a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_X(\mathscr{E},\mathscr{K}(y)) \longrightarrow \mathrm{Hom}_X(\mathscr{E},\mathscr{I}(y)) \xrightarrow{\beta_y\circ -} \mathrm{Hom}_X(\mathscr{E},\mathscr{J}(y)) \longrightarrow \mathrm{Ext}^1_X(\mathscr{E},\mathscr{K}(y)) \longrightarrow \mathrm{Ext}^1_X(\mathscr{E},\mathscr{I}(y)) = 0\,.$$

The $\mathcal{I}(Y)$-precover property of the map $\mathscr{I}(y) \to \mathscr{J}(y)$ guarantees that the map $\beta_y \circ - : \mathrm{Hom}_X(\mathscr{E},\mathscr{I}(y)) \to \mathrm{Hom}_X(\mathscr{E},\mathscr{J}(y))$ is surjective (since $\mathscr{E} \in \mathcal{I}(Y)$), hence it follows that $\mathrm{Ext}^1_X(\mathscr{E},\mathscr{K}(y)) = 0$.

□

Lemma 3.23. *Consider a point $y \in Y$ and a quasi-coherent sheaf $\mathscr{F}$ such that $y \in \mathrm{Ass}\,\mathscr{F}$. Then $\mathrm{Ext}^1_X(\mathscr{F},\mathscr{K}(y)) \neq 0$.*

Proof. Consider a coherent subsheaf $\mathscr{G} \subseteq \mathscr{F}$ with the properties $\mathrm{Supp}\,\mathscr{G} = \overline{\{y\}}$ and $\mathrm{Ass}\,\mathscr{G} = \{y\}$, as in Lemma 2.14.

Firstly, observe that $\mathrm{Hom}_X(\mathscr{G},\mathscr{I}(y)) = 0$. This follows from the fact that $\mathscr{G} \in \mathcal{T}(Y)$ and $\mathscr{I}(y) \in \mathcal{F}(Y)$.

Next we claim that $\mathrm{Ext}^1_X(\mathscr{G},\mathscr{K}(y)) \neq 0$. Indeed, consider the short exact sequence

$$0 \longrightarrow \mathscr{K}(y) \xrightarrow{\alpha_y} \mathscr{I}(y) \xrightarrow{\beta_y} \mathscr{J}(y) \longrightarrow 0\,.$$

Applying $\mathrm{Hom}_X(\mathscr{G},-)$, we obtain a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_X(\mathscr{G},\mathscr{K}(y)) \longrightarrow \mathrm{Hom}_X(\mathscr{G},\mathscr{I}(y)) \longrightarrow \mathrm{Hom}_X(\mathscr{G},\mathscr{J}(y)) \longrightarrow \mathrm{Ext}^1_X(\mathscr{G},\mathscr{K}(y)) \longrightarrow \mathrm{Ext}^1_X(\mathscr{G},\mathscr{I}(y))\,.$$

From Corollary 2.16 it follows that $\mathrm{Ass}\,E(\mathscr{G}) = \mathrm{Ass}\,\mathscr{G} = \{y\}$ and by the structure theorem for injective quasi-coherent sheaves (Theorem 1.32) it follows that $E(\mathscr{G}) \simeq \mathscr{J}(y)^{\oplus I}$ for some set I. That is, there is a nonzero (mono-) morphism

$$\mathscr{G} \hookrightarrow \mathscr{J}(y)^{\oplus I} \hookrightarrow \mathscr{J}(y)^{\times I}$$

In particular, we necessarily have that $\mathrm{Hom}_X(\mathscr{G},\mathscr{J}(y)) \neq 0$ (this is because

$$0 \neq \mathrm{Hom}_X(\mathscr{G},\mathscr{J}(y)^{\times I}) \simeq \mathrm{Hom}_X(\mathscr{G},\mathscr{J}(y))^{\times I}).$$

However, $\mathrm{Ext}^1_X(\mathscr{G},\mathscr{I}(y)) = 0$ and we also have that $\mathrm{Hom}_X(\mathscr{G},\mathscr{I}(y)) = 0$ by the first part. Thus, we obtain an isomorphism

$$\mathrm{Ext}^1_X(\mathscr{G},\mathscr{K}(y)) \simeq \mathrm{Hom}_X(\mathscr{G},\mathscr{J}(y)) \neq 0.$$

Finally, we show that $\mathrm{Ext}^1_X(\mathscr{F},\mathscr{K}(y)) \neq 0$. We start with a short exact sequence

$$0 \longrightarrow \mathscr{G} \xrightarrow{\subseteq} \mathscr{F} \longrightarrow \mathscr{H} \longrightarrow 0\,.$$

(with $\mathscr{H} = \operatorname{Coker}(\mathscr{G} \hookrightarrow \mathscr{F})$) and apply $\operatorname{Hom}_X(-, \mathscr{K}(y))$ to obtain a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{K}(y)) \longrightarrow \operatorname{Ext}^1_X(\mathscr{G}, \mathscr{K}(y)) \longrightarrow \operatorname{Ext}^2_X(\mathscr{H}, \mathscr{K}(y)) .$$

Since $\operatorname{injdim} \mathscr{K}(y) \leq 1$ (it is a kernel of an epimorphism between injective sheaves), $\operatorname{Ext}^2_X(\mathscr{H}, \mathscr{K}(y)) = 0$. Thus, we have obtained a group epimorphism

$$\operatorname{Ext}^1_X(\mathscr{F}, \mathscr{K}(y)) \longrightarrow\!\!\!\!\!\rightarrow \operatorname{Ext}^1_X(\mathscr{G}, \mathscr{K}(y))$$

with $\operatorname{Ext}^1_X(\mathscr{G},\mathscr{K}(y)) \neq 0$, which proves that $\operatorname{Ext}^1_X(\mathscr{F},\mathscr{K}(y)) \neq 0$. □

The following lemma is a partial remedy to the fact that products are not exact in the category of quasi-coherent sheaves in general. Here we use fully the fact that the class $\mathcal{I}(Y)$ is precovering. The reader is encouraged to compare the given argument with the proof of Proposition 3.4.

Lemma 3.24. *For every $y \in Y$, consider the exact sequence*

$$0 \longrightarrow \mathscr{K}(y) \xrightarrow{\alpha_y} \mathscr{I}(y) \xrightarrow{\beta_y} \mathscr{J}(y) \longrightarrow 0,$$

from Construction 3.19. Given any family $(y_j \mid j \in J)$ of points from Y, the sequence

$$0 \longrightarrow \prod_{j\in J} \mathscr{K}(y_j) \xrightarrow{\prod_{j\in J} \alpha_{y_j}} \prod_{j\in J} \mathscr{I}(y_j) \xrightarrow{\prod_{j\in J} \beta_{y_j}} \prod_{j\in J} \mathscr{J}(y_j) \longrightarrow 0$$

is also exact.

Proof. The product functor is left exact, since it preserves limits, in particular, it preserves kernels. We need to show that $\prod_{j\in J} \beta_{y_j}$ is an epimorphism.

For $j \in J$, in the exact sequence

$$0 \longrightarrow \mathscr{K}(y_j) \xrightarrow{\alpha_{y_j}} \mathscr{I}(y_j) \xrightarrow{\beta_{y_j}} \mathscr{J}(y_j) \longrightarrow 0,$$

the morphism β_{y_j} is an $\mathcal{I}(Y)$-cover. We claim that

$$\prod_{j\in J} \beta_{y_j} : \prod_{j\in J} \mathscr{I}(y_j) \longrightarrow \prod_{j\in J} \mathscr{J}(y_j)$$

is an $\mathcal{I}(Y)$-precover. Firstly, the sheaf $\prod_{j\in J} \mathscr{I}(y_j)$ belongs to $\mathcal{I}(Y)$ as the class $\mathcal{I}(Y)$ is closed under direct products. Consider an arbitrary morphism

$$f : \mathscr{E} \longrightarrow \prod_{j\in J} \mathscr{J}(y_j)$$

with $\mathscr{E} \in \mathcal{I}(Y)$. For each $j \in J$, we obtain, using the $\mathcal{I}(Y)$-precover property of β_{y_j}, a commutative diagram

$$\begin{array}{ccc} \mathscr{I}(y_j) & \xrightarrow{\beta_{y_j}} & \mathscr{J}(y_j) \\ {\scriptstyle\gamma_j}\nwarrow & & \nearrow{\scriptstyle\pi_j f} \\ & \mathscr{E} & \end{array},$$

where $\pi_j : \prod_{i\in J} \mathscr{J}(y_i) \to \mathscr{J}(y_j)$ is the canonical projection (coming with the product). Using the universal property of product, it is easy to see that we have the commutative diagram

$$\begin{array}{ccc} \prod\limits_{j\in J} \mathscr{I}(y_j) & \xrightarrow{\prod_{j\in J} \beta_{y_j}} & \prod\limits_{j\in J} \mathscr{J}(y_j) \\ {\scriptstyle\prod_{j\in J}\gamma_j}\nwarrow & & \nearrow{\scriptstyle f} \\ & \mathscr{E} & \end{array}$$

as well. Thus, the $\mathcal{I}(Y)$-precover property for $\prod_{j\in J} \beta_{y_j}$ is satisfied.

By Corollary 3.14, it follows that the map $\prod_{j\in J} \beta_{y_j}$ is an epimorphism. □

Proposition 3.25. $\operatorname{injdim} \mathscr{C}_Y = 1$.

Proof. Let us first show that $\mathscr{C}_Y$ is not injective. Consider a point $y \in Y$ and a quasi-coherent sheaf $\mathscr{F}$ with $y \in \operatorname{Ass} \mathscr{F}$. Denote by $\mathscr{G}$ the quasi-coherent sheaf

$$\mathscr{G} = \prod_{z\in Y\setminus\{y\}} \mathscr{K}(z) \times \prod_{x\in X\setminus Y} \mathscr{J}(x)$$

so that we have

$$\mathscr{C}_Y \simeq \mathscr{K}(y) \oplus \mathscr{G}.$$

Then we obtain

$$\operatorname{Ext}^1_X(\mathscr{F},\mathscr{C}_Y) \simeq \operatorname{Ext}^1_X(\mathscr{F},\mathscr{K}(y)\oplus\mathscr{G}) \simeq \operatorname{Ext}^1_X(\mathscr{F},\mathscr{K}(y)) \oplus \operatorname{Ext}^1_X(\mathscr{F},\mathscr{G}) \neq 0,$$

since $\operatorname{Ext}^1_X(\mathscr{F},\mathscr{K}(y)) \neq 0$ by Lemma 3.23.

Next we prove that $\operatorname{injdim} \mathscr{C}_Y \leq 1$. Using the product complex

$$0 \longrightarrow \prod_{y\in Y} \mathscr{K}(y) \xrightarrow{\prod_{y\in Y}\alpha_y} \prod_{y\in Y} \mathscr{I}(y) \xrightarrow{\prod_{y\in Y}\beta_y} \prod_{y\in Y} \mathscr{J}(y) \longrightarrow 0\,,$$

which is a short exact sequence by Lemma 3.24, we obtain an exact sequence

$$0 \longrightarrow \mathscr{C}_Y \longrightarrow \prod_{y\in Y} \mathscr{I}(y) \oplus \prod_{x\in X\setminus Y} \mathscr{J}(x) \longrightarrow \prod_{y\in Y} \mathscr{J}(y) \longrightarrow 0$$

by adding the injective direct summand $\prod_{x\in X\setminus Y} \mathscr{J}(x)$ to the first two terms. This is an injective coresolution of $\mathscr{C}_Y$ of the length 2. Thus, we infer that $\operatorname{injdim} \mathscr{C}_Y \leq 1$. □

Corollary 3.26. *The class* ${}^{\perp}\mathscr{C}_Y$ *is closed under subobjects.*

Proof. Consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

with $\mathscr{G} \in {}^{\perp}\mathscr{C}_Y$. Our aim is to prove that $\mathscr{F} \in {}^{\perp}\mathscr{C}_Y$.

The long exact sequence arising from application of $\operatorname{Hom}_X(-, \mathscr{C}_Y)$ contains the terms

$$\cdots \longrightarrow \operatorname{Ext}^1_X(\mathscr{G}, \mathscr{C}_Y) \longrightarrow \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{C}_Y) \longrightarrow \operatorname{Ext}^2_X(\mathscr{H}, \mathscr{C}_Y) \longrightarrow \cdots$$

However, $\operatorname{Ext}^1_X(\mathscr{G}, \mathscr{C}_Y) = 0$ by the assumptions and $\operatorname{Ext}^2_X(\mathscr{H}, \mathscr{C}_Y) = 0$, since $\operatorname{injdim} \mathscr{C}_Y = 1$. Thus, $\operatorname{Ext}^1_X(\mathscr{F}, \mathscr{C}_Y) = 0$. □

Finally, we are prepared for proof of the second equality.

Proposition 3.27. ${}^{\perp}\mathscr{C}_Y = \mathcal{F}(Y)$.

Proof. First we prove that ${}^{\perp}\mathscr{C}_Y \subseteq \mathcal{F}(Y)$. Suppose for contradiction that there is a quasi-coherent sheaf $\mathscr{F} \in {}^{\perp}\mathscr{C}_Y \setminus \mathcal{F}(Y)$. That is, there is a point $y \in Y$ such that $y \in \operatorname{Ass} \mathscr{F}$. From Lemma 3.23 it follows that $\operatorname{Ext}^1_X(\mathscr{F}, \mathscr{K}(y)) \neq 0$. Denote by $\mathscr{G}$ the quasi-coherent sheaf

$$\mathscr{G} = \prod_{z \in Y \setminus \{y\}} \mathscr{K}(z) \times \prod_{x \in X \setminus Y} \mathscr{J}(x).$$

That is, we have

$$\mathscr{C}_Y \simeq \mathscr{K}(y) \oplus \mathscr{G}.$$

Then, as $\operatorname{Ext}^1_X(-,-)$ is an additive functor in each variable, we have that

$$\operatorname{Ext}^1_X(\mathscr{F}, \mathscr{C}_Y) \simeq \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{K}(y) \oplus \mathscr{G}) \simeq \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{K}(y)) \oplus \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{G}) \neq 0.$$

This is a contradiction.

Next we prove that $\mathcal{F}(Y) \subseteq {}^{\perp}\mathscr{C}_Y$. In order to do this, it is enough to show that $\mathcal{I}(Y) \subseteq {}^{\perp}\mathscr{C}_Y$, since $\mathcal{F}(Y)$ is closed under injective envelopes (i.e. $\mathcal{F}(Y)$ consists precisely of all subobjects of $\mathcal{I}(Y)$) and ${}^{\perp}\mathscr{C}_Y$ is closed under subobjects by Corollary 3.26.

Let us choose $\mathscr{E} \in \mathcal{I}(Y)$ and compute that $\operatorname{Ext}^1_X(\mathscr{E}, \mathscr{C}_Y) = 0$. Firstly, we have

$$\begin{aligned}
\operatorname{Ext}^1_X\Big(\mathscr{E}, \mathscr{C}_Y\Big) &\simeq \operatorname{Ext}^1_X\Big(\mathscr{E}, \prod_{y \in Y} \mathscr{K}(y) \oplus \prod_{x \in X \setminus Y} \mathscr{J}(x)\Big) \\
&\simeq \operatorname{Ext}^1_X\Big(\mathscr{E}, \prod_{y \in Y} \mathscr{K}(y)\Big) \oplus \operatorname{Ext}^1_X\Big(\mathscr{E}, \prod_{x \in X \setminus Y} \mathscr{J}(x)\Big) \\
&\simeq \operatorname{Ext}^1_X\Big(\mathscr{E}, \prod_{y \in Y} \mathscr{K}(y)\Big)
\end{aligned}$$

(the sheaf $\prod_{x \in X \setminus Y} \mathscr{J}(x)$ is injective). To prove that $\operatorname{Ext}^1_X\Big(\mathscr{E}, \prod_{y \in Y} \mathscr{K}(y)\Big) = 0$, by Proposition 3.4 it is enough to check that $\operatorname{Ext}^1_X(\mathscr{E}, \mathscr{K}(y)) = 0$ for every $y \in Y$. This holds by Proposition 3.22, so the proof is complete. □

Once again, we summarize the main result of this section.

Theorem 3.28. *Let X be a Noetherian scheme and $Y \subseteq X$ be a specialization closed subset.*

(1) If X has the resolution property or is separated and the set Y satisfies $\operatorname{Ass}\mathcal{O}_X \cap Y = \emptyset$, the construction 3.19 yields a 1-cotilting quasi-coherent sheaf $\mathscr{C}_Y$ such that the associated 1-cotilting class is equal to $\mathscr{F}(Y)$.

(2) More generally, if there is a generator $\mathscr{G}$ for QCoh_X satisfying $\operatorname{Ass}\mathscr{G} \cap Y = \emptyset$, the construction 3.19 again yields a 1-cotilting quasi-coherent sheaf $\mathscr{C}_Y$ such that the associated 1-cotilting class is equal to $\mathscr{F}(Y)$.

We close the chapter with an additional corollary of Lemma 3.24, interesting by itself.

Corollary 3.29. *Let $\mathscr{G}$ be any quasi-coherent sheaf. Given a family $(y_j \mid j \in J)$ of points from Y and any $i \geq 0$, we have*

$$\operatorname{Ext}^i_X\Big((\mathscr{G}, \prod_{j\in J} \mathscr{K}(y_j)\Big) \simeq \prod_{j\in J} \operatorname{Ext}^i_X(\mathscr{G}, \mathscr{K}(y_j)).$$

In particular, for any set I we have

$$\operatorname{Ext}^i_X\Big(\mathscr{G}, \mathscr{C}_Y^{\times I}\Big) \simeq \operatorname{Ext}^i_X\Big(\mathscr{G}, \mathscr{C}_Y\Big)^{\times I}$$

if $i \geq 1$.

Proof. For any $j \in J$, the exact sequence

$$0 \longrightarrow \mathscr{K}(y_j) \xrightarrow{\alpha_{y_j}} \mathscr{I}(y_j) \xrightarrow{\beta_{y_j}} \mathscr{J}(y_j) \longrightarrow 0,$$

provides an injective coresolution of $\mathscr{K}(y_j)$.

By Lemma 3.24, the product sequence

$$0 \longrightarrow \prod_{j\in J} \mathscr{K}(y_j) \xrightarrow{\prod_{j\in J}\alpha_{y_j}} \prod_{j\in J} \mathscr{I}(y_j) \xrightarrow{\prod_{j\in J}\beta_{y_j}} \prod_{j\in J} \mathscr{J}(y_j) \longrightarrow 0$$

is also exact, and since $\prod_{j\in J} \mathscr{I}(y_j), \prod_{j\in J} \mathscr{J}(y_j)$ are injective, it provides an injective coresolution of $\prod_{j\in J} \mathscr{K}(y_j)$.

Applying $\operatorname{Hom}_X(\mathscr{G}, -)$, we obtain a complex

$$0 \longrightarrow \operatorname{Hom}_X\Big(\mathscr{G}, \prod_{j\in J} \mathscr{K}(y_j)\Big) \xrightarrow{\tilde{\alpha}} \operatorname{Hom}_X\Big(\mathscr{G}, \prod_{j\in J} \mathscr{I}(y_j)\Big) \xrightarrow{\tilde{\beta}} \operatorname{Hom}_X\Big(\mathscr{G}, \prod_{j\in J} \mathscr{J}(y_j)\Big) \longrightarrow 0,$$

where $\tilde{\alpha} = \big(\prod_{j\in J} \alpha_{y_j}\big) \circ -$, $\tilde{\beta} = \big(\prod_{j\in J} \beta_{y_j}\big) \circ -$. Since $\operatorname{Hom}_X(\mathscr{G},-)$ preserves direct products, this complex is isomorphic to

$$0 \longrightarrow \prod_{j\in J} \operatorname{Hom}_X(\mathscr{G}, \mathscr{K}(y_j)) \xrightarrow{\alpha'} \prod_{j\in J} \operatorname{Hom}_X(\mathscr{G}, \mathscr{I}(y_j)) \xrightarrow{\beta'} \prod_{j\in J} \operatorname{Hom}_X(\mathscr{G}, \mathscr{J}(y_j)) \longrightarrow 0,$$

with $\alpha' = \prod_{j\in J} (\alpha_{y_j} \circ -)$, $\beta' = \prod_{j\in J} (\beta_{y_j} \circ -)$(that is, $\prod_{j\in J} \operatorname{Hom}_X(\mathscr{G}, \alpha_{y_j})$, $\prod_{j\in J} \operatorname{Hom}_X(\mathscr{G}, \beta_{y_j})$, resp.).

By taking the i-th homology of the second (deleted) complex instead of the first one, the first claim follows.

The second part follows from the first one by observing that (for $i \geq 1$)

$$\begin{aligned}\mathrm{Ext}^i_X(\mathscr{G},\mathscr{C}_Y) &= \mathrm{Ext}^i_X(\mathscr{G},\mathscr{K}_Y \oplus \mathscr{J}_Y) \simeq \mathrm{Ext}^i_X(\mathscr{G},\mathscr{K}_Y) \oplus \mathrm{Ext}^i_X(\mathscr{G},\mathscr{J}_Y)\\ &= \mathrm{Ext}^i_X\left(\mathscr{G},\prod_{y\in Y}\mathscr{K}(y)\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ext}^i_X\left(\mathscr{G},(\mathscr{C}_Y)^{\times I}\right) &= \mathrm{Ext}^i_X\left(\mathscr{G},(\mathscr{K}_Y \oplus \mathscr{J}_Y)^{\times I}\right)\\ &= \mathrm{Ext}^i_X\left(\mathscr{G},\left(\mathscr{K}_Y^{\times I} \oplus \mathscr{J}_Y^{\times I}\right)\right)\\ &\simeq \mathrm{Ext}^i_X\left(\mathscr{G},\mathscr{K}_Y^{\times I}\right) \oplus \mathrm{Ext}^i_X\left(\mathscr{G},\mathscr{J}_Y^{\times I}\right)\\ &= \mathrm{Ext}^i_X\left(\mathscr{G},\left(\prod_{y\in Y}\mathscr{K}(y)\right)^{\times I}\right)\end{aligned}$$

since $\mathscr{J}_Y$, $\mathscr{J}_Y^{\times I}$ are injective, hence the respective summands vanish. Using the first part it easily follows that both $\mathrm{Ext}^i_X(\mathscr{G},\mathscr{C}_Y)^{\times I}$ and $\mathrm{Ext}^i_X(\mathscr{G},\mathscr{C}_Y^{\times I})$ are isomorphic to

$$\mathrm{Ext}^i_X\left(\mathscr{G},\prod_{y\in Y}\mathscr{K}(y)^{\times I}\right) \simeq \prod_{y\in Y}\mathrm{Ext}^i_X\left(\mathscr{G},\mathscr{K}(y)\right)^{\times I}.$$

□

Appendix

In this chapter, additional supporting facts are presented, mostly without proofs or with just sketches of proofs. In each section we refer the reader to appropriate literature.

A Sheaf on a topological space

The general references for this section and the next one are [GW10], [Har77].

Before we begin, let us introduce the following terminology regarding non-Hausdorff topological spaces:

(1) A topological space X is called *quasi-compact* if every open cover admits a finite subcover. That is, a quasi-compact space is a space that enjoys the compactness property but is not necessarily Hausdorff.

(2) If a point $x \in X$ is in the topological closure of a point $y \in X$, we call x *specialization* of y, and the point y *generization* of x.

(3) A topological space X is *Noetherian* if every strictly descending chain of closed subsets terminates (i.e. is finite).

Definition A.1. Let X be a topological space and $\mathcal{C}$ be a complete and co-complete concrete category. Denote by Ouv_X the category of open subsets of X (i.e. objects are open subsets of X and morphisms are set-theoretic inclusions).

A *presheaf with values in* $\mathcal{C}$ is a functor $\mathscr{F} : \mathsf{Ouv}_X^{\mathrm{op}} \to \mathcal{C}$.

If $U \subseteq X$ is an open set and $s \in \mathscr{F}(U)$, s is called a *section of* $\mathscr{F}$ *over* U.

If $V \subseteq U$ are two open subsets in X, we will denote the image of the inclusion under $\mathscr{F}$ by res^U_V, and call it *restriction of* $\mathscr{F}$ *from* U *to* V. That is, the restrictions of $\mathscr{F}$ satisfy

$$\mathrm{res}^U_W = \mathrm{res}^V_W \circ \mathrm{res}^U_V, \quad \mathrm{res}^U_U = 1_{\mathscr{F}(U)}, \quad W \subseteq V \subseteq U \subseteq X \text{ open}.$$

If there is a danger of confusion (for example, when one considers several presheaves on X), we will denote the restrictions $\mathscr{F}^U_V$ instead. If $\mathcal{C}$ is a concrete category and $s \in \mathscr{F}(U)$, we may occasionally write $s \restriction_V$ instead of $\mathrm{res}^U_V(s)$.

A *morphism of presheaves* $\mathscr{F}, \mathscr{G}$ is a natural transformation $f : \mathscr{F} \Rightarrow \mathscr{G}$. That is, it is given as a collection of morphisms (in $\mathcal{C}$)

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U), \;\; U \subseteq X \text{ open},$$

such that for every $V \subseteq U \subseteq X$ open, the square

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \big\downarrow{\scriptstyle \mathscr{F}^U_V} & & \big\downarrow{\scriptstyle \mathscr{G}^U_V} \\ \mathscr{F}(V) & \xrightarrow{f_V} & \mathscr{G}(V) \end{array}$$

is commutative.

A presheaf $\mathscr{F} : \mathsf{Ouv}_X^{\mathrm{op}} \to \mathcal{C}$ on X is called a *sheaf* if for any open subset $U \subseteq X$ and any its open cover $U = \bigcup_{i \in I} U_i$, the sequence

$$\mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \overset{\beta}{\underset{\beta'}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j)$$

is an equalizer sequence, where

$$\begin{aligned} \alpha : s &\mapsto (s \restriction_{U_i})_{i\in I} \ , \\ \beta : (s_i)_{i\in I} &\mapsto (s_i \restriction_{U_i\cap U_j})_{(i,j)\in I\times I} \ , \\ \beta' : (s_i)_{i\in I} &\mapsto (s_j \restriction_{U_i\cap U_j})_{(i,j)\in I\times I} \ . \end{aligned}$$

A morphism of sheaves $\mathscr{F}, \mathscr{G}$ is just a morphism of presheaves. That is, the category of sheaves with values in $\mathcal{C}$ is a full subcategory of the category of presheaves with values in $\mathcal{C}$.

Denote by $\mathcal{C}_X^{\mathrm{pre}}$ the category of $\mathcal{C}$-valued presheaves on X and by $\mathcal{C}_X$ the category of $\mathcal{C}$-valued sheaves on X.

Remark A.2. The axiom in Definition A.1 imposed on a presheaf in order to be a sheaf is called the *gluing axiom*. There are several equivalent restatement of the condition, each of them useful in some contexts. We mention additional two such reformulations.

The first is geometrically motivated and justifies the name "gluing axiom":

For any open set $U \subseteq X$ and any open cover $U = \bigcup_{i\in I} U_i$, given a collection $s_i \in \mathscr{F}(U_i)$, $i \in I$ such that it is compatible in the sense that

$$\forall i, j \in I: \ s_i \restriction_{U_i\cap U_j} = s_j \restriction_{U_i\cap U_j} \ ,$$

there is a unique $s \in \mathscr{F}(U)$ such that

$$\forall i \in I : s \restriction_{U_i} = s_i.$$

That is, a sheaf is a presheaf such that collections of sections over smaller open sets U_i uniquely "glue together" to a section over the union $\bigcup_{i\in I} U_i$ provided that they "agree on overlaps" – that is, if they pairwise restrict to the same section over the relevant intersections.

The second reformulation is as follows.

For every open set $U \subseteq X$ and every open cover $U = \bigcup_{i\in I} U_i$, consider the diagram $\mathcal{D}$ consisting of all the restrictions

$$\mathscr{F}(U_i) \overset{\mathrm{res}^{U_i}_{U_i\cap U_j}}{\longrightarrow} \mathscr{F}(U_i \cap U_j), \ \ i, j \in I.$$

Then $\mathscr{F}(U)$ together with the restrictions $\mathrm{res}^U_{U_i}, \mathrm{res}^U_{U_i\cap U_j}$ $i, j \in I$ is the limit cone for $\mathcal{D}$.

Note that this version of the gluing axiom includes instructions how to define sections and restrictions over larger open sets (U) out of the same data on smaller open sets (U_i's and their intersections). This is useful e.g. for constructing a sheaf from its prescription on a base of open sets.

There are two instances of the concrete category $\mathcal{C}$ that are of our interest. Namely, if $\mathscr{F}$ is an Ab-valued (pre)sheaf, we talk about a *(pre)sheaf of Abelian groups*. Similarly, a CRing-valued[3] (pre)sheaf is called *(pre)sheaf of commutative rings*.

Remark A.3. The full category $\mathcal{C}_X$ of $\mathcal{C}$-valued sheaves on X is in fact a reflective subcategory of $\mathcal{C}_X^{\text{pre}}$. The reflector is usually called a *sheafification functor*, denoted by

$$(-)^{\text{sh}} : \mathcal{C}_X^{\text{pre}} \longrightarrow \mathcal{C}_X \; .$$

See e.g. [GW10, (2.7)] for its explicit description.

Remark A.4.

1. Note that the gluing axiom for the situation when $U = \emptyset$ and the cover of U is an empty cover translates to

$$\mathscr{F}(\emptyset) \text{ is the terminal object of } \mathcal{C}.$$

2. If $X = \{x\}$, the above condition is equivalent to the gluing axiom. In fact, in this case the category $\mathcal{C}_X^{\text{pre}}$ is clearly equivalent to the category of arrows $\mathcal{C}^{\to}$, and the category $\mathcal{C}_X$ is equivalent to $\mathcal{C}$ itself.

Definition A.5. Let $\mathcal{C}$ be a complete and co-complete concrete category and X be a topological space. For an open set $U \subseteq X$, the *functor of sections on U* $\Gamma(U, -)$ is defined as follows.

1. If $\mathscr{F} \in \mathcal{C}_X$ is a $\mathcal{C}$-valued sheaf, put

$$\Gamma(U, \mathscr{F}) = \mathscr{F}(U).$$

2. If $f : \mathscr{F} \to \mathscr{G}$ is a morphisms of $\mathcal{C}$-valued sheaves, put

$$\Gamma(U, f) = f_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

(that is, the U-th component of the natural transformation f).

If $U = X$, the functor $\Gamma(X, -)$ is called the *global sections functor*.

Definition A.6. Let $\mathcal{C}$ be a complete and co-complete concrete category and $\pi : X \to Y$ be a continuous map between topological spaces X, Y.

Define the functor $\pi_* : \mathcal{C}_X \to \mathcal{C}_Y$ as follows.

(1) Given a $\mathcal{C}$-valued sheaf $\mathscr{F}$ on X, for an open set $U \subseteq Y$ put

$$\pi_*\mathscr{F}(U) := \mathscr{F}(\pi^{-1}(U)) \; ,$$

and for a pair of open subsets $V \subseteq U \subseteq X$, define the restriction from U to V as

$$(\pi_*\mathscr{F})^U_V := \mathscr{F}^{\pi^{-1}(U)}_{\pi^{-1}(V)} : \mathscr{F}(\pi^{-1}(U)) \longrightarrow \mathscr{F}(\pi^{-1}(V)) \; .$$

This is indeed a sheaf on Y, called the *pushforward* (or *direct image*) of $\mathscr{F}$ along the map π.

[3] CRing denotes the category of commutative rings and ring homomorphisms.

(2) Similarly, if $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X, define a morphism of sheaves on X $\pi_* f : \pi_*\mathscr{F} \to \pi_*\mathscr{G}$ by

$$(\pi_* f)_U := f_{\pi^{-1}(U)} : \mathscr{F}(\pi^{-1}(U)) \longrightarrow \mathscr{G}(\pi^{-1}(U)), \ \ U \subseteq Y \text{ open}.$$

The functor $\pi_* : \mathcal{C}_X \to \mathcal{C}_Y$ is called the *pushforward functor* (or *direct image functor*) *along* π.

We now define a functor $\pi^{-1} : \mathcal{C}_Y \to \mathcal{C}_X$ going in the opposite direction.

Definition A.7. Let $\mathcal{C}$ be a complete and co-complete concrete category and $\pi : X \to Y$ be a continuous map between topological spaces X, Y.

(1) Given a $\mathcal{C}$-valued sheaf $\mathscr{F}$ on Y, for an open set $U \subseteq Y$ put

$$\pi'\mathscr{F}(U) := \varinjlim_{W} \mathscr{F}(W)\ ,$$

where W goes over all open sets $W \subseteq Y$ containing $\pi(U)$, and all the possible restrictions between such W's. Given open subsets $V \subseteq U \subseteq X$, define the restriction from U to V as the induced morphism $\pi'\mathscr{F}(U) \to \pi'\mathscr{F}(V)$ from the universal property of the direct limit $\pi'\mathscr{F}(U) = \varinjlim_W \mathscr{F}(W)$ (note that $\pi(V) \subseteq \pi(U)$, hence if W contains $\pi(U)$, then it contains $\pi(V)$). The resulting collection $\pi'\mathscr{F}$ is then easily seen to be a presheaf on X.

(2) If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on Y, define a morphism of presheaves on X $\pi' f : \pi'\mathscr{F} \to \pi'\mathscr{G}$ by the universal property of direct limits again. More precisely, if $U \subseteq X$ is an open subset, the collection of the compositions

$$\mathscr{F}(W) \xrightarrow{f_W} \mathscr{G}(W) \longrightarrow \pi'\mathscr{G}(U),$$

where W runs over all $W \subseteq Y$ open containing $\pi(U)$ (the second map comes from the universal co-cone of the direct limit) gives a co-cone for the directed system defining $\pi'\mathscr{F}(U)$, and so they induce a map

$$(\pi' f)_U : \pi'\mathscr{F}(U) \longrightarrow \pi'\mathscr{G}(U).$$

the resulting collection $\pi' f : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves. Altogether, $\pi' : \mathcal{C}_Y \to \mathcal{C}_X^{\mathrm{pre}}$ is a functor, called the *presheaf pullback functor along* π.

Define the functor π^{-1} by

$$\pi^{-1} = (-)^{sh} \circ \pi'.$$

That is, $\pi^{-1} : \mathcal{C}_Y \to \mathcal{C}_X$ which assigns to a $\mathcal{C}$-valued sheaf $\mathscr{F}$ the sheafification of presheaf the presheaf pullback of $\mathscr{F}$ along π. The functor π^{-1} is called the *pullback functor along* π.

Theorem A.8. *Let $\mathcal{C}$ be a complete and co-complete concrete category and $\pi : X \to Y$ be a continuous map between topological spaces X, Y. Then the pair of functors (π^{-1}, π_*) is an adjoint pair. That is, π^{-1} is the left adjoint to π_* (and thus, π_* is the right adjoint to π^{-1}).*

There are two situations when the pullback functor is especially important for our purposes.

Definition A.9. Let $\mathcal{C}$ be a complete and co-complete concrete category and X be a topological space. Let $\mathscr{F}$ be a $\mathcal{C}$-valued sheaf on X.

(1) If $x \in X$ is a point and $i_x : \{x\} \to X$ the inclusion of $\{x\}$ into X, from Remark A.4 it follows easily that the presheaf $i'_x\mathscr{F}$ is already a sheaf. Thus, we have $i_x^{-1}\mathscr{F} = i'_x\mathscr{F}$ and this sheaf can be further identified with the object $i_x^{-1}\mathscr{F}(\{x\})$, which is given by

$$i_x^{-1}\mathscr{F}(\{x\}) = \varinjlim_{\substack{x \in U \subseteq X \\ U \text{ open}}} \mathscr{F}(U).$$

We call this object of $\mathcal{C}$ (or a sheaf over $\{x\}$) the *stalk of $\mathscr{F}$ at the point x* and denote it by $\mathscr{F}_x$. Given an open neighbourhood U of x there is a map $\mathscr{F}(U) \to \mathscr{F}_x$ obtained from the co-cone of the above direct limit. If $s \in \mathscr{F}(U)$ is a section, call the image s_x of s under this map a *germ of s at x*.

(2) Consider $U \subseteq X$ an open subset and $i_U : U \to X$ the open embedding of U into X. Since for any open subset $W \subseteq U$, the set $i_U(W) = W$ is open in X, it follows that

$$\forall W \subseteq U \text{ open:} \quad i_U^{-1}\mathscr{F}(W) = \mathscr{F}(W)$$

(and the restrictions of $i_U^{-1}\mathscr{F}$ agree with agree with the restrictions of $\mathscr{F}$ as well). We call the sheaf $i_U^{-1}\mathscr{F}$ the *restriction sheaf of $\mathscr{F}$ to U* (or *sheaf $\mathscr{F}$ restricted to U*) and denote it by $\mathscr{F}_{\restriction U}$.

B Locally ringed spaces and schemes

Definition B.1. A *locally ringed space* $(X, \mathcal{O}_X)$ consists of a topological space X together with a sheaf of commutative rings $\mathcal{O}_X$ on X such that for each point $x \in X$, the stalk $\mathcal{O}_{X,x} = (\mathcal{O}_X)_x$ is a local ring. We call X the *underlying (topological) space* and $\mathcal{O}_X$ the *structure sheaf* of the locally ringed space $(X, \mathcal{O}_X)$.

Given two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a morphism of locally ringed spaces $(\pi, \pi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of

(1) a continuous map $\pi : X \to Y$, and

(2) a morphism of sheaves of rings $\pi^\sharp : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$,

such that for every point $x \in X$, the induced ring homomorphism on stalks

$$(\pi^\sharp)_x : \mathcal{O}_{Y,\pi(x)} \longrightarrow (\pi_*\mathcal{O}_X)_{\pi(x)} = \mathcal{O}_{X,x}$$

takes the maximal ideal of $\mathcal{O}_{Y,\pi(x)}$ onto the maximal ideal of $\mathcal{O}_{X,x}$.

Note that if $(X, \mathcal{O}_X)$ is a locally ringed space and $U \subseteq X$ is an open subset, U has a natural structure of locally ringed space given by restriction of the structure sheaf $\mathcal{O}_X$ to U. In other words, $(U, \mathcal{O}_{X\restriction U})$ is a locally ringed space as well.

Construction B.2. Let R be a commutative ring. We construct a locally ringed space $\operatorname{Spec} R = (X, \mathcal{O}_X)$ as follows:

1. The topological space $X = \operatorname{Spec} R$ consists of all prime ideals of R endowed with the Zariski topology. That is, the closed sets are the sets of the form
$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\},$$
where I runs over the set of all ideals of R. Alternatively, the topology is given by a basis of open sets $\{D_f \mid f \in R\}$, where
$$D_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\} = \operatorname{Spec} R \setminus V((f)).$$
Note that $D_{fg} = D_f \cap D_g$. Call the open sets of the form D_f *distinguished open sets* in $\operatorname{Spec} R$.

2. When $U = D_f$ is a distinguished open set, put
$$\mathcal{O}_X(U) = R_f,$$
the localisation of R with respect to the multiplicative set $S_f = \{f^k \mid k \in \mathbb{N}\}$ (in particular, if $U = D_1 = X$, then $\mathcal{O}_X(X) = R$). Note that by the correspondence theorem for ideals under localization, there is a bijection between $\operatorname{Spec} R_f$ and U. Also observe that $D_f = D_g$ for some $f, g \in R$ if and only if $\sqrt{(f)} = \sqrt{(g)}$. In that case, there is a canonical isomorphism $R_f \simeq R_g$, i.e. the unique isomorphism making the diagram
$$\begin{array}{ccc} & R & \\ {}^{\mathrm{loc}_f}\swarrow & & \searrow^{\mathrm{loc}_g} \\ R_f & \xrightarrow{\ \simeq\ } & R_g \end{array}$$
commutative (here $\mathrm{loc}_f, \mathrm{loc}_g$ are the localization morphisms). That is, the ring $\mathcal{O}_X(U)$ depends on the distinguished open set U only, and not on its representation by an element $f \in R$.

3. If D_f, D_g is a pair of distinguished open sets such that $D_f \subseteq D_g$, we have $D_f = D_f \cap D_g = D_{gf}$, hence $R_f \simeq R_{gf} \simeq (R_g)_f$ can be treated as the localization of R_g whith respect to the multiplicative set $\{\left(\frac{f}{1}\right)^k \mid k \in \mathbb{N}\}$. Put
$$\left(\mathcal{O}_X(D_g) \xrightarrow{\mathrm{res}^{D_g}_{D_f}} \mathcal{O}_X(D_f)\right) := \left(R_g \xrightarrow{\mathrm{loc}_f} R_{gf}\right).$$
(Note that this is again uniquely determined up to a canonical isomorphism.)

4. Given an open set $U \subseteq X$, define $\mathcal{O}_X(U)$ according to the gluing axiom as described in Remark A.2. That is, consider all open sets $U_i,\ i \in I$ distinguished in X and contained in U, the rings $\mathcal{O}_X(U_i)$ and all restriction among them (these were defined in the previous two steps). Then put
$$\mathcal{O}_X(U) := \lim_{i \in I} \mathcal{O}_X(U_i).$$

For any pair of open sets $V \subseteq U \subseteq X$, from the universal property of the limit (defining $\mathcal{O}_X(V)$) we obtain a unique morphism

$$\mathrm{res}^U_V : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V) \ .$$

The result is a locally ringed space – it is not difficult to verify that the stalk $\mathcal{O}_{X,x}$ at the point x corresponding to a prime ideal $\mathfrak{p} \subseteq R$ is isomorphic to the localization at the prime ideal $R_\mathfrak{p}$, and the canonical map to the stalk $\mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ is the localization morphism $R \to R_\mathfrak{p}$.

Definition B.3. A locally ringed space $(X, \mathcal{O}_X)$ is called an *affine scheme* if it is isomorphic to a locally ringed space of the form $\operatorname{Spec} R$ for some commutative ring R.

An open subset U of a locally ringed space $(X, \mathcal{O}_X)$ is called *affine* if the induced locally ringed space $(U, \mathcal{O}_{X\restriction U})$ is an affine scheme.

A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* if X can be covered by affine open sets.

Remark B.4. Note that if $U = \operatorname{Spec} R$ is an affine scheme and $f \in \mathcal{O}_U(U)$ is a function, then the open subscheme $(D_f, \mathcal{O}_{U\restriction D_f})$ is again affine. Consequently, the set of all affine open sets of a scheme X forms a basis for the topology on X.

Moreover, observe that if U, V are affine open subsets of a scheme X, the intersection $U \cap V$ can be covered by affine open sets that are distinguished both in U and in V. This can be inferred as follows: given a point $x \in U \cap V$, choose $f \in \mathcal{O}_X(U)$ so that we have

$$x \in D_f \subseteq U \cap V.$$

Now choose $g \in \mathcal{O}_X(V)$ such that

$$x \in D_g \subseteq D_f.$$

Consider the restriction $g' = g\mathrm{res}_{D_f}$. Interpreting $\mathcal{O}_X(D_f)$ as $\mathcal{O}_X(U)_f$, g' is of the form $g' = g''/f^k$ for some $g'' \in \mathcal{O}_X(U)$ and some k. Now it is enough to observe that

$$D_g = D_{g''},$$

ie the above neighbourhood of x is distinguished both in U and in V.

Notation B.5. Let X be a scheme. We use the following notation:

(1) Given a point $x \in X$ and an affine open set U containing x, denote the prime ideal of $\mathcal{O}_X(U)$ corresonding to x by $\mathfrak{p}_x$ or $\mathfrak{q}_x$. Conversely, the point of x corresponding to a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_X(U)$ is denoted by $[\mathfrak{p}]$.

(2) Given a point $x \in X$, denote the maximal ideal of the stalk $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$. The field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is called *the residue field at* x and denoted by $\kappa(x)$.

Remark B.4 leads to a notion of "affine localness", intrinsic to schemes. The following (meta-)lemma makes this notion precise.

Lemma B.6 (Affine Communication Lemma)**.** *Let (P) be a property that can be stated about an affine open set of a scheme X. Suppose that the following holds:*

(1) If an affine open set U enjoys the property (P) and V is a distinguished open set in U, then V has the property (P).

(2) If an affine open set U is covered by a collection of its distinguished open sets

$$U = \bigcup_{i \in I} V_i$$

and each V_i has the property (P), then U has the property (P).

Suppose that there is an affine open cover $X = \bigcup_{i \in I} U_i$ by affine open sets such that each U_i has the property (P). Then every affine open subset of X enjoys the property (P).

Proof. If U is an affine open subset of X, we have

$$U = X \cap U = \bigcup_{i \in I} (U_i \cap U).$$

For each i, cover $U_i \cap U$ by affine open sets

$$U_i \cap U = \bigcup_j V_{i,j}$$

distinguished both in U and in U_i. The sets $V_{i,j}$ enjoy the property (P) by assertion (1), since they are distinguished in U_i. On the other hand, all the sets $V_{i,j}$ are distinguished in U and they cover U, hence U has the property (P) by the assumption (2). □

We use this affine localness to define locally Noetherian schemes in particular. Recall the fact that the property "being a Noetherian commutative ring" satisfies the assumptions (1) and (2) of Lemma B.6. Algebraically, this means that

(1) Whenever R is a commutative Noetherian ring and $f \in R$, the ring R_f is again Noetherian, and

(2) given $f_1, f_2, \ldots, f_k \in R$ such that $(f_1, f_2, \ldots, f_k) = R$[4] and each R_{f_i} is Noetherian, then R is Noetherian.

Definition B.7. A scheme X is called *locally Noetherian* if either one of the two following conditions holds:

(1) There is an affine open cover $X = \bigcup_i U_i$ such that each of the rings $\mathcal{O}_X(U_i)$ is Noetherian.

(2) For every affine open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is Noetherian.

A scheme X is called *Noetherian* if it is locally Noetherian and quasi-compact.

Remark B.8. It is easily seen that the underlying space X of a Noetherian scheme is Noetherian. Note that the fact that X is a Noetherian topological space implies that every subset $Y \subseteq X$ is quasi-compact.

[4] This is just a restatement of $\operatorname{Spec} R = \bigcup_{i=1}^k D_{f_i}$.

Example B.9. As an example of a non-affine scheme, let us now describe the construction of a projective line $X = \mathbb{P}^1_k$ where k is an algebraically closed field.

Consider two affine schemes $\operatorname{Spec} k[x_0]$, $\operatorname{Spec} k[x_1]$ (that is, two copies of the scheme $\mathbb{A}^1_k$, an affine line over k) and their affine open subsets

$$U = D_{x_0} = \operatorname{Spec} k[x_0] \setminus \{[(x_0)]\} \subseteq \operatorname{Spec} k[x_0],$$
$$V = D_{x_1} = \operatorname{Spec} k[x_1] \setminus \{[(x_1)]\} \subseteq \operatorname{Spec} k[x_1].$$

By Hilbert's Nullstellensatz, we have

$$D_{x_i} = \{[(0)]\} \cup \{[(x_1 - a)] \mid a \in k, a \neq 0\}.$$

The space of points of X is the topological space obtained from the union of spaces $\operatorname{Spec} k[x_0]$, $\operatorname{Spec} k[x_1]$ after identification of U and V as follows:

$$[(0)] = [(0)] \ \text{(the generic points of } \operatorname{Spec} k[x_0], \operatorname{Spec} k[x_1] \text{ are identified)},$$

$$[(x_0 - a)] = [(x_1 - a^{-1})], \ \ a \in k, a \neq 0 \ .$$

That is, the space is obtained as the "adjunction space" $\operatorname{Spec} k[x_0] \amalg_\varphi \operatorname{Spec} k[x_0]$, where $\varphi : U \to V$ is the homeomorphism given by $\varphi([(0)]) = [(0)]$ and $\varphi([(x_0 - a)]) = [(x_1 - a^{-1})]$ for each nonzero element $a \in k$. Denote the open subset of X consisting of all the points originating from U, V by W.

To prescribe the structure sheaf on X, it is enough to proclaim the sets $U' = W \cup \{[(x_0)]\}$, $V' = W \cup \{[(x_1)]\}$ and W affine open, and set

$$\mathcal{O}_X(U') = k[x_0], \quad \mathcal{O}_X(V') = k[x_1], \quad \mathcal{O}_X(W) = k[y,y^{-1}],$$

$$\operatorname{res}^{U'}_W : f(x_0) \mapsto f(y), \ \ f(x_0) \in k[x_0],$$
$$\operatorname{res}^{V'}_W : g(x_1) \mapsto g(y^{-1}), \ \ g(x_1) \in k[x_1]$$

(where y denotes a new indeterminate). In other words, the maps $\operatorname{res}^{U'}_W, \operatorname{res}^{V'}_W$ are localizations with respect to x_0, x_1, resp., and in the common localized ring, x_0 and x_1 are inverse to each other. The rest of the information is uniquely (and indeed, correctly) obtained using the procedure analogous to Construcion B.2.

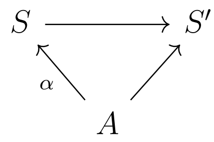
C Preenvelopes, precovers and injectives in an Abelian category

The definitions from this section and their further developement can be found e.g. in [GT12] or [EJ11].

Definition C.1. Let $\mathcal{A}$ be an Abelian category and $\mathcal{S} \subseteq \mathcal{A}$ a class of objects.

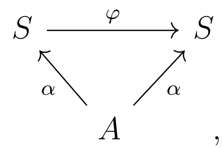
(1) An $\mathcal{S}$*-preenvelope* of an object $A \in \mathcal{A}$ is a morphism $\alpha : A \to S$ with $S \in \mathcal{S}$ such that the map $- \circ \alpha : \operatorname{Hom}_{\mathcal{A}}(S, S') \to \operatorname{Hom}_{\mathcal{A}}(A, S')$ is surjective for every $S' \in \mathcal{S}$.

That is, given any morphism $A \to S'$ with $S' \in \mathcal{S}$, there is a morphism $S \to S'$ such that the diagram



is commutative.

(2) An $\mathcal{S}$-preenvelope $\alpha : A \to S$ is called an $\mathcal{S}$-*envelope* if additionally, whenever we have a commutative diagram of the form



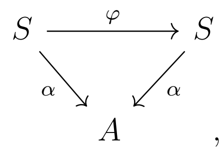
the morphism φ is an automorphism of S.

(3) The class $\mathcal{S}$ is said to be *preenveloping* if every object $A \in \mathcal{A}$ admits an $\mathcal{S}$-preenvelope; $\mathcal{S}$ is called *enveloping* if every object $A \in \mathcal{A}$ admits an $\mathcal{S}$-envelope.

Definition C.2. Let $\mathcal{A}$ be an Abelian category and $\mathcal{S} \subseteq \mathcal{A}$ a class of objects.

(1) An $\mathcal{S}$-*precover* of an object $A \in \mathcal{A}$ is a morphism $\alpha : S \to A$ with $S \in \mathcal{S}$ such that the map $\alpha \circ - : \mathrm{Hom}_{\mathcal{A}}(S', S) \to \mathrm{Hom}_{\mathcal{A}}(S', A)$ is surjective for every $S' \in \mathcal{S}$.

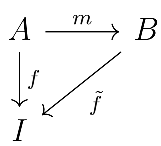
(2) An $\mathcal{S}$-precover $\alpha : S \to A$ is called an $\mathcal{S}$-*cover* if additionally, whenever we have a commutative diagram of the form



the morphism φ is an automorphism of S.

(3) The class $\mathcal{S}$ is said to be *precovering* if every object $A \in \mathcal{A}$ admits an $\mathcal{S}$-precover; $\mathcal{S}$ is called *covering* if every object $A \in \mathcal{A}$ admits an $\mathcal{S}$-cover.

Definition C.3. Let $\mathcal{A}$ be an Abelian category. An object $I \in \mathcal{A}$ is called *injective* if for every monomorphism $m : A \to B$ in $\mathcal{A}$ and every morphism $f : A \to I$ there is a morphism $\tilde{f} : B \to I$ such that the following diagram commutes:



A monomorphism $m : A \to C$ is called *essential* if for every nonzero monomorphism $n : B \to C$, the pullback of m and n

$$\begin{array}{ccc} A & \xrightarrow{\quad m \quad} & C \\ \uparrow & & {\scriptstyle n}\uparrow \\ A \cap B & \longrightarrow & B \end{array}$$

is nonzero.

Let $A \in \mathcal{A}$ be an object. An *injective hull of* A is an injective object $E(A) \in \mathcal{A}$ together with an essential monomorphism $m : A \to E(A)$.

We say that $\mathcal{A}$ *has enough injectives* if every object can be embedded into an injective object.

We say that $\mathcal{A}$ *has injective hulls* if for every object of $\mathcal{A}$, there exists an injective hull.

Proposition C.4. *Let $\mathcal{A}$ be an Abelian category. Denote by $\mathcal{I}$ the class of all injective objects in $\mathcal{A}$ and by $\mathcal{P}$ the class of all projective objects in $\mathcal{A}$.*

(1) Assume that $\mathcal{A}$ has enough injectives. Then the class $\mathcal{I}$ is preenveloping. In fact, a morphism $\iota : A \to E$ with E injective is an $\mathcal{I}$-preenvelope if and only if ι is monic.

(2) Dually, assume that $\mathcal{A}$ has enough projectives. The class $\mathcal{P}$ is precovering. More precisely, a morphism $P \to A$ with $p \in \mathcal{P}$ is a $\mathcal{P}$-precover if and only if it is epic.

Proof. Let us prove (1) only, since (2) is obtained by dualization.

Consider an object $A \in \mathcal{A}$ and a monomorphism $\iota : A \hookrightarrow E$ with E injective. If E' is another injective object and $\alpha : A \to E'$ morphism, by the injectivity of E' it follows that ι can be extended so that we have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle \iota}\nwarrow & & \nearrow{\scriptstyle \alpha} \\ & A & \end{array}$$

Thus, the preenvelope property of ι holds.

Conversely, assume that $\iota : A \to E$ is an $\mathcal{I}$-preenvelope. Since there are enough injectives, consider a monomorphism $\alpha : A \hookrightarrow E'$ with E' injective. By the $\mathcal{I}$-preenvelope property of ι there is a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle \iota}\nwarrow & & \nearrow{\scriptstyle \alpha} \\ & A & \end{array}$$

and since α is monic, it follows that ι is monic as well.

□

D Yoneda Ext and derived Hom functors

A very thorough reference of the results presented here is given in the Mitchell's book [Mit65, Chapter 7].

Definition D.1. Let $\mathcal{A}$ be an Abelian category and n be an integer.

For a pair of objects $A, B \in \mathcal{A}$, an *n-fold extension of A by B* (or an *n-extension of A by B*) is an exact sequence of the form

$$(\varepsilon) \qquad 0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow A \longrightarrow 0\,.$$

Given two $n-$fold extensions of A by B (ε) and (ε'), we write $(\varepsilon) \sim (\varepsilon')$ if there is a commutative diagram of the form

$$\begin{array}{lccccccccccccc}
(\varepsilon) & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & & & \downarrow & & \| & & \\
(\varepsilon') & 0 & \longrightarrow & B & \longrightarrow & X'_n & \longrightarrow & \cdots & \longrightarrow & X'_1 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

and denote by $\approx$ the equivalence generated by $\sim$. If $n = 1$ and

$$(\varepsilon) \qquad 0 \longrightarrow B \xrightarrow{\alpha} X \xrightarrow{\beta} A \longrightarrow 0$$

$$(\varepsilon') \qquad 0 \longrightarrow B \xrightarrow{\alpha'} X' \xrightarrow{\beta'} A \longrightarrow 0$$

is a pair of extensions, consider the pullback P of β and β'. P (together with the pair of associated canonical morphisms γ, γ') induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & A & \longrightarrow & 0 \\
 & & \| & & \gamma\uparrow & & \beta'\uparrow & & \\
0 & \longrightarrow & B & \xrightarrow{\delta'} & P & \xrightarrow{\gamma'} & X' & \longrightarrow & 0 \\
 & & & & \delta\uparrow & & \alpha'\uparrow & & \\
 & & & & B & = & B & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Put $Y = \operatorname{Coker}(\delta - \delta')$ and denote the cokernel morphism by π. Then it is easy to see that we obtain an exact sequence

$$(\varepsilon) + (\varepsilon') \qquad 0 \longrightarrow B \xrightarrow{\pi\delta = \pi\delta'} Y \xrightarrow{\gamma\beta = \gamma'\beta'} A \longrightarrow 0\,.$$

We call the result the *Baer sum of* (ε) *and* (ε').

Suppose $n \geq 2$. Consider a pair of n-extensions

$$(\varepsilon) \qquad 0 \longrightarrow B \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow A \longrightarrow 0$$

$$(\varepsilon') \qquad 0 \longrightarrow B \longrightarrow X'_n \longrightarrow \cdots \longrightarrow X'_1 \longrightarrow A \longrightarrow 0$$

Let Z be the pushout of $B \to X_n$ and $B \to X'_n$ and Y be the quotient of the pullback of $X_1 \to A$ and $X'_1 \to A$, as above. Define then $(\varepsilon) + (\varepsilon')$ as the n-extension

$$0 \longrightarrow B \longrightarrow Z' \longrightarrow X_{n-1} \oplus X'_{n-1} \longrightarrow \cdots \longrightarrow X_2 \oplus X'_2 \longrightarrow Y' \longrightarrow A \longrightarrow 0\,.$$

Proposition D.2. *The addition of n-extensions is well-defined on the $\approx$-equivalence classes of n-extensions of A by B. The set of $\approx$-equivalence classes of n-extensions of A by B is an Abelian group under this addition, denoted by $\mathrm{YExt}^n_{\mathcal{A}}(A,B)$. The neutral element is represented by the split exact sequence*

$$0 \longrightarrow B \xrightarrow{\iota_B} B \oplus A \xrightarrow{\pi_A} A \longrightarrow 0$$

in case when $n = 1$, and by the n-extension

$$0 \longrightarrow B \xrightarrow{1_B} B \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow 0$$

if $n \geq 2$. More generally, any n-fold extension

$$0 \longrightarrow B \xrightarrow{\beta} X_n \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\alpha} A \longrightarrow 0$$

where β is split mono or α is split epi, represents the trivial class in $\mathrm{YExt}^n_{\mathcal{A}}(A,B)$.

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