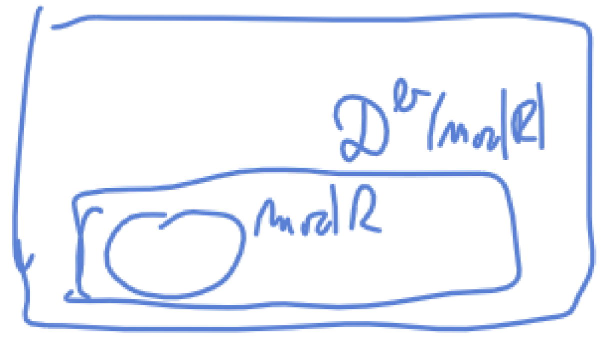

$\mathcal{D}(R) = \mathcal{D}(\mathrm{Mod}\,R)$

DERIVED EQUIVALENCES

(HAPPEL '87, RICKARD '89, KELLER '94)

- R A RIGHT NOETHERIAN RING, $\mathrm{mod}\,R$ = CATEGORY OF FIN. GEN. RIGHT R-MODULES (ABELIAN CATEGORY)

- $\mathcal{D}^b(R) := \mathcal{D}^b(\mathrm{mod}\,R)$ ... BOUNDED DERIVED CATEGORY OF R, I.E.

$\left(\begin{array}{c} \cup\!\mathsf{I} \\ \mathrm{mod}\text{-}R \end{array} \right)$

$$C^b(\mathrm{mod}\,R) = \{ \dots \to 0 \to 0 \to X^m \to \dots \to X^n \to 0 \to \dots \},$$

$$\mathcal{D}^b(\mathrm{mod}\,R) = C^b(\mathrm{mod}\,R)\,[\{\text{quasi-iso}\}^{-1}]$$

- DEF (MIYASHITA, GOES BACK TO AUSLANDER-PLATZECK-REITEN, HAPPEL-RINGEL)

A MODULE $T \in \mathrm{mod}\,R$ IS TILTING IF:

(T COMPACT IN $\mathcal{D}(R)$) (1) $\mathrm{proj\,dim}\, T = n < \infty$ (I.E. $\exists\ 0 \to \underbrace{P_n \to P_{n-1} \to \dots \to P_0 \to T}_{\in \mathrm{mod}\,R} \to 0$)

(T RIGID) (2) $\mathrm{Ext}^i_R(T,T) = 0 \quad \forall i > 0$

(T GENERATOR FOR $\mathcal{D}(R)$) (3) $\exists\ 0 \to R \to T^0 \to T^1 \to \dots \to T^{m-1} \to T^m \to 0$ EXACT IN $\mathrm{mod}\,R$ WITH EACH $T^i \in \mathrm{add}(T)$

- THE POINT (BY HAPPEL): MIYASHITA TILTING MODULES INDUCE DERIVED EQUIVALENCES:

- LET $T \in \operatorname{mod} R$ BE TILTING, DENOTE $S := \operatorname{End}_R(T)$ (I.E. ${}_S T_R$ IS A BIMODULE) | R BE MODULE FINITE ALGEBRA OVER A COMMUTATIVE NOETHERIAN RING k

- THEN WE HAVE EQUIVALENCES:

$$D^b(\operatorname{mod} R) \xleftarrow[\simeq]{- \overset{\mathbb{L}}{\underset{S}{\otimes}} T} D^b(\operatorname{mod} S) \ni X_S^{\cdot}$$

- $X^{\cdot} \overset{\mathbb{L}}{\underset{S}{\otimes}} T$: $\quad \cdots \to P^0 \underset{S}{\otimes} T_R \to P^1 \underset{S}{\otimes} T_R \to \cdots$

(TAKE $P^{\cdot} \to X^{\cdot}$ QUASI ISO, $P^{\cdot}$ COMPLEX OF PROJECTIVES, BOUNDED ABOVE)

- NOTE: $K^{-,b}(\operatorname{proj} S) \xrightarrow{\simeq} D^b(\operatorname{mod} S)$

BOUNDED ABOVE   BOUNDED COHOMOLOGY

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X^n \arrow[r] & X^{n+1} \arrow[r] & \cdots \arrow[r] & X^{m-1} \arrow[r] & X^{m-1} \arrow[r] & X^m \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\arrow[r] & P^{n-3} \arrow[r] \arrow[u] & P^{n-2} \arrow[r] \arrow[u] & P^{n-1} \arrow[r] \arrow[u] & P^n \arrow[r, "d^n"] \arrow[u] & P^{n+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & P^{m-2} \arrow[r] \arrow[u] & P^{m-1} \arrow[r] \arrow[u] & P^m \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

$\ker d^n$ Y^{m-2} Y^{m-1}

- TILTING COMPLEXES (RICKARD)

- **DEF:** R AS BEFORE. A COMPLEX $T \in D^b(R)$ IS TILTING IF

  1. T IS ISOMORPHIC IN $D^b(R)$ TO A BOUNDED COMPLEX OF FIN. GEN. PROJECTIVES.
  2. $\mathrm{Hom}_{D^b(R)}(T, T(i)) = 0 \quad \forall i \neq 0$.
  3. R_R CAN BE OBTAINED FROM T IN $D^b(R)$ USING DIRECT SUMS, SUMMANDS, SHIFTS $[?]$ AND MAPPING CONES

- **T:** (RICKARD) LET $T \in D^b(R)$ BE TILTING AND $S = \mathrm{End}_{D^b(R)}(T)$. THEN $D^b(R) \simeq D^b(S)$ ARE EQUIVALENT AS TRIA. CATEGORIES.

- WHAT IS WRONG WITH HAPPEL'S PROOF?

$$T = (\cdots \to 0 \to \underbrace{T_R^m \to \cdots \to T_R^n}_{\text{WLOG PROJECTIVE}} \to 0 \to \cdots), \quad S = \mathrm{End}_{K^b(\mathrm{proj}\,R)}(T)$$

- WHAT IS $- \otimes_S T$??
- WE WOULD NEED A COMPLEX OF S-R BIMODULES

$$\cdots \to 0 \to {}_S T_R^m \to \cdots \to {}_S T_R^n \to 0 \to \cdots$$

- HOW TO FIX THAT?

DG ALGEBRAS (KELLER)

- DG = DIFFERENTIAL GRADED
- LET $\mathbb{k}$ BE A COMMUTATIVE RING
- DEF: A DG-ALGEBRA R OVER $\mathbb{k}$ IS:

(1) A $\mathbb{Z}$-GRADED $\mathbb{k}$-ALGEBRA $R = \bigoplus_{n \in \mathbb{Z}} R^n$

(2) WHICH IS AT THE SAME TIME A COMPLEX

$$\cdots \to R^{-2} \xrightarrow{d} R^{-1} \xrightarrow{d} \underset{\substack{\Psi \\ 1}}{R^0} \xrightarrow{d} R^1 \xrightarrow{d} R^2 \to \cdots$$

(3) AND THE GRADED LEIBNIZ RULE IS SATISFIED:

$$\forall n, m \in \mathbb{Z} \quad \forall x \in R^n \quad \forall y \in R^m : \qquad d(xy) = d(x) \cdot y + (-1)^{n \cdot 1} \underbrace{x \cdot d}(y)$$

CHAGE OF ORDER OF SYMBOLS

KOSZUL SIGN

- KOSZUL SIGN: IF WE REPLACE THE ORDER $\alpha\beta \rightsquigarrow \beta\alpha$, WE INTRODUCE THE SIGN $(-1)^{\deg\alpha \cdot \deg\beta}$

– THE ENDOMORPHISM DG-ALGEBRA OF A COMPLEX:

– LET $X^\cdot = (\quad \cdots \xrightarrow{d} X^0 \xrightarrow{d} X^1 \xrightarrow{d} X^2 \longrightarrow \cdots \quad), Y^\cdot$ BE COMPLEXES IN AN ADDITIVE CATEGORY $\mathcal{C}$

– THEN DEFINE THE COMPLEX $\mathcal{H}om(X^\cdot, Y^\cdot) \in \mathcal{C}(ab)$ AS FOLLOWS:

$$i \in \mathbb{Z}: \quad \mathcal{H}om^i(X^\cdot, Y^\cdot) = \prod_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^n, Y^{n+i})$$

$$\begin{tikzcd}
\cdots & X^0 \arrow[r, "d"] \arrow[drr] & X^1 \arrow[r, "d"] \arrow[drr] & X^2 \arrow[r, "d"] \arrow[drr] & X^3 \arrow[drr] & \cdots & \\
\cdots & Y^0 \arrow[r, "d"] & Y^1 \arrow[r, "d"] & Y^2 \arrow[r, "d"] & Y^3 & & \cdots
\end{tikzcd}$$

E.G.: $i=2$

– THE DIFFERENTIAL:

$$\mathcal{H}om^i(X^\cdot, Y^\cdot) \xrightarrow{d} \mathcal{H}om^{i+1}(X^\cdot, Y^\cdot)$$

$$d(f) = \left(d_Y f^n - (-1)^i f^{n+1} d_X \right)_{n \in \mathbb{Z}}$$

$$f = (f^n)_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^n, Y^{n+i})$$

– IF $X^\cdot = Y^\cdot$, THEN $\mathcal{E}nd(X^\cdot) := \mathcal{H}om(X^\cdot, X^\cdot)$ IS A DG-ALGEBRA

- WHY DG-ALGEBRAS?

  - LET R BE A RING (ANY RING IS TRIVIALLY A DGA)

    AND $X^\bullet \in \mathcal{C}(\mathrm{Mod}(R))$

  $\rightsquigarrow$ $S := \mathcal{E}nd(X^\bullet)$

  $\rightsquigarrow$ $X^\bullet$ BECOMES NATURALLY A LEFT DG-MODULE OVER S,
  IN FACT ${}_S X^\bullet_R$ WILL BE A DG-BIMODULE

- A DG MODULE OVER A DGA $S = \bigoplus_{n \in \mathbb{Z}} S^n$ IS
  - A GRADED S MODULE $M_S = \bigoplus_{n \in \mathbb{Z}} M^n$
  - WHICH IS ALSO A COMPLEX $(\dots \to M^{n-1} \xrightarrow{d} M^n \xrightarrow{d} M^{n+1} \to \dots)$
  - AND THE LEIBNIZ RULE IS SATISFIED:

$$\forall n, k \in \mathbb{Z} \quad \forall m \in M^n \quad \forall s \in S^k : \quad d(m \cdot s) = d(m) \cdot s + (-1)^n m \cdot d(s)$$

- IF S IS CONCENTRATED IN DEGREE 0, THEN A DG-MODULE OVER S IS JUST A COMPLEX OF S-MODULES

– A HINT ON KELLER'S PROOF OF RICKARD'S THEOREM

- LET R BE RING, $T^\cdot \in \mathcal{D}(\mathrm{Mod}\,R)$ A TILTING COMPLEX
- WLOG $T^\cdot$ IS A BOUNDED CPX OF FIN. GEN. PROJECTIVES
- PUT $U := \mathrm{End}(T^\cdot)$ DGA, SO ${}_U T^\cdot_R$ IS A DG-BIMODULE
- WE CONSIDER $\mathcal{D}(U) = \mathrm{Mod}_{DG}\text{-}U\,[\{\text{QUASI-ISO}\}^{-1}]$
- THEN

$$\mathcal{D}(\mathrm{Mod}\,R) \xleftarrow[\simeq]{- \overset{L}{\otimes}_U T_R} \mathcal{D}(U)$$

- ONE HAS THAT $H^n(U) \cong \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\,R)}(T, T[n])$,

SO $H^n(U) = 0$ UNLESS $n = 0$ AND $H^0(U) = \mathrm{End}_{\mathcal{D}(\mathrm{Mod}\,R)}(T^\cdot) =: S$

- WE HAVE

$$\begin{array}{lll} & (\cdots \to U^{-2} \to U^{-1} \to U^0 \xrightarrow{d} U^1 \to U^2 \to \cdots) & \\ \text{QUASI-ISO} \to & \quad\;\uparrow \qquad\;\; \uparrow \qquad\;\; \uparrow \qquad\;\; \uparrow \qquad \uparrow & \\ & (\cdots \to U^{-2} \to U^{-1} \xrightarrow{d} \mathrm{Ker}\,d \to 0 \to 0 \to \cdots) & \text{DG-SUBALG.} \\ \text{QUASI-ISO} & \quad\;\downarrow \qquad\;\; \downarrow \qquad\;\; \downarrow \qquad\;\; \downarrow & \\ & (\cdots \to 0 \to 0 \to \underset{\simeq H^0(U)}{S} \to 0 \to 0 \to \cdots) & \text{QUOTIENT DGA} \end{array}$$

- **USE:** IF $U \to V$ IS A QUASI-ISO OF DGA'S, THEN $\mathcal{D}(U) \simeq \mathcal{D}(V)$