DERIVED CATEGORIES

① LOCALIZATION OF CATEGORIES: ([GABRIEL-ZISMAN])

- LET $\mathcal{C}$ BE A CATEGORY AND Σ A CLASS OF MORPHISMS
- DEF: A LOCALIZATION OF $\mathcal{C}$ AT Σ IS A FUNCTOR $Q: \mathcal{C} \longrightarrow \mathcal{D}$ SUCH THAT:
  1. Q IS Σ-INVERTING, I.E. $Q(\sigma)$ IS ISO IN $\mathcal{D}$ $\forall \sigma: X \to Y$ IN Σ
  2. Q IS UNIVERSAL Σ-INVERTING, I.E.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "\forall F" ', "\Sigma\text{-INVERTING}" '] & \mathcal{D} =: \mathcal{C}[\Sigma^{-1}] \arrow[d, dashed, "\exists ! F"] \\ & \mathcal{E} \end{tikzcd}$$

- UP TO A SET-THEORETIC DISCUSSION, LOCALIZATIONS ALWAYS EXIST!
  - WE CONSTRUCT $\mathcal{C}[\Sigma^{-1}]$ AS FOLLOWS:
  - OBJECTS: $\mathrm{ob}\, \mathcal{C}[\Sigma^{-1}] := \mathrm{ob}\, \mathcal{C}$
  - MORPHISMS: EQUIVALENCE CLASSES OF ZIGZAGS MODULO RELATIONS INDUCED BY $\forall \sigma \in \Sigma$: $\sigma\sigma^{-1} = \mathrm{id} = \sigma^{-1}\sigma$
  - COMPOSITION GIVEN BY CONCATENATION AN COMPOSITION IN $\mathcal{C}$

$$X \xrightarrow{f_1 \in \mathcal{C}} X_1 \xleftarrow{\sigma_1 \in \Sigma} X_2 \xrightarrow{f_2} X_3 \xleftarrow{\sigma_2 \in \Sigma} \cdots \xleftarrow{\sigma_n \in \Sigma} Y \;\;\xrightarrow[\text{IN } \mathcal{C}[\Sigma^{-1}]]{}\;\; \sigma_n^{-1} \cdots \sigma_2^{-1} f_2 \sigma_1^{-1} f_1$$

$$[\cdot \xrightarrow{\sigma} \cdot \xleftarrow{\sigma} \cdot] = [\mathrm{id}] \circ [\cdot \xleftarrow{\sigma} \cdot \xrightarrow{\sigma} \cdot]$$

- REM: IF $\mathcal{C} \xrightarrow{Q_1} \mathcal{D}_1$ AND $\mathcal{C} \xrightarrow{Q_2} \mathcal{D}_2$ ARE TWO LOC'S OF $\mathcal{C}$ AT Σ, THEN:

$$\exists \quad \begin{tikzcd} & \mathcal{D}_1 \arrow[d, "\exists !\ \cong"] \\ \mathcal{C} \arrow[ur, "Q_1"] \arrow[dr, "Q_2"'] & \mathcal{D}_2 \end{tikzcd}$$

ISOMORPHISM OF CATEGORIES

$\leadsto$ OTHER VERSIONS OF THE UNIVERSAL PROPERTY IN THE LANGUAGE OF 2-CATEGORIES:

- $\mathcal{C}at$, THE CATEGORY OF SMALL CATEGORIES, IS A 2-CATEGORY:
  - $\forall \mathcal{C}, \mathcal{D} \in \mathcal{C}at$, WE NATURALLY HAVE A CATEGORY $[\mathcal{C}, \mathcal{D}]$:
    - OBJECTS OF $[\mathcal{C}, \mathcal{D}]$:= FUNCTORS $\mathcal{C} \xrightarrow{F} \mathcal{D}$
    - MORPHISM IN $[\mathcal{C}, \mathcal{D}]$:= NATURAL TRANSFORMATIONS $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$
  - $\forall \mathcal{C}, \mathcal{D}, \mathcal{E}$ IN $\mathcal{C}at$, THE COMPOSITION IS A FUNCTOR

$$\circ : [\mathcal{C}, \mathcal{D}] \times [\mathcal{D}, \mathcal{E}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

INCLUDED HERE

- THEN: THE UNIVERSAL PROPERTY FROM THE DEF. OF LOC. OF AT Σ CAN BE EQUIVALENTLY RESTATED AS FOLLOWS:

$Q : \mathcal{C} \to \mathcal{D}$ SATISFIES THAT $\forall \mathcal{E}$ THE FUNCTOR

$$Q^* : [\mathcal{D}, \mathcal{E}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$
$$F : \mathcal{D} \to \mathcal{E} \longmapsto F \circ Q : \mathcal{C} \to \mathcal{E}$$

RESTRICTS TO A CATEGORY ISO:

$$[\mathcal{D}, \mathcal{E}] \longrightarrow \{ \begin{smallmatrix} F \\ \mathcal{C} \\ \downarrow \\ \mathcal{E} \end{smallmatrix} \mid F \ \Sigma\text{-INVERT} \} \quad \Big(\underset{\text{FULL}}{\subseteq} [\mathcal{C}, \mathcal{E}] \Big)$$

- OFTEN, IT IS BETTER TO WEAKEN THIS CONDITION TO

$Q : \mathcal{C} \to \mathcal{D}$ SATISFIES $\forall \mathcal{E}$ THAT

$$Q^* : [\mathcal{D}, \mathcal{E}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

IS $\underbrace{\text{FULLY FAITFUL}}_{\text{Q^* INDUCES EQUIV. TO ITS ESS. IMAGE}}$ AND $\underbrace{\mathrm{Im}(Q^*)}_{\text{ESS. IMAGE}} = \{ F : \mathcal{C} \to \mathcal{E} \mid F \text{ IS } \Sigma\text{-INVERTING} \}$.

REMINDER ① $\mathcal{A}$ ABELIAN $\rightsquigarrow$ CONSTRUCT $D(\mathcal{A})$ AS A LOC. OF $C(\mathcal{A})$ AT $\Sigma = \{ f : X \to Y \mid f \text{ QUASI-ISO} \}$.

② WE'LL DIVIDE THIS LOC. INTO TWO SEPARATE STEPS

$$C(\mathcal{A}) \xrightarrow[\text{LOC. AT HTPY ISO}]{①} K(\mathcal{A}) \xrightarrow[\text{LOC. AT HTPY CLASSES OF QUASI-ISOS}]{②} D(\mathcal{A}).$$

- THE HOMOTOPY CATEGORY $K(\mathcal{A})$ (SOMETIMES DENOTED BY $\mathcal{H}(\mathcal{A})$)

- $K(\mathcal{A})$ HAS THE SAME OBJECTS AS $C(\mathcal{A})$, BUT MORPHISMS ARE THE HOMOTOPY CLASSES OF COCHAIN COMPLEX MAPS
- TWO MORPHISMS $f, g : X^\bullet \to Y^\bullet$ IN $C(\mathcal{A})$ ARE HOMOTOPIC IF $\exists$ A HOMOTOPY $s^\bullet = (s^n : X^n \to Y^{n-1})_{n \in \mathbb{Z}}$ SUCH THAT

$\forall n \in \mathbb{Z}$:
$$f^n - g^n = d_Y \circ s^n + s^{n+1} \circ d_X$$

$$\begin{tikzcd}
\arrow[r] & X^{n-1} \arrow[r, "d_X"] \arrow[d, "f^{n-1}-g^{n-1}"] \arrow[dl, red] & X^n \arrow[r, "d_X"] \arrow[d, "f^n-g^n"] \arrow[dl, red, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}-g^{n+1}"] \arrow[dl, red, "s^{n+1}"'] & {} \arrow[dl, red] \\
\arrow[r] & Y^{n-1} \arrow[r, "d_Y"'] & Y^n \arrow[r, "d_Y"'] & Y^{n+1} \arrow[r] & {}
\end{tikzcd}$$

- FACTS: 1) THERE IS AN OBVIOUS FULL FUNCTOR

$$\begin{aligned} P: \mathcal{C}(\mathcal{A}) &\longrightarrow \mathcal{K}(\mathcal{A}) \\ X &\longmapsto X \\ f^\bullet: X^\bullet \to Y^\bullet &\longmapsto [f^\bullet]_{HTPY} \end{aligned}$$

WANT TO SHOW THAT P IS A LOC. OF $\mathcal{C}(\mathcal{A})$ AT SOME CLASS OF MORPHISMS.

TO DO SO, WE NEED TO IDENTIFY THAT CLASS AND PROVE THE UNIV. PROPERTY FROM PG.1.

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2) CALL A MAP $f^\bullet: X^\bullet \to Y^\bullet$ IN $\mathcal{C}(\mathcal{A})$ A **HOMOTOPY ISOMORPHISM** IF $P(f^\bullet) = [f^\bullet]_{HTPY}$ IS AN ISO IN $\mathcal{K}(\mathcal{A})$.

WE CAN WELL TAKE THE CLASS Σ_H OF ALL HTPY ISO FOR

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3) IF $f^\bullet, g^\bullet: X^\bullet \to Y^\bullet$ ARE HOMOTOPIC, THEN $H^*(f^\bullet) = H^*(g^\bullet): H^*(X^\bullet) \longrightarrow H^*(Y^\bullet)$

THAT IS, $H^n: \mathcal{C}(\mathcal{A}) \to \mathcal{A}$ LIFT OVER

$$\begin{tikzcd} \mathcal{C}(\mathcal{A}) \arrow[r, "P"] \arrow[dr, "H^n"'] & \mathcal{K}(\mathcal{A}) \arrow[d, dashed, "\exists!\, H^n"] \\ & \mathcal{A} \end{tikzcd}$$

CONSEQUENTLY, ANY HOMOTOPY ISO IS QUASI-ISO ($\Sigma_H \subseteq \Sigma$).

• EXPL: PUT $\mathcal{A} = \mathrm{Mod}\,R$, $M \in \mathrm{Mod}\,R$, CONSIDER A PROJ. RES.

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

⇝ WE HAVE A QUASI-ISO:

$$\begin{tikzcd} P_\bullet \arrow[d, "\sigma"'] & \cdots \arrow[r] & P_3 \arrow[r] \arrow[d, "0"'] & P_2 \arrow[r] \arrow[d, "0"'] & P_1 \arrow[r] \arrow[d, "0"'] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[d, "0"] \\ M & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

– BUT σ IS A HOMOTOPY ISO IFF M IS PROJECTIVE!

– IT REMAINS TO PROVE THE UNIVERSAL PROPERTY:

$$\begin{tikzcd} \mathcal{C}(\mathcal{A}) \arrow[r, "P"] \arrow[dr, "\forall F \ \Sigma_H\text{-INVERTING}"'] & \mathcal{K}(\mathcal{A}) \arrow[d, dashed, "?\exists!\, F"] \\ & \mathcal{E} \end{tikzcd}$$

– OUTLINE:

- A COMPLEX $X^\bullet \in \mathcal{C}(\mathcal{A})$ IS CONTRACTIBLE IF $P(X^\bullet) \cong 0$ IN $\mathcal{K}(\mathcal{A})$.

EQUIVALENTLY:

(1) $1_{X^\bullet}$ IS HOMOTOPIC TO $0_{X^\bullet}$. OR

(2) $X \cong \bigoplus \left(\dots \to 0 \to 0 \to Y = Y \to 0 \to 0 \to \dots \right)$ IN $\mathcal{C}(\mathcal{A})$

- EXERCISE: $f : X^\bullet \to Y^\bullet$ IN $\mathcal{C}(\mathcal{A})$ IS NULLHOMOTOPIC (IE HTPIC TO $0 : X^\bullet \to Y^\bullet$) IFF f FACTORS AS

$$\begin{tikzcd} X^\bullet \arrow[rr, "f"] \arrow[dr] & & Y^\bullet \\ & C^\bullet \arrow[ur] & \end{tikzcd}$$

CONTRACTIBLE

- IN FACT:

$$\begin{tikzcd} X^\bullet \arrow[d, "i"] \arrow[dd, bend right, "f"'] \\ CX^\bullet \arrow[d, dashed] \\ Y^\bullet \end{tikzcd}
\qquad
\begin{tikzcd} \dots \arrow[r] & X^{n-1} \arrow[r, "d"] \arrow[d, "\binom{1}{d}"] & X^n \arrow[r, "d"] \arrow[d, "\binom{1}{d}"] & X^{n+1} \arrow[r] \arrow[d, "\binom{1}{d}"] & \dots \\ \arrow[r] & X^{n-1} \oplus X^n \arrow[r, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[d, "(:)"] & X^n \oplus X^{n+1} \arrow[r, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[d] & X^{n+1} \oplus X^{n+2} \arrow[r] \arrow[d] & {} \\ \dots \arrow[r] & Y^{n-1} \arrow[r, "d"'] & Y^n \arrow[r, "d"'] & Y^{n+1} \arrow[r] & \dots \end{tikzcd}$$