# NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 1—February 24, 2022

Our goal today is to review the notion of equivalence of categories using examples in Section I.2 in [1] and some material in Section I.3 in [1]. Aside from equivalence of categories, the other main topic which we will cover is the Wedderburn-Artin theorem.

## Equivalence of categories

**Definition 1.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. We say that it is an equivalence of categories if there exists a functor $G : \mathsf{D} \to \mathsf{C}$ such that $FG$ and $GF$ is naturally isomorphic to the identity functor on $\mathsf{C}$ and $\mathsf{D}$, respectively.*

**Definition 2.** *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. It is an equivalence of categories if and only if it is full ($F(-) : \mathrm{Hom}_{\mathsf{C}}(C, C') \to \mathrm{Hom}_{\mathsf{D}}(F(C), F(C'))$ is surjective for any $C, C' \in \mathsf{C}$), faithful ($F(-) : \mathrm{Hom}_{\mathsf{C}}(C, C') \to \mathrm{Hom}_{\mathsf{D}}(F(C), F(C'))$ is injective for any $C, C' \in \mathsf{C}$), and essentially surjective (for each $D \in \mathsf{D}$, there exists $C \in \mathsf{C}$ such that $D$ is isomorphic to $F(C)$)*

*Exercise* 1 (Modules over the Kronecker algebra; Example 2.5 in Section I.2 in [1]). Show that the category of right modules over the Kronecker algebra:

$$K_2 = \begin{pmatrix} k & 0 \\ k \oplus k & k \end{pmatrix}$$

is equivalent to a category with objects of form:

$$X_1 \overset{\varphi_1}{\underset{\varphi_2}{\leftleftarrows}} X_2,$$

where $X_1, X_2$ are vector spaces over $k$ and $\varphi_1, \varphi_2$ are linear maps, endowed with suitable morphims.

*Exercise* 2 (Modules over the algebra of polynomials in one variable; Example 2.5 in Section I.2 in [1]). Show that the category of right modules over $k[t]$ is equivalent to a category with objects of form $(X, \varphi)$, where $X$ is a vector space over $k$ and $\varphi$ is its linear endomorphism, equipped with suitable morphisms.

## Wedderburn-Artin theorem

**Definition 3** (Semisimple modules and rings)**.** *A non-zero module $M$ in* $\mathrm{Mod} - R$ *is called simple if it has no proper submodules (other than zero submodule and itself). It is called semisimple (or completely reducible) if it is a direct sum of simple $R-$modules. Finally, a ring $S$ is semisimple if it is semisimple as a module over itself.*

**Definition 4** (Socle of a module)**.** *Let $M$ in* $\mathrm{Mod} - R$ *be a module. Then,* $\mathrm{soc}(M)$ *is the submodule of $M$ generated by all simple submodules of $M$. It is referred to as the socle of $M$.*

*Exercise* 3. Prove that, for $M$ and $N$ right modules over $R$:

(i) $M$ is semisimple if and only if $\operatorname{soc}(M) = M$. (Hint: Use Zorn lemma.)

(ii) Let $f : M \to N$ be an $R-$module homomorphism. Then, $f(\operatorname{soc}(M)) \subseteq \operatorname{soc}(N)$.

(iii) Epimorphic image of a semisimple module is semisimple.

(iv) $R$ is semisimple if and only if all right modules over $R$ are semisimple.

*Exercise* 4 (Schur lemma; 3.1 in chapter I in [1]). Let $S_1, S_2 \in \operatorname{Mod} - R$, and $f : S_1 \to S_2$ be a non-zero homomorphism between them. Then, prove the following:

(i) If $S_1$ is simple, $f$ is a monomorphism.

(ii) If $S_2$ is simple, $f$ is an epimorphims.

(iii) If both are simple, $f$ is an isomorphism.

*Exercise* 5. Find a simple example of a ring $R$ (preferrably a finite-dimensional algebra over a field $k$) and an $R-$module $M$ such that $M$ is not simple, yet $\operatorname{End}_R(M)$ is a division ring.

*Exercise* 6 (Corollary 3.2 in chapter I in [1]). Let $R$ be a finite-dimensional algebra over an *algebraically closed* field $k$. Then, for every $S$, a simple module over $R$, prove that $\operatorname{End}_R(S) \cong k$.

*Exercise* 7 (Wedderburn-Artin theorem; 3.4 in chapter I in [1]). Let $R$ be a ring. Then, prove that the following propositions are equivalent:

(i) $R$ is semisimple.

(ii) $R$ is isomorphic to $M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$ for $m_1, \ldots, m_n \in \mathbb{N}$ and division rings $D_1, \ldots, D_n$.

*Exercise* 8 (Wedderburn-Artin theorem continued; 3.4 in chapter I in [1]). Let $A$ be a finite-dimensional algebra. Then, prove that the following propositions are equivalent:

(i) $A$ is semisimple.

(ii) $\operatorname{rad} A = 0$.

(Hint: Show that every right ideal in $A$ splits.)

## References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

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