- PLAN: 1) TRIANGULATED STRUCTURE ON $\mathcal{D}(\mathcal{A})$

  2) DERIVED EQUIVALENCES

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1) TRIANGULATED CATEGORIES

- PROBLEM: $\mathcal{D}(\mathcal{A})$ NOT WELL-BEHAVED AS A CATEGORY
  (OFTEN $\mathcal{D}(\mathcal{A})$ HAS $\prod$'s AND $\coprod$'s, BUT NOT Ker, Coker)
- SOLUTIONS: a) CONSIDER ADDITIONAL STRUCTURE ON $\mathcal{D}(\mathcal{A})$
  b) CONSIDER ENHANCEMENTS (MORPHISMS IN $\mathcal{D}(\mathcal{A})$ PROMOTED TO BE COMPLEXES OR SIMPLICIAL SET)
- CHOSE a) HERE!

- A TRIANGULATED CATEGORY IS AN ADDITIVE CATEGORY $\mathcal{T}$ TOGETHER WITH AN AUTOEQUIVALENCE $\mathcal{T} \to \mathcal{T}$, $X \mapsto X[1]$ AND A CLASS Δ OF DIAGRAMS OF THE FORM

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \qquad \left(\begin{tikzcd} & Y \arrow[dr, "v"] & \\ X \arrow[ur, "u"] & & Z \arrow[ll, "w"'] \end{tikzcd} \right)$$

- THIS DATUM $(\mathcal{T}, [1], \Delta)$ IS REQUIRED TO SATISFY THE AXIOMS:

(TR1)
- $\forall X \in \mathcal{T}$, $\quad (X \xrightarrow{1_X} X \to 0 \to X[1]) \in \Delta$
- Δ IS CLOSED UNDER ISOMORPHISMS
- IF $(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]) \in \Delta \iff (Y \xrightarrow{-v} Z \xrightarrow{-w} X[1] \xrightarrow{-u[1]} Y[1])$

$$\begin{tikzcd} Y \arrow[r, "-v"] \arrow[d, "1"] & Z \arrow[r, "-w"] \arrow[d, "-1"] & X[1] \arrow[r, "-u[1]"] \arrow[d, "1"] & Y[1] \arrow[d, "1"] \\ Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \arrow[r, "-u[1]"] & Y[1] \end{tikzcd}$$

(TR2) $\forall X \xrightarrow{u} Y \quad \exists (X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1])$

(TR3) $\forall$ SOLID PART WE CAN FIND A DOTTED MORPHISM WHICH MAKES IT COMMUTATIVE:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "="] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "="] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] \arrow[dr, phantom, "="] & X[1] \arrow[d, "f[1]"] & \leftarrow \in \Delta \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & X'[1] & \end{tikzcd}$$

(TR4) (OCTAHEDRAL AXIOM)

$\forall$ SOLID PART $\exists$ DOTTED ARROWS :

$$
\begin{tikzcd}
 & X'[1] \arrow[r, equal] \arrow[d, "i"] & X'[1] \arrow[d, dotted] & & \\
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "j"] \arrow[d, "v"] & Z' \arrow[r, "k"] \arrow[d, dotted] & X[1] \arrow[d, equal] & \arrow[l] \in \Delta \\
X \arrow[r, "vu"] & Z \arrow[r, "m"] \arrow[d, "l"] & Y' \arrow[r, "n"] \arrow[d, dotted] & X[1] \arrow[d, "u[1]"] & \arrow[l] \in \Delta \\
 & X' \arrow[r, equal] & X' \arrow[r, "i[1]"] & Y[1] & \\
 & \Delta \arrow[u] & \Delta \arrow[u] & &
\end{tikzcd}
$$

- **L:** IF $\mathcal{T}$ IS A TRIANGULATED CATEGORY, $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X(1)$ IS A TRIANGLE AND $W \in \mathcal{T}$, THEN

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(W, Z(-1)) \to \mathrm{Hom}_{\mathcal{T}}(W,X) \to \mathrm{Hom}_{\mathcal{T}}(W,Y) \to \mathrm{Hom}_{\mathcal{T}}(W,Z) \to \mathrm{Hom}_{\mathcal{T}}(W,X(1)) \to ,$$

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(X(1),W) \to \mathrm{Hom}_{\mathcal{T}}(Z,W) \to \mathrm{Hom}_{\mathcal{T}}(Y,W) \to \mathrm{Hom}_{\mathcal{T}}(X,W) \to \mathrm{Hom}_{\mathcal{T}}(Z(-1),W) \to$$

ARE EXACT IN Ab.

- **PF:** (1)

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, dashed, "\exists"'] & Y \arrow[r, "v"] \arrow[d, "v"] & Z \arrow[r, "w"] \arrow[d, "1_Y"] & X(1) \arrow[d, dashed] \\
0 \arrow[r] & Y \arrow[r, "1_Y"'] & Y \arrow[r] & 0(1)=0
\end{tikzcd}$$

$\Longrightarrow$ $v \circ u = 0$

(2)

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "0"'] & Y \arrow[r] \arrow[d, "f"] & Z \arrow[r] \arrow[d, dashed, "\exists"] & X(1) \arrow[d, "0"] \\
0 \arrow[r] & W \arrow[r, "1_W"'] & W \arrow[r] & 0(1)=0
\end{tikzcd}$$

$\Longrightarrow$ EXACTNESS AT $\mathrm{Hom}(Y,W)$

- **COROLLARY:** ("5-LEMMA") IF WE HAVE

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X(1) \arrow[d, "f(1)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'(1)
\end{tikzcd}$$

AND 2 OF f, g, h ARE ISO, SO IS THE THIRD.

– **P**: IN THE SITUATION OF

$$
\begin{tikzcd}
 & X'[-1] \arrow[r, equal] \arrow[d, "i"] & X'[-1] \arrow[d, dashed] & \\
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "j"] \arrow[d, "v"] & Z' \arrow[r, "k"] \arrow[d, dashed, "w"] & X[1] \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "m"] \arrow[d, "l"] & Y' \arrow[r, "n"] \arrow[d, dashed, "t"] & X[1] \arrow[dd, "u[1]"] \\
 & X' \arrow[r, equal] & X' \arrow[dr, "{i[1]}"'] & \\
 & & & Y[1]
\end{tikzcd}
$$

,

WE HAVE ANOTHER TRIANGLE:

$$Y \xrightarrow{\binom{v}{j}} Z \oplus Z' \xrightarrow{(m,\,-w)} Y' \xrightarrow[u(1)\circ n]{i(1)\circ t \;=} Y(1)$$

– <u>LITERATURE</u>: VERDIÉR, NEEMAN, OTHER BOOKS ON DERIVED CATEGORIES ...

– THE NATURAL TRIA. STRUCTURE ON $\mathcal{D}(\mathcal{A})$:

- SHIFT AUTOEQUIVALENCE:

$$X^\bullet = (\cdots \to \underset{\substack{\uparrow \\ \deg i}}{X^i} \xrightarrow{d} X^{i+1} \xrightarrow{d} X^{i+1} \to \cdots) \longmapsto X^\bullet(1) = (\cdots \to \underset{\substack{\uparrow \\ \deg i}}{X^{i+1}} \xrightarrow{-d} X^{i+2} \xrightarrow{-d} X^{i+3} \to \cdots)$$

- TRIANGLES:

$$\underbrace{X^\bullet \xrightarrow{f^\bullet} Y^\bullet}_{\text{MAP IN } \mathcal{C}(\mathcal{A})} \xrightarrow{i^\bullet} \underbrace{C_{f^\bullet}}_{(C_{f^\bullet})^n = Y^n \oplus X^{n+1}} \xrightarrow{\pi^\bullet} X^\bullet(1) \quad (\text{A DIAGRAM IN } \mathcal{C}(\mathcal{A}))$$

AND ALL ISOMORPHIC DIAGRAMS

– L: IF $0 \to X^\bullet \xrightarrow{u^\bullet} Y^\bullet \xrightarrow{v^\bullet} Z^\bullet \to 0$ IS A S.E.S. OF COMPLEXES IN $\mathcal{C}(\mathcal{A})$, THEN THERE IS A TRIANGLE $X^\bullet \xrightarrow{u^\bullet} Y^\bullet \xrightarrow{v^\bullet} Z^\bullet \to X^\bullet(1)$.

– PF:

$$\begin{tikzcd}
X^\bullet \arrow[r, "u^\bullet"] \arrow[d, equal] & Y^\bullet \arrow[r, "v^\bullet"] \arrow[d, equal] & Z^\bullet = \mathrm{Coker}\, u^\bullet \\
X^\bullet \arrow[r, "u^\bullet"] & Y^\bullet \arrow[r, "i"] & C_{u^\bullet} \arrow[u, dashed, "\sigma^\bullet"] \arrow[r, "\pi"] & X^\bullet(1)
\end{tikzcd}$$

in $\mathcal{C}(\mathcal{A})$, σ QUASI-ISO

$\rightsquigarrow$ TRIANGLE IN $\mathcal{D}(\mathcal{A})$: $X^\bullet \xrightarrow{u^\bullet} Y^\bullet \xrightarrow{v^\bullet} Z^\bullet \xrightarrow{\pi\sigma^{-1}} X(-1)$.

– Relation between Hom's in $D(\mathcal{A})$ and Yoneda Ext:

– Say $[\varepsilon : 0 \to Y \to E_1 \to \dots \to E_n \xrightarrow{h} X \to 0] \in \mathrm{Ext}^n_{\mathcal{A}}(X,Y)$

⇝ we get the following map in $D(\mathcal{A})$

$$
\begin{tikzcd}
X \arrow[d] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & \dots \\
 & \dots \arrow[r] & Y \arrow[r] \arrow[u, "0"] \arrow[d, "1_Y"'] & E_1 \arrow[r] \arrow[u, "0"'] & \dots \arrow[r] & E_{n-1} \arrow[r] \arrow[u, "0"] \arrow[d] & E_n \arrow[r] \arrow[u, "h"'] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \dots \\
Y[n] & \dots \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}
$$

← quasi-iso

= degree 0

= degree $-n$

⇝ one can check that this procedure defines

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \mathrm{Hom}_{D(\mathcal{A})}(X, Y[n])$$

– Relation between Hom's in $D(\mathcal{A})$ and Ext via proj. resolutions

(assume enough projectives, $X, Y \in \mathcal{A}$)

$$
\begin{tikzcd}
X \arrow[d] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & \dots \\
 & \dots \arrow[r] & P_{n+1} \arrow[r] \arrow[u] \arrow[d] & P_n \arrow[r] \arrow[u] \arrow[d] & P_{n-1} \arrow[r] \arrow[u] \arrow[d, "0"] & \dots \arrow[r] & P_2 \arrow[r] \arrow[u] \arrow[d, "0"] & P_1 \arrow[r] \arrow[u] \arrow[d, "0"] & P_0 \arrow[r] \arrow[u, "h"'] \arrow[d, "0"] & 0 \arrow[r] & \dots \\
Y[n] & \dots \arrow[r] & 0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}
$$

– **Fact**: If $P^\bullet \in C(\mathcal{A})$ is a bounded above complex of projectives and $Z^\bullet \in C(\mathcal{A})$, then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Z^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, Z^\bullet)$$

- **TILTING OBJECT**: $\mathcal{A}$ ABELIAN, R RING

$$\text{(1)} \circlearrowleft D(\mathcal{A}) \simeq D(\text{mod}R) \circlearrowright \text{(1)} \quad \text{TRIANGLE EQUIVALENCE}$$
$$T^\cdot \longleftarrow\!\shortmid R$$

$T^\cdot$ IS (BY DEFINITION) A TILTING COMPLEX

- FOR WHAT COMES NOW, SEE:
  - [Kra 07]
  - Keller's PAPER IN "HANBOOK OF TILTING THEORY", EDITED BY ANGELERI-HÜGEL, HAPPEL, KRAUSE
  - PAPERS BY RICKARD

- PROPERTIES OF R IN $D(\text{mod}R)$ (AND HENCE OF $T^\cdot$ IN $D(\mathcal{A})$). R IS

1. COMPACT (A BIT LATER)
2. RIGID: $\text{Hom}_{D(\text{mod}R)}(R, R(n)) = 0 \quad \forall n \neq 0$.
3. GENERATOR: $\underbrace{\text{Hom}_{D(\text{mod}R)}(R(n), X^\cdot)}_{\cong \text{Hom}_{K(\text{mod}R)}(R(n), X^\cdot) \cong H^{-n}(X^\cdot)} = 0 \quad \forall n \in \mathbb{Z} \Rightarrow X^\cdot = 0 \text{ in } D(\mathcal{A})$

- COMPACTNESS:

- NOTE: A DIRECT SUM OF QUASI-ISO'S IN $D(\mathrm{mod}\,R)$ IS QUASI-ISO.

$\Rightarrow$ $D(\mathrm{mod}\,R)$ HAS COPRODUCTS COMPUTED AS $\oplus$

- DEF: LET $\mathcal{T}$ BE TRIANGULATED WITH $\coprod$'s.
THEN $X \in \mathcal{T}$ IS COMPACT IF $\forall (Y_i)_{i \in I}$ IN $\mathcal{T}$,

$$\bigoplus \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \xrightarrow{\text{can}} \mathrm{Hom}_{\mathcal{T}}(X, \coprod_{i \in I} Y_i) \quad \text{IS ISO.}$$

(I.E. $\mathrm{Hom}_{\mathcal{T}}(X, -)$ PRESERVES COPRODUCTS)

- T (RICKARD): $X \in D(\mathrm{mod}\,R)$ IS COMPACT $\Leftrightarrow$ X ISO IN $D(\mathrm{mod}\,R)$ TO A BOUNDED COMPLEX OF FIN. GEN. PROJECTIVES