– WHAT ARE THE GOALS OF [I18]?

① EXACT CATEGORIES ([BÜHLER])

- LET $\mathcal{A}$ BE ABELIAN, $\mathcal{E} \subseteq \mathcal{A}$ CLOSED UNDER EXTENSIONS
  (IF $0 \to X \to Y \to Z \to 0$ EXACT IN $\mathcal{A}$, $X, Z \in \mathcal{E} \Rightarrow Y \in \mathcal{E}$)
- CONSIDER $(\mathcal{E}, \{0 \to X \to Y \to Z \to 0 \text{ EXACT IN } \mathcal{A} \mid X, Y, Z \in \mathcal{E}\})$ IS AN EXACT CATEGORY AND EVERY SMALL EXACT CATEGORY ARISES LIKE THAT (GABRIEL-QUILLEN THEOREM).

– **DEF:** AN EXACT CATEGORY IS A PAIR $(\mathcal{E}, \mathbb{E})$ SUCH THAT

- $\mathcal{E}$ IS ADDITIVE CATEGORY,
- $\mathbb{E}$ IS A CLASS OF KERNEL-COKERNEL PAIRS $0 \to X \rightarrowtail Y \twoheadrightarrow Z \to 0$ IN $\mathcal{E}$

  ADMISSIBLE MONO, INFLATION

  ADMISSIBLE EPI, DEFLATION

  (THESE PAIRS ARE CALLED ADMISSIBLE S.E.S. OR CONFLATIONS)
- THESE DATA ARE REQUIRED TO SATISFY:

  (E0) $\forall E \in \mathcal{E}$: 1_E IS ADMISSIBLE MONO & EPI

  (E1) BOTH ADMISSIBLE MONO'S & ADMISSIBLE EPI'S ARE CLOSED UNDER COMPOSITION

  (E2) $\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[d, dashed] \\ \cdot \arrow[r, tail, dashed] & \cdot \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$ ADMISSIBLE MONO'S ARE CLOSED UNDER PUSHOUTS

  ($E2^{op}$) ADMISSIBLE EPI'S ARE CLOSED UNDER PULLBACKS.

② ALGEBRAIC TRIANGULATED CATEGORIES

- EXPL:
  - $\mathcal{B}$ ADDITIVE CATEGORY
  - $\mathcal{C}(\mathcal{B}) = \{ \cdots \to B^{-1} \to B^0 \to B^1 \to \cdots \mid$ COMPLEX OVER $\mathcal{B} \}$
  - $\mathbb{E} = \{ 0 \to B \to C \to D \to 0 \mid \forall n \in \mathbb{Z}: \; 0 \to B^n \xrightarrow{\oplus} C^n \xrightarrow{\oplus} D^n \to 0$ IS SPLIT EXACT $\}$
  - FACT: $(\mathcal{C}(\mathcal{B}), \mathbb{E})$ IS AN EXACT CATEGORY, EVEN FROBENIUS

- DEF: AN EXACT CATEGORY $\mathcal{E}$ IS **FROBENIUS** IF
  1. IT HAS ENOUGH PROJECTIVE AND INJECTIVE OBJECTS ($\forall X \in \mathcal{E} \; \exists P \twoheadrightarrow X$ WITH P PROJ., $\forall X \in \mathcal{E} \; \exists X \rightarrowtail I$ WITH I INJ.)
  2. $\forall X \in \mathcal{E}$: X PROJ. $\iff$ X INJECTIVE.

AN OBJECT $P \in \mathcal{E}$ IS **PROJECTIVE** IF

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d, "\forall"] \\ X \arrow[r, two heads, "\forall \text{ ADMIS. EPI}"'] & Y \end{tikzcd}$$

, INJECTIVES ARE DEF'D DUALLY

- EXPL:
  - a) $(\mathcal{C}(\mathcal{B}), \mathbb{E})$ $\forall \mathcal{B}$ ADDITIVE CATEGORY, PROJECTIVES-INJECTIVES ARE PRECISELY THE CONTRACTIBLE COMPLEXES.
  - b) G FINITE GROUP, k A FIELD $\rightsquigarrow$ $(\mathrm{Mod}\, kG, \{\text{ALL S.E.S.}\})$ IS FROBENIUS, SO IS $\mathrm{mod}\, kG$
  - c) R COMM. NOETH. LOCAL RING WHICH IS GORENSTEIN, $\mathcal{E} = \{$MAX. C.M. F.G. MODULES$\}$

— CONSTRUCTION OF TRIANGULATED CATEGORIES FROM FROB. EXACT CAT.:

— <u>EXPL</u>: $\mathcal{B}$ ADDITIVE, $\mathcal{E} = \mathcal{C}(\mathcal{B})$, $\underline{\mathcal{E}} := \mathcal{K}(\mathcal{B}) = \mathcal{C}(\mathcal{B}) / [\text{CONTRACTIBLE COMPLEXES}]$

$= \mathcal{E} / [\text{PROJ.-INJ. OBJECTS}]$

- $\mathcal{K}(\mathcal{B})$ IS TRIANGULATED:
  - SHIFT FUNCTOR: $[1]$ $\quad (0 \to X^\cdot \rightarrowtail CX^\cdot \to X^\cdot[1] \to 0)$
  - TRIANGLES: $X^\cdot \xrightarrow{f} Y^\cdot \to C_f^\cdot \to X^\cdot(1)$. $\quad \left(\begin{array}{ccc} X^\cdot & \rightarrowtail & CX^\cdot \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y^\cdot & \to & C_f^\cdot \end{array}\right)$

[Ha'88] — A TRIANGULATED CATEGORY $\mathcal{T}$ IS ALGEBRAIC IF THERE IS A FROB. EXACT CATEGORY $\mathcal{E}$ SUCH THAT $\mathcal{T} \simeq \mathcal{E}/(\text{PROJ. INJ.})$ AND THE TRIANGULATED STRUCTURE IS AS FOLLOWS:

- <u>SHIFT AUTOEQUIVALENCE</u>: $X \in \mathcal{E}$, TAKE $0 \to X \rightarrowtail \overset{\text{INJ.}}{I} \twoheadrightarrow X[1] \to 0$ IN $\mathcal{E}$

$\rightsquigarrow \underline{\mathcal{E}} := \mathcal{E}/(\text{PROJ. INJ.}) \circlearrowleft^{\simeq}_{[1]}$

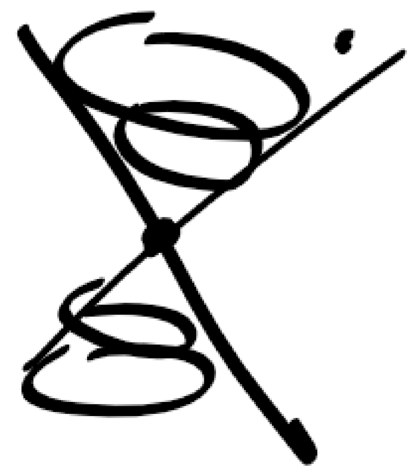
- <u>TRIANGLES</u>:

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "f"'] & E \arrow[r, two heads] \arrow[d] & X(1) \arrow[d, equal] \\ Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & X(1) \end{tikzcd}$$

IN $\mathcal{E}$ $\rightsquigarrow$ $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X(1)$

ARE THE TRIANGLES (UP TO ISO) IN $\underline{\mathcal{E}} \simeq \mathcal{T}$

③ A BIT OF COMMUTATIVE ALGEBRA



- R COMM. NOETHERIAN LOCAL ($\Rightarrow$ FINITE KRULL DIM.)
- WANT TO DO HOM. ALGEBRA IN $\operatorname{mod} R$
- WHY IS THAT USEFUL:

  - T (SERRE): R REGULAR RING $\iff$ GL. DIM $(\operatorname{mod} R) < \infty$

    (IN THAT CASE GL. DIM. $(\operatorname{mod} R)$ = KRULL DIM. OF R)

- HOW DO WE COMPUTE $\operatorname{proj.dim} M$ FOR $M \in \operatorname{mod} R$?
- T (AUSLANDER-BUCHSBAUM): IF $M \in \operatorname{mod} R$ AND $\operatorname{proj.dim} M < \infty$, THEN

$$\operatorname{proj.dim} M = \operatorname{depth}(R) - \operatorname{depth}(M).$$

- HERE: $\operatorname{depth}(M)$ = A MAX. LENGTH OF AN M-REGULAR SEQUENCE

- WE KNOW: $\operatorname{depth}(R) \leq \operatorname{Krull\,dim}(R)$, DEF: R IS COHEN-MACAULAY IF =
- DEF: R IS GORENSTEIN IF $\operatorname{inj}. R < \infty$.
- FACTS: R REGULAR $\Rightarrow$ R GORENSTEIN $\Rightarrow$ R COHEN-MACAULAY (CM)
- DEF: $0 \neq M \in \operatorname{mod} R$ IS MAX. C.M. (MCM) IF $\operatorname{depth}(M) = \operatorname{Krull\,dim}(R)$.
- IF R GORENSTEIN, THEN MCM(R) IS FROBENIUS, PROJECTIVES-INJECTIVES ARE $\operatorname{proj}(R)$ AND $\underline{\mathrm{MCM}}(R) := \mathrm{MCM}(R)/[\operatorname{proj} R]$ IS TRIANGULATED.

(4) APPETIZER FOR THE ROLE OF TRIA EQUIV. IN COMM. ALGEBRA

- THERE IS A CLASS OF NICE SINGULARITIES WHICH ARE KNOWN UNDER SEVERAL NAMES: DU VAL, KLEINIAN, SIMPLE, ...
  (M8) DOES NOT MATCH

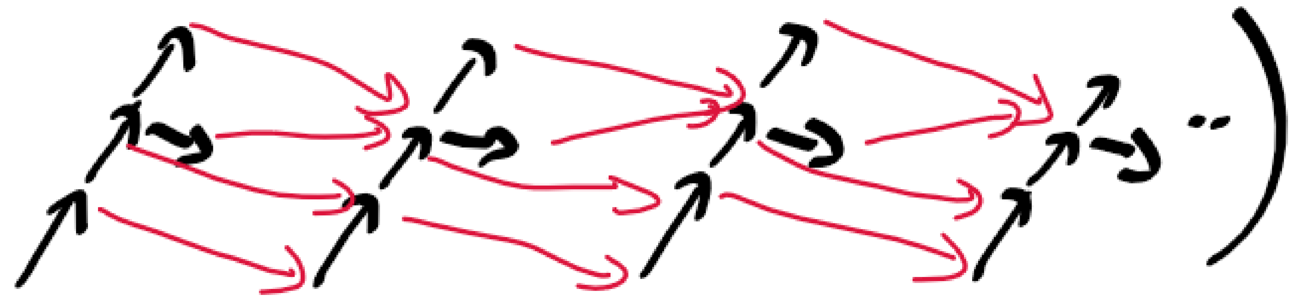
- THEY ARE OF THE FORM $R = \dfrac{\mathbb{C}[x,y,z]}{(f)}$

- TO MAKE OUR LIFE EASIER:

  ↗ WORK WITH $R = \dfrac{\mathbb{C}[[x,y,z]]}{(f)}$

  ↘ CONSIDER GRADING ON $\mathbb{C}[x,y,z]$ SO THAT f IS HOMOGENEOUS AND CONSIDER $R = \dfrac{\mathbb{C}[x,y,z]}{(f)}$ AS A GRADED LOCAL RING ($\deg(x), \deg(y), \deg(z) > 0$)

- T: IN THE GRADED CASE: $\mathrm{MCM}^{gr}(R) \underset{\text{TRIANGLE}}{\simeq} \mathcal{D}^b(kQ)$, Q DYNKIN QUIVER

- KNOWN (CH'28?], AUSLANDER-REITEN THEORY): $\mathrm{ind}\,\mathcal{D}^b(kQ) = k\Big(\cdots$  $\cdots \Big) \Big/ \big($MESH RELATION$\big)$, Q DYNKIN