

- REMINDER: (A ABELIAN)

① WE WANT TO PROVE THE UNIVERSAL PROPERTY FOR $P: \mathcal{C}(A) \rightarrow \mathcal{K}(A)$:

$\mathcal{C}(\mathcal{A}) \xrightarrow[\text{FULL}]{\mathcal{P}} \mathcal{K}(\mathcal{A})$, WHERE $\Sigma_H = \{f \mid f \text{ HTPY ISO}\} =$
 $= \{f \mid P(f) \text{ IS ISO IN } \mathcal{K}(\mathcal{A})\}$
 $\mathcal{C}(\mathcal{A}) \xrightarrow[\Sigma_H\text{-INVGRING}]{\forall F} \Sigma$
 $\mathcal{K}(\mathcal{A}) \xrightarrow[\text{ANY CATEGORY}]{\exists F! F} \Sigma$

② RECALL THAT $f: X \rightarrow Y$ IN $\mathcal{C}(W)$ IS NULLHOMOTOPIC
(IE HTPIC TO $0: X \rightarrow Y$) IFF f FACTORS

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow & \nearrow \\
 & C(X) &
 \end{array}$$

CONTRACTIBLE

[illegible]

- SKETCH PROOF OF ①:

- SAY $F: \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{E}$ IS Σ_H -INVERTING
- NEED TO PROVE: F IDENTIFIES HOMOTOPIC MAPS
- SAY $f \sim g: X' \rightarrow Y'$, SO

$$\begin{array}{ccc} X' & \xrightarrow{i} & CX' \\ & \searrow f-g & \downarrow \rho \\ & & Y' \end{array}$$

$$\begin{array}{ccc} X' & \xrightarrow{\begin{pmatrix} id \\ 0 \end{pmatrix}} & X' \oplus CX' \\ X' & \xrightarrow{\begin{pmatrix} id \\ i \end{pmatrix}} & \downarrow (f, s) \\ & & Y' \end{array}$$

$f + s \circ i = g$

$(id, 0) \xrightarrow{\quad} X'$

$$F \rightsquigarrow$$

$$\begin{array}{ccc} Fx' & \xrightarrow{id} & Fx' \\ Fx' & \xrightarrow{\sim} & F(X' \oplus CX') \xrightarrow{\sim} Fx' \end{array}$$

 = HTPY ISO, I.E. $P(\text{---})$ IS ISO

$$\Rightarrow F\left(\begin{pmatrix} id \\ 0 \end{pmatrix}\right) = F\left(\begin{pmatrix} id \\ 0 \end{pmatrix}\right)^{-1} = F\left(\begin{pmatrix} id \\ i \end{pmatrix}\right)$$

$$\Rightarrow F|f| = F|g|$$

□

- SUMMARY: $\Sigma_H = \{ \text{HTPY ISO'S IN } \mathcal{O}(A) \}$
 $\cap$
 $\overline{W} = \{ \text{QUASI-ISO'S IN } \mathcal{O}(A) \}$

$$\leadsto \mathcal{O}(A) \xrightarrow{P} \mathcal{K}(A) \xrightarrow{Q} \mathcal{O}(A)[\overline{w}^{-1}] = \mathcal{K}(A)[\overline{w}^{-1}] = \mathcal{D}(A)$$

$\mathcal{O}(A)[\Sigma_H^{-1}]$

HOW TO UNDERSTAND THIS?

- THE POINT: $\overline{W}$ ADMITS CALCULUS OF BOTH LEFT & RIGHT FRACTIONS IN $\mathcal{O}(A)$.

(BACK TO ORE)

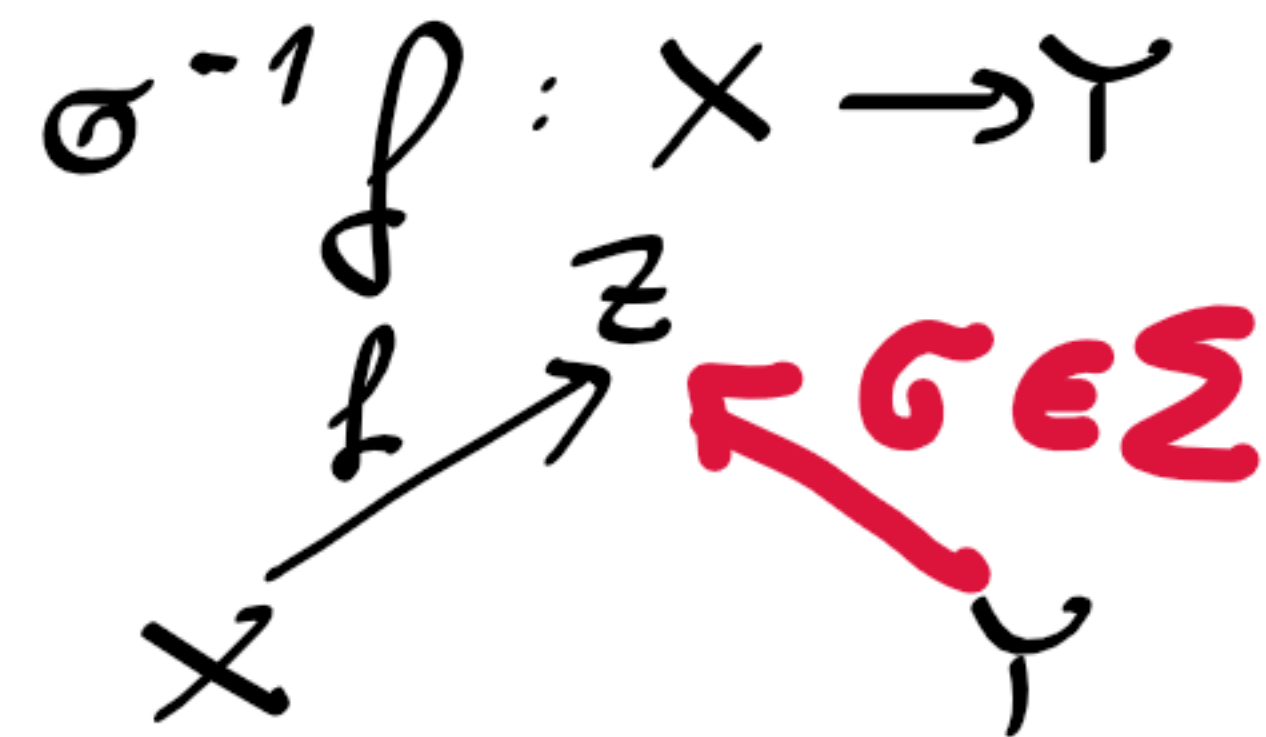
- 2 THINGS TO EXPLAIN:

- WHAT IS THE CALCULUS OF FRACTIONS
- WHY DOES $\overline{W}$ ADMIT SUCH A CALCULUS

- CALCULUS OF LEFT FRACTIONS :

- $\mathcal{C}$ A CATEGORY, Σ A CLASS OF MAPS

- A LEFT FRACTION $\sigma^{-1} f : X \rightarrow Y$ IS A DIAGRAM IN $\mathcal{C}$ OF THE FORM



- Σ ADMITS CALCULUS OF LEFT FRACTIONS IF IT SATISFIES

(LF1) Σ CONTAINS $\text{id}_x \forall x \in \mathcal{C}$ AND IS CLOSED UNDER COMPOSITIONS

(LF2) EACH DIAGRAM $W \xrightarrow{f} X$ IN $\mathcal{C}$ CAN BE COMPLETED TO $W \xrightarrow{f} X$
 $\quad \quad \quad \sigma \downarrow \quad \quad \quad \sigma \downarrow \parallel \downarrow \tau$
 $\quad \quad \quad Y \xrightarrow{g} Z$

(LF3) WHENEVER WE HAVE A DIAGRAM $Y \xrightarrow{\sigma} X \xrightarrow{f} Z$ WITH $f \sigma = g \sigma$,
 THEN THERE EXIST $X \xrightarrow{f} Z \xrightarrow{\tau} W$ SUCH THAT $\tau f = \tau g$

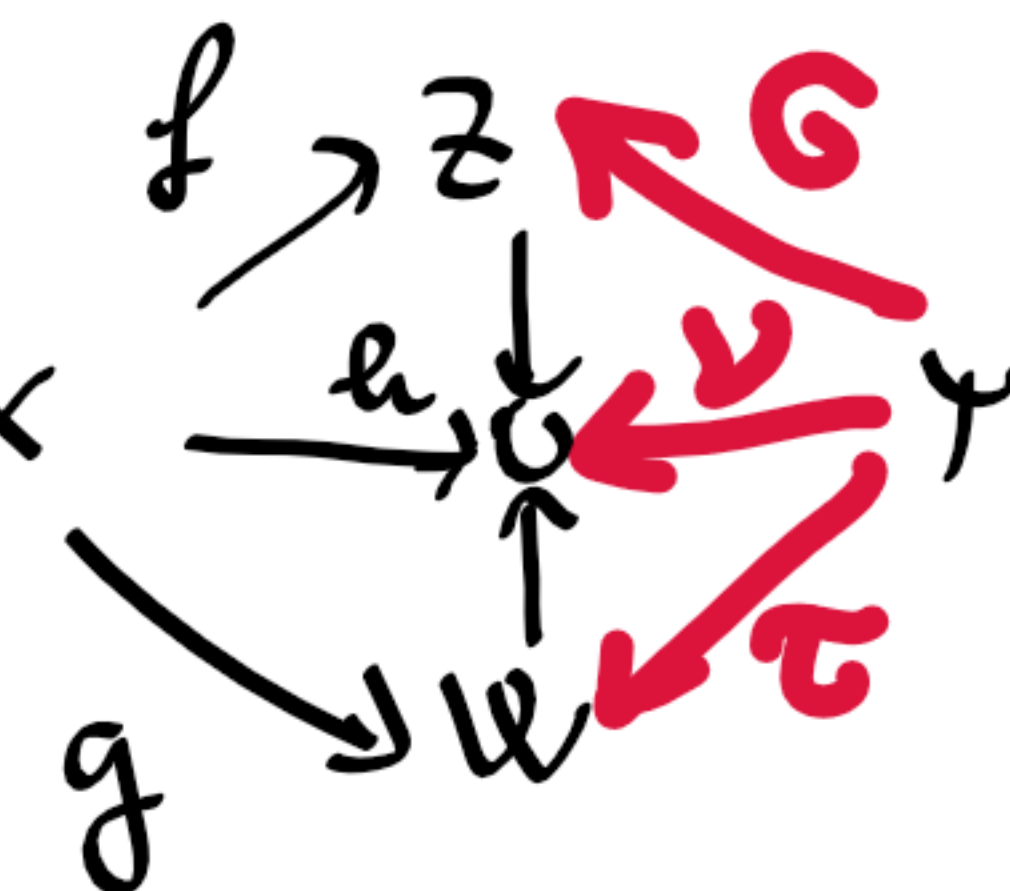
- WHY CALCULUS OF LEFT FRACTIONS?

- THEN WE CAN DEFINE THE CATEGORY OF LEFT FRACTIONS $\Sigma^{-1}\mathcal{C}$:

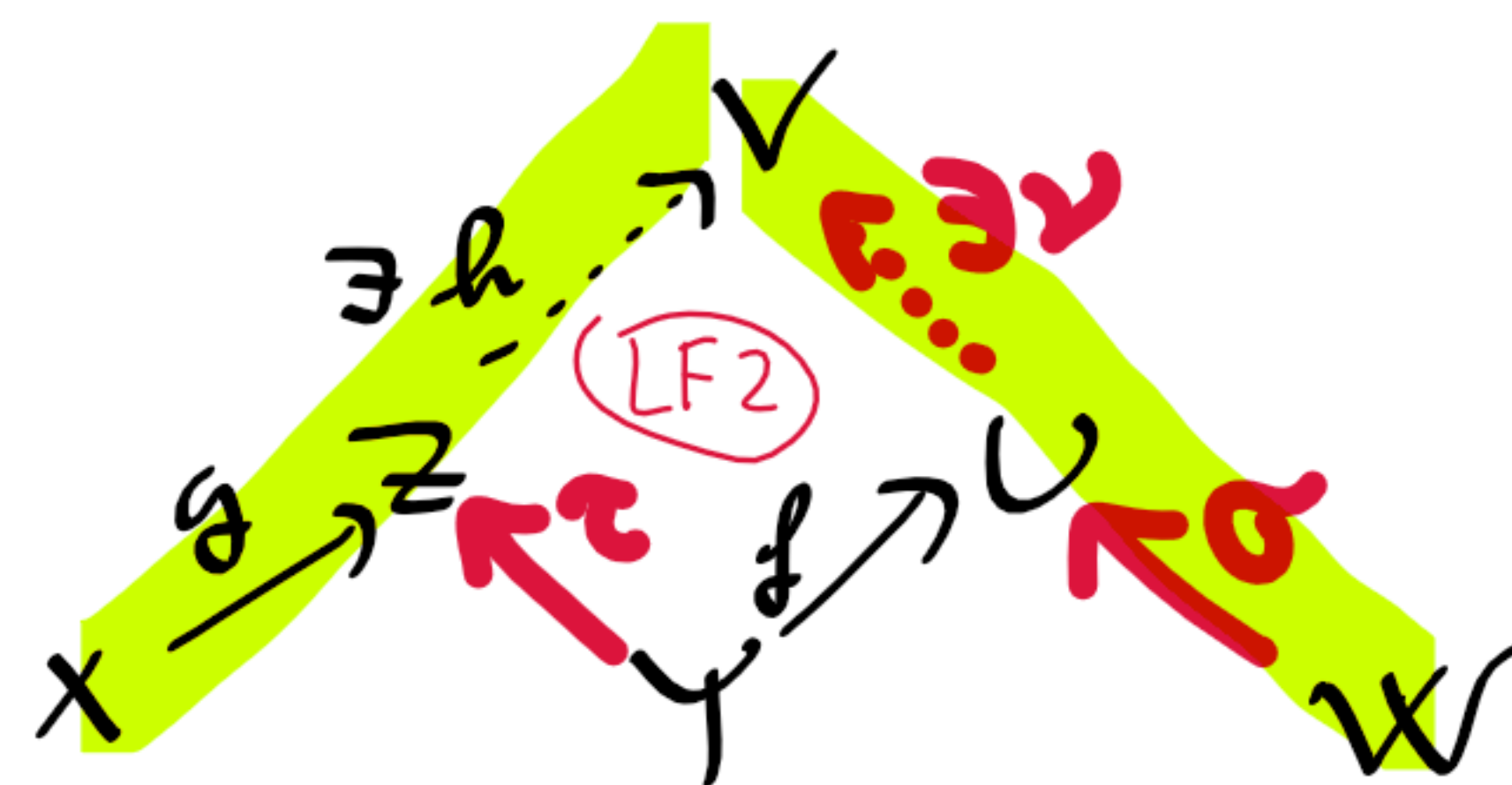
• OBJECTS: THE SAME AS IN $\mathcal{C}$

• MORPHISMS: $\sigma^{-1}f = \left[\begin{array}{c} x \xrightarrow{f} z \xleftarrow{\sigma} y \end{array} \right]_{\sim} : x \rightarrow y$ WHERE

$$\begin{array}{c} x \xrightarrow{f} z \xleftarrow{\sigma} y \\ \sim \\ x \xrightarrow{g} w \xleftarrow{\tau} y \end{array}$$

IF $\exists$ 

• COMPOSITION OF FRACTIONS: $(\sigma^{-1}f) \circ (\tau^{-1}g) = (\nu\sigma)^{-1}(hg)$



- FACTS: • $\Sigma^{-1}\mathcal{C}$ IS A WELL DEFINED CATEGORY

• $Q: \mathcal{C} \rightarrow \Sigma^{-1}\mathcal{C}$ $(f: x \rightarrow y) \mapsto id_y^{-1} \cdot f = \left[\begin{array}{c} x \xrightarrow{f} y \xleftarrow{id} y \end{array} \right]$
 $(\Rightarrow \Sigma^{-1}\mathcal{C} \cong \mathcal{C}[\Sigma^{-1}])$

SATISFIES UNIV. PROP. OF LOCALIZATION

- MOREOVER, IF $\mathcal{C}$ IS (PRE)ADDITIVE, THEN $\Sigma^{-1}\mathcal{C}$ IS SUCH AS WELL:

$$\left[\begin{array}{c} \downarrow j \\ x \nearrow f \\ z \end{array} \begin{array}{c} \uparrow i \\ \nwarrow g \\ y \end{array} \right] + \left[\begin{array}{c} \downarrow k \\ x \nearrow h \\ w \end{array} \begin{array}{c} \uparrow l \\ \nwarrow i \\ y \end{array} \right] = \left[\begin{array}{c} \downarrow j+k \\ x \nearrow f+h \\ v \end{array} \begin{array}{c} \uparrow l \\ \nwarrow i \\ y \end{array} \right]$$

- USE (LF3) & (LF1): $\begin{array}{ccc} y & \xrightarrow{g} & z \\ \tau \downarrow & & \downarrow \exists j \\ w & \dashrightarrow_{\exists k} & v \end{array}$ AND $v = jg = k\tau \in \Sigma$

$$\text{AND } \sigma^{-1}f = v^{-1}(jf) \quad \text{AND} \quad \tau^{-1}g = v^{-1}(zg)$$

- THIS IS WELL DEFINED & IF $\mathcal{C}$ HAS FINITE DIRECT SUMS, SO HAS $\Sigma^{-1}\mathcal{C}$ AND $Q: \mathcal{C} \rightarrow \Sigma^{-1}\mathcal{C}$ PRESERVES THEM

- ALL THIS IS IN GABRIEL-ZIMMERMAN OR IS ELEMENTARY

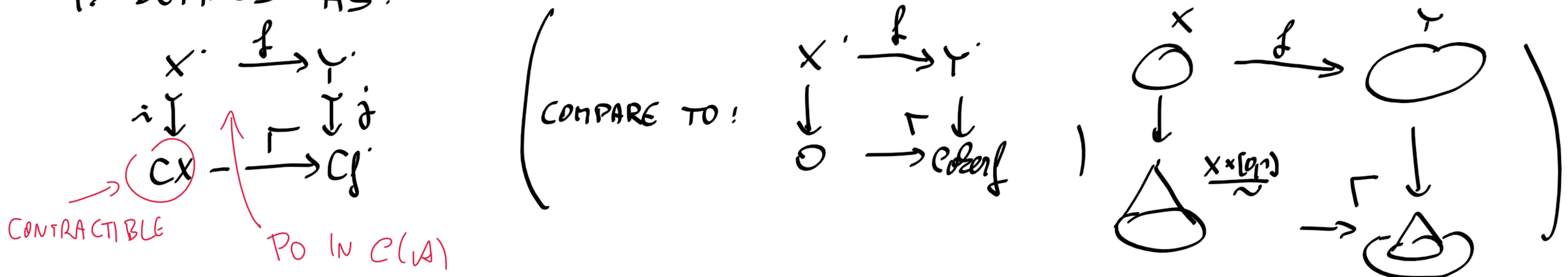
- TERMINOLOGY: @ CATEGORY, THEN Σ IS A MULTIPLICATIVE
CLASS OF MORPHISMS IF IT ADMITS BOTH
 THE CALCULUS OF LEFT & RIGHT FRACTIONS

-THN: $\wedge$ ABELIAN, THEN

$\overline{W} = \{ f: X' \rightarrow Y' \text{ in } \mathcal{K}(\mathcal{A}) \mid f \text{ QUASI-ISO} \}$
IS A MULTIPLICATIVE SET IN $\mathcal{K}(\mathcal{A})$.

- TO PROVE THAT, WE NEED TO CONSIDER MAPPING CONES OF MAPS IN $\mathcal{C}(W)$

- LET $f': X' \rightarrow Y'$ BE A MAP OF COMPLEXES. THEN $C_{f'}$, THE MAPPING CONE OF f' , IS DEFINED AS:



- EXPLICIT CONSTRUCTION OF Cf :

$$\begin{array}{c}
 X' \\
 \downarrow f \\
 Y' \\
 \downarrow j \\
 Cf \\
 \downarrow q \\
 X'[1]
 \end{array}$$

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & X^n & \xrightarrow{d_X} & X^{n+1} & \xrightarrow{d_X} & X^{n+2} \rightarrow \cdots \\
 & & \downarrow f & & \downarrow f & & \downarrow f \\
 \cdots & \rightarrow & Y^n & \xrightarrow{d_Y} & Y^{n+1} & \xrightarrow{d_Y} & Y^{n+2} \rightarrow \cdots \\
 & & \downarrow \oplus & & \downarrow \oplus & & \downarrow \oplus \\
 \cdots & \rightarrow & Y^n \oplus X^{n+1} & \rightarrow & Y^{n+1} \oplus X^{n+2} & \rightarrow & Y^{n+2} \oplus X^{n+3} \rightarrow \cdots \\
 & & \downarrow \begin{pmatrix} d_Y f \\ 0 - d_X \end{pmatrix} & & \downarrow & & \downarrow \\
 \cdots & \rightarrow & X^{n+1} & \xrightarrow{-d_X} & X^{n+2} & \xrightarrow{-d_X} & X^{n+3} \rightarrow \cdots
 \end{array}$$

$$\begin{array}{c}
 \curvearrowright \\
 \mathcal{C}(\mathcal{A}) \\
 \text{AUTO-} \\
 \text{MORPHISM}
 \end{array}$$

- FACTS: (EASY)

① $f: X' \rightarrow Y'$ IS A QUASI-ISO $\Leftrightarrow Cf'$ IS ACYCLIC

(BECAUSE WE HAVE A S.E.S. $0 \rightarrow X' \rightarrow Y' \oplus CX' \rightarrow Cf' \rightarrow 0$ IN $\mathcal{C}(\mathcal{A})$)

② WE HAVE A CANONICAL MAP $Cf \rightarrow C\text{coker } f$ WHICH IS QUASI-ISO IF f IS MONO

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & X \rightarrow Y \\
 \downarrow & & \downarrow & & \downarrow \\
 CX & \rightarrow & Cf & & 0 \rightarrow C\text{coker } f
 \end{array}$$

- SKETCH PROOF THAT $\overline{W}$ IS A MULT. CLASS OF MORPHISMS IN $\mathcal{K}(\mathcal{A})$

(LF1) $\overline{W}$ CONTAINS $\text{id}_X \forall X \in \mathcal{K}(\mathcal{A})$ AND IS CLOSED UNDER COMPOSITIONS

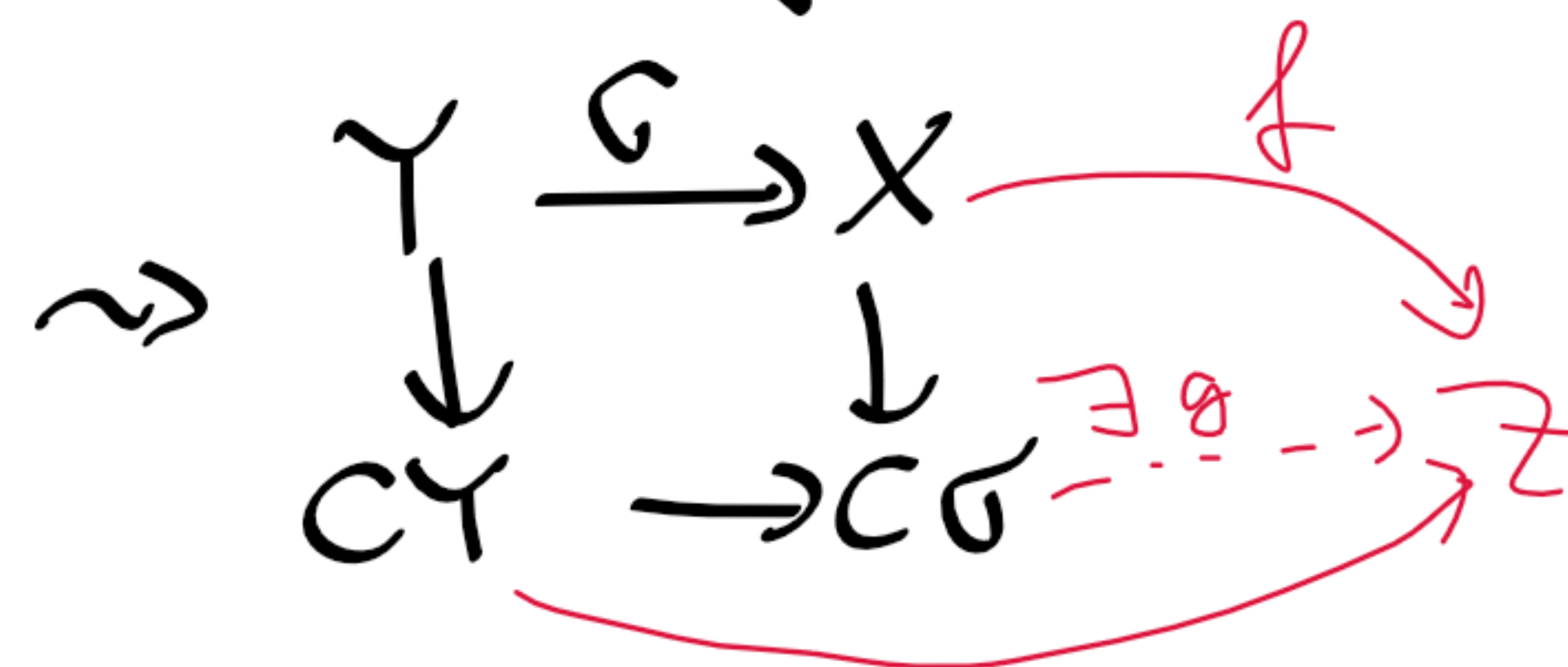
(LF2) EACH DIAGRAM $\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow \sigma & & \\ Y & & \end{array}$ IN $\mathcal{K}(\mathcal{A})$ CAN BE COMPLETED TO $\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow \sigma & \parallel & \downarrow \tau \\ Y & \xrightarrow{g} & Z \end{array}$

(LF3) WHENEVER WE HAVE A DIAGRAM $Y \xrightarrow{\sigma} X \xrightarrow[f=0]{} Z$ WITH $f \circ \sigma = 0$, THEN THERE EXIST $X \xrightarrow[g=0]{} Z \xrightarrow{\tau} W$ SUCH THAT $\tau \circ g = 0$

- (LF1) IS TRIVIAL (WE HAVE $H^*: \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{A}$)

- (LF2): $\begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow \sigma & & \\ Y & & \end{array} \rightsquigarrow W \xrightarrow{\left(\begin{smallmatrix} f \\ \sigma \end{smallmatrix}\right)} X \oplus Y \xrightarrow{(\pi, g)} C(f) \rightsquigarrow \begin{array}{ccc} W & \xrightarrow{f} & X \\ \downarrow \sigma & \parallel & \downarrow \tau \\ Y & \xrightarrow{g} & C(f) \end{array}$ HOMOLOGY PUSHOUT

- (LF3) SUPPOSE: $f \circ \sigma = 0$ IN $\mathcal{K}(\mathcal{A})$, SO $f \circ \sigma$ FACTORS THROUGH $\hat{\sigma}: Y \rightarrow CY$ IN $\mathcal{C}(\mathcal{A})$



$\rightsquigarrow$ IN $\mathcal{K}(\mathcal{A})$: $\begin{array}{ccc} Y & \xrightarrow{\sigma} & X \\ \downarrow \hat{\sigma} & & \downarrow f \\ CY & \xrightarrow{\hat{f}} & C\sigma \end{array}$ ACYCLIC AND $\tau \circ g = 0 \Rightarrow \tau \circ f = 0$

- TILTING : k A FIELD

o-o-o

$$\begin{array}{ccc}
 \mathcal{D}(k(\cdot \rightarrow \cdot \rightarrow)) & \xrightarrow[\mathbb{F}]{\mathbb{Z}} & \mathcal{D}(k(\cdot \leftarrow \cdot \rightarrow)) \\
 k_0(\text{mod } k(\rightarrow \rightarrow)) & \xrightarrow[k_0(\mathbb{F} \dots)]{\mathbb{Z}^3 \text{ GROUP HOM}} & k_0(\text{mod } k(\rightarrow \rightarrow))
 \end{array}$$

↑ CHRONOLOGICALLY

↑ POSITIVE



TILTED