- AXIOMS FOR $\mathrm{Ext}^n_{\mathcal{A}}$'S:

- VIEW $\mathrm{Ext}^n_{\mathcal{A}}(-, Y) : \mathcal{A}^{op} \longrightarrow \mathcal{A}b$ AS THE n-TH RIGHT DERIVED FUNCTOR OF $\mathrm{Hom}(-, Y) : \mathcal{A}^{op} \to \mathcal{A}b$ (LEFT EXACT FUNCTOR).
- ASSUME THAT $\mathcal{A}$ HAS ENOUGH PROJECTIVES ($\Leftrightarrow$ $\mathcal{A}^{op}$ HAS ENOUGH INJECTIVES)

- AIM: AXIOMATIZE RIGHT DERIVED FUNCTORS, FIRST IF OUR ABELIAN CATEGORY HAS ENOUGH INJECTIVES

(REFERENCE: ROTMAN'S INTRO TO HOMOLOGICAL ALGEBRA, Thm 6.36, p. 360)

- LET $\mathcal{B}$ BE ABELIAN WITH ENOUGH INJECTIVES

- DEF: IF $\mathcal{C}$ IS ALSO AN ABELIAN CATEGORY, THEN A COHOMOLOGICAL ∂-FUNCTOR IS

  - A SEQUENCE OF FUNCTORS $(F^n : \mathcal{B} \to \mathcal{C})_{n \geq 0}$

SUCH THAT F^n ARE ALL ADDITIVE AND

- $\forall$ ANY S.E.S. $\varepsilon : 0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ IN $\mathcal{A}$ WE HAVE A LONG EXACT SEQ.:

$$0 \to F^0X \xrightarrow{F^0 i} F^0Y \xrightarrow{F^0\pi} F^0Z \xrightarrow{\partial^0_\varepsilon} F^1X \xrightarrow{F^1 i} F^1Y \xrightarrow{F^1\pi} F^1Z \xrightarrow{\partial^1_\varepsilon} F^2X \xrightarrow{F^2 i} \cdots$$

- THE LONG EXACT SEQUENCES ARE NATURAL IN ε, I.E. IF WE HAVE $\varepsilon' : 0 \to X' \xrightarrow{i'} Y' \xrightarrow{\pi'} Z' \to 0$ AND A COMMUTATIVE DIAGRAM

$$\begin{array}{ccccccccc}
\varepsilon: & 0 & \to & X & \to & Y & \to & Z & \to 0 \\
 & & & \downarrow u & & \downarrow v & & \downarrow w & \\
\varepsilon': & 0 & \to & X' & \to & Y' & \to & Z' & \to 0
\end{array}$$

THEN WE OBTAIN A COMMUTATIVE DIAGRAM!

$$\begin{array}{ccccccccc}
0 & \to & F^0X & \to & F^0Y & \to & F^0Z & \xrightarrow{\partial_\varepsilon} & F^1X \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow F^0w & & \downarrow F^1u \\
0 & \to & F^0X' & \to & F^0Y' & \to & F^0Z' & \xrightarrow[\partial_{\varepsilon'}]{} & F^1X' \to \cdots
\end{array}$$

– T (DUAL TO ROTHAN'S Thm 6.36)

LET $\mathcal{B}, \mathcal{C}$ BE ABELIAN, $\mathcal{B}$ HAS ENOUGH INJECTIVES.

SUPPOSE WE HAVE $H = (H^n, \partial^n)$, $F = (F^n, \partial^n)$ BE COHOMOLOGICAL ∂-FUNCTORS.

ASSUME: $H^n(E) = 0 \quad \forall E \in \mathcal{B}$ INJECTIVE $\forall n > 0$

THEN:

UNIVERSAL PROPERTY

(1) ANY NATURAL TRANSFORMATION $\tau^0 : H^0 \to F^0$ UNIQUELY EXTENDS TO A NATURAL TRANSFORMATION $\tau : H \to F$ OF COHOMOLOGICAL ∂-FUNCTORS. IF τ^0 IS INVERTIBLE, SO IS τ.

(2) IF $H^0 : \mathcal{B} \to \mathcal{C}$ IS LEFT EXACT, THEN THERE EXIST UNIQUE (UP TO NATURAL EQUIVALENCE) $H = (H^n, \partial^n)_{n \geq 0}$ SUCH THAT $H^n(E) = 0 \ \forall E$ INJ. $\forall n > 0$.

– PROOF (SKETCH):

– SAY WE HAVE CONSTRUCTED $\tau^n : H^n \to F^n$, $n \geq 0$

WANT $\tau^{n+1}_B : H^{n+1}B \to F^{n+1}B$ FOR SOME $B \in \mathcal{B}$ /

$\rightsquigarrow$ CONSIDER: $0 \to B \xrightarrow{\text{INJ.}} E \xrightarrow{\pi} C \to 0$ AND

$$\begin{array}{ccccccc} H^n E & \xrightarrow{H^n \pi} & H^n C & \xrightarrow{\partial^n} & H^{n+1}B \overset{\in \{0\}}{=} 0 & \longrightarrow & H^{n+1}E \overset{=}{=} 0 \\ \downarrow \tau^n_E & = & \downarrow \tau^n_C & & \vdots \exists!\, \tau^{n+1}_K & & \\ F^n E & \longrightarrow & F^n C & \xrightarrow[\partial^n]{} & F^{n+1}B & & \end{array}$$

EXACT ROWS

$\Rightarrow H^{n+1}K = \operatorname{Coker} H^n \pi$

— WHAT IF $\mathcal{B}$ DOES NOT HAVE ENOUGH INJECTIVES?

- WE CAN USE THE UNIVERSAL PROPERTY FROM Thm 6.36 (1):

  $H = (H^n, \partial^n)$ IS A UNIVERSAL COHOMOLOGICAL ∂-FUNCTOR (FOR H^0) IF ANY NATURAL TRANSFORMATION $\tau^0 : H^0 \to F^0$ UNIQUELY EXTENDS TO A NATURAL TRANSFORMATION $\tau : H \to F$ OF COHOMOLOGICAL ∂-FUNCTORS.
  IF τ^0 IS INVERTIBLE, SO IS τ.

  (MAC LANE'S HOMOLOGY, §XII.7, §XII.8)

- FOR $\mathcal{B} = \mathcal{A}^{\mathcal{C}}$, $\mathcal{C} = \mathcal{A}b$, $H^0 = \mathrm{Hom}(-, T) : \mathcal{B} \to \mathcal{A}b$, WE HAVE THE FOLLOWING PROPERTY: (*) $\forall X \in \mathcal{B}$ $\forall n > 0$ $\forall \xi \in H^n(X)$ $\exists i_\xi : X \to Y$ SUCH THAT $H^n i_\xi(\xi) = 0$.

NEEDS TO BE CHECKED...

- MODIFICATION OF ROTMAN'S Thm 6.36(1): IF $H = (H^n, \partial^n)$ IS A COHOMOLOGICAL ∂-FUNCTOR & SATISFIES (*), THEN IT IS UNIVERSAL (IN THE SENSE ABOVE).

- Q1: WHY DOES YONEDA Ext SATISFY (4)?

$$H^n := Ext^n_{\mathcal{A}}(-, Y) : \mathcal{A}^{op} \longrightarrow Ab$$

$$X \longmapsto \xi = [0 \to Y \xrightarrow{i} E_1 \to \dots \to E_n \xrightarrow{\pi} X \to 0]_\sim \quad \text{IN } \mathcal{A}$$

- CLAIM: IF WE TAKE $\pi : X \to E_n$ IN $\mathcal{A}^{op}$, THEN $H^n(\pi) : Ext^n_{\mathcal{A}}(X, Y) \to Ext^n_{\mathcal{A}}(E_n, Y)$

- PF: - FOR $n = 1$

$$\begin{array}{ccccccc} 0 & \to & Y & \to & F & \xrightarrow{\oplus} & E_n \to 0 \\ & & \| & & \downarrow & & \downarrow \pi \\ 0 & \to & Y & \to & E_1 & \xrightarrow{\pi} & X \to 0 \end{array}$$

- THE UPPER ROW SPLITS!

- Q2 : BALANCEDNESS ?

2 WAYS TO LOOK AT $Ext^n_{\mathcal{A}}(X,Y)$:

$$Ext^n_{\mathcal{A}}(X,Y) = \underbrace{Ext^n_{\mathcal{A}}(-,Y)}_{n\text{-TH RIGHT DERIVED FUNCTOR OF } Hom(-,Y):\mathcal{A}^{op}\to Ab}(X)$$

$$Ext^n_{\mathcal{A}}(X,Y) = \underbrace{Ext^n_{\mathcal{A}}(X,-)}_{n\text{-TH RIGHT DERIVED FUNCTOR OF } Hom(X,-):\mathcal{A}\to Ab}(Y)$$

- ONE MUST PROVE THAT THESE ARE NATURALLY ISOMORPHIC !
- FOR Ext^1's, ONE CAN USE YONEDA Ext.
- OR PLAY WITH BICOMPLEXES (IF $\mathcal{A}$ HAS ENOUGH INJECTIVES PROJECTIVES) $Ext^n_{\mathcal{A}}(X,Y)$, TAKE $P_\bullet \to X$, $Y \to E^\bullet$, CONSIDER $Hom(P_\bullet, I^\bullet)$
- OR USE RESOLUTIONS AND LOOK AT arXiv : 1802.09989

– TOWARDS THE DERIVED CATEGORY

- $\mathcal{A}$ ABELIAN, $\mathcal{C}(\mathcal{A})$ = COMPLEXES OVER $\mathcal{A}$,

$$\Sigma = \{ G : X^\bullet \to Y^\bullet \text{ IN } \mathcal{C}(\mathcal{A}) \mid H^\bullet(G) \text{ ISO} \}$$

– DEF: $\mathcal{D}(\mathcal{A}) = \mathcal{C}(\mathcal{A})[\Sigma^{-1}]$.

(UNIVERSAL FUNCTOR $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$
GABRIEL-ZISMAN, THE CALCULUS OF FRACTIONS)

– WE DIVIDE THE LOCALIZATION IN 2 STEPS:

$$\mathcal{C}(\mathcal{A}) \xrightarrow{\text{(1)}} \mathcal{R}(\mathcal{A}) \xrightarrow{\text{(2)}} \mathcal{D}(\mathcal{A})$$

(1): LOCALIZATION AT THE CLASS OF HTPY ISO'S (THESE ARE ALL IN Σ)

$\mathcal{R}(\mathcal{A})$: THE HOMOTOPY CATEGORY OF COMPLEXES (SAME OBJECTS, IDENTIFY HOMOTOPIC MORPHISMS)

(2): LOCALIZE AT Σ

→ THIS LOCALIZATION IS A SO CALLED ORE LOCALIZATION, WE HAVE FRACTION ($\sigma^{-1} f$, $g \tau^{-1}$, $\sigma, \tau \in \Sigma$)