# NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 2—March 10, 2022

Our goal today is to focus also on some exercises about idempotents, local algebras, and projective covers. We will also calculate endomorphims rings for some modules over finite-dimensional algebras.

Let $k$ be a field.

## Idempotents, local algebras, and projective covers

*Exercise* 1 (Inspired by Example 4.9, I.4 in [1]). Let us have an algebra over $k$:

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \alpha_{21} & \lambda & 0 \\ \alpha_{31} & \alpha_{32} & \lambda \end{pmatrix} ; \lambda, \alpha_{21}, \alpha_{31}, \alpha_{32} \in k \right\}$$

Show that:

(i) $B$ is indeed a well-defined algebra.

(ii) $B$ is local.

(iii) Show that $\mathsf{Mod} - B$ is equivalent to a category with objects of form $(X, \varphi, \psi)$, where $X$ is a vector space over $k$ and $\varphi, \psi$ are its linear endomorphism subject to some relations, equipped with suitable morphisms.

*Exercise* 2. Given a commutative finite-dimensional algebra over $k$, find a decomposition thereof.

*Exercise* 3. Exhibit a $k$-algebra with no non-trivivial idempotents that is not local.

*Exercise* 4 (Exercise 7 in I.6 in [1]). Show that $k[t]/(t^3)$ as a module over $k[t]$ has no projective cover.

## Case study: modules over the Kronecker algebra

Last time, we showed that the category of right modules over the Kronecker algebra:

$$K_2 = \begin{pmatrix} k & 0 \\ k \oplus k & k \end{pmatrix}$$

is equivalent to a category with objects of form:

$$X_1 \underset{\varphi_2}{\overset{\varphi_1}{\leftleftarrows}} X_2,$$

where $X_1, X_2$ are vector spaces over $k$ and $\varphi_1, \varphi_2$ are linear maps, endowed with suitable morphims.

*Exercise* 5. Represent $K_2$ as right module over itself and its right submodules in their equivalent forms $X_1 \overset{\varphi_1}{\underset{\varphi_2}{\leftleftarrows}} X_2$.

*Exercise* 6. Find all indecomposable right projectives over $K_2$ in the form from the equivalence. Do you see any pattern in them?

*Exercise* 7. Find a decomposition into a direct sum of indecomposables of the right module over $K_2$ represented as $k^{n+1} \overset{\psi_1}{\underset{\psi_2}{\leftleftarrows}} k^n$ where $\psi_1$ is the inclusion on the first $n$ coordinates and $\psi_2$ is the inclusion on the last $n$.

*Exercise* 8. Find injective envelopes for simple right modules over $K_2$ and contrast them with the indecomposable projectives. (Hint: Use Baer's criterion and minimality.)

## References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

Feel free to reach me at `jakub.kopriva@mff.cuni.cz`. Also, I am available for short consultations on problems from the exercise sessions after previous arrangment via e-mail.