— PULLBACKS, PUSHOUTS AND (CO)HOMOLOGIES OF COMPLEX

• $\mathcal{A}$ ABELIAN CATEGORY

$\rightsquigarrow$ $\mathcal{A}$ HAS FINITE PRODUCTS & COPRODUCTS & $\underbrace{\text{KERNELS \& COKERNELS}}$

EQUIV. EQUALIZERS AND COEQUALIZERS

$$\begin{tikzcd} K \arrow[r, dashed] & X \arrow[r, shift left] \arrow[r, shift right] & Y \\ L \arrow[u, dashed, "\exists!"] \arrow[ur, dashed, "\forall"'] & & \end{tikzcd}$$

$\Rightarrow$ $\mathcal{A}$ HAS ALL FINITE LIMITS AND COLIMITS

$\Rightarrow$ $\mathcal{A}$ HAS ALL PULLBACKS & PUSHOUTS

$$\begin{tikzcd} \forall Q \arrow[r, dashed, "\exists!"'] \arrow[rr, dashed, bend left] \arrow[dr, dashed, bend right] & P \arrow[r, dashed, "i"] \arrow[d, dashed, "j"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ & Z \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\binom{i}{-j}"] \arrow[rr, dashed, bend right] & X \oplus Z \arrow[r, "(f{,}g)"] & Y \arrow[r] & 0 \end{tikzcd}$$

$(f,g)\binom{i}{-j} = f \circ i - g \circ j$

—REM: IF R IS A RING AND $\mathcal{A} = \mathrm{Mod}\, R$, AND WE HAVE $\begin{tikzcd} & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$

$\rightsquigarrow$ THE PULL BACK IS

$$\begin{tikzcd} \{(x,z) \mid g(x) = f(z)\} = P \arrow[r, "i"] \arrow[d, "j"] \arrow[dr, phantom, "="] & X \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

$\{(x,z) \mid g(x) = f(z)\} \subseteq X \oplus Z$

$j(x,z) := z$, $i(x,z) := x$

- PUSHOUTS IN $\mathrm{Mod}\,R$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "i"] \\ Z \arrow[r, "j"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad = \frac{X \oplus Z}{\{(-g(y), f(y)) \in X \oplus Z \mid y \in Y\}}$$

$$\begin{tikzcd} y \in Y \arrow[r, mapsto] \arrow[d, mapsto] & g(y) \arrow[d, mapsto] \\ f(y) \arrow[r, mapsto] & (g(y), 0) + \{\ldots\} = (0, f(y)) + \{\ldots\} \end{tikzcd}$$

$i : x \longmapsto (x, 0) + \{\ldots\}$

$j : z \longmapsto (0, z) + \{\ldots\}$

– PROPERTIES OF PULLBACKS AND PUSHOUTS

- SUPPOSE $\mathcal{A}$ IS ABELIAN AND WE HAVE

$$\begin{tikzcd} & X \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

⇝ WE HAVE :

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker} f \arrow[r, "\ell"] \arrow[d, equal] & P \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "g \circ c"] \arrow[d, "g"] & \mathrm{Coker} f \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Ker} f \arrow[r, "k"'] & Z \arrow[r, "f"'] & Y \arrow[r, "c"'] & \mathrm{Coker} f \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Ker} f \arrow[dr, dashed, "\exists ! \ell"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "k"'] & & \\ & P \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ & Z \arrow[r, "f"'] & Y \end{tikzcd}$$

$= 0$

⇝ FACT: UPPER (HIGHLIGHTED) ROW IS EXACT

(I.E. $0 \to \mathrm{Ker} f \xrightarrow{\ell} P \to \mathrm{Im}\, i \to 0$ & $0 \to \mathrm{Im}\, i \to X \xrightarrow{g \circ c} \mathrm{Coker} f$)

ARE EXACT

⇝ A POSSIBLE PROOF : (1) PROVE FOR $\mathcal{A} = \mathrm{Mod} R$ USING EXPLICIT CONSTRUCTION OF THE PULLBACK

(2) USE MITCHELL'S THEOREM FOR A GENERAL $\mathcal{A}$

- FACT: A DUAL PROPERTY HOLD FOR PUSHOUTS.
  - POSSIBLE PROOFS
    - USE THAT:
      - $\mathcal{A}$ ABELIAN $\Rightarrow$ $\mathcal{A}^{op}$ ABELIAN
      - PULLBACKS IN $\mathcal{A}$ $\longleftrightarrow$ PUSHOUTS IN $\mathcal{A}^{op}$
    - SAME STRATEGY AS BEFORE:
      1. PROVE FOR $\mathrm{Mod}\,R$
      2. THEN USE MITCHELL'S THEOREM
    - DIRECTLY FROM AXIOMS

– CONSEQUENCES:

- PULLBACK ALONG A MONOMORPHISM IS A MONOMORPHISM

$$\begin{tikzcd} P \arrow[r, dashed, tail] \arrow[d, dashed] & X \arrow[d, "g"] \\ Z \arrow[r, tail, "f"'] & Y \end{tikzcd}$$

- PULLBACK ALONG AN EPIMORPHISM IS AN EPI., EVEN BETTER

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & P \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\circledast"] & X \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & K \arrow[r] & Z \arrow[r, "f"'] & Y \arrow[r] & 0 \end{tikzcd}$$

AND ⊛ IS ALSO A PUSHOUT

$\left(0 \to P \to X \oplus Z \xrightarrow{(g,g)} Y \to 0\right)$ IS EXACT

- DUALLY FOR PUSHOUTS

– REFERENCES: TEXTBOOKS ABOUT ABELIAN CATEGORIES

- CONSEQUENCES FOR YONEDA Ext IN $\mathcal{A}$:

- SAY WE HAVE A S.E.S $\varepsilon: 0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ AND $M \in \mathcal{A}$

$\rightsquigarrow$ LONG EXACT SEQUENCE:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M,X) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(M,Y) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathcal{A}}(M,Z) \xrightarrow{\partial_\varepsilon} \mathrm{Ext}^1_{\mathcal{A}}(M,X) \xrightarrow{i_*} \mathrm{Ext}^1_{\mathcal{A}}(M,Y) \xrightarrow{\pi_*} \cdots$$

$$(f: M \to X) \longmapsto i \circ f$$

$$(g: M \to Y) \longmapsto \pi \circ g$$

- LOOK AT $\partial_\varepsilon: \mathrm{Hom}_{\mathcal{A}}(M,Z) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(M,X)$

$$(h: M \to Z) \longmapsto [\tilde{\varepsilon}: 0 \to X \to E \to M \to 0], \text{ WHERE}$$

$$\begin{tikzcd}
\tilde{\varepsilon}: 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[r] \arrow[d, "h"] & 0 \\
\varepsilon: 0 \arrow[r] & X \arrow[r, "i"] & Y \arrow[r, "\pi"] & Z \arrow[r] & 0
\end{tikzcd}$$

(PROOFS: MACLANE, HOMOLOGY, §XII.5 (Ext W/O PROJECTIVES))

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M,X) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(M,Y) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathcal{A}}(M,Z) \xrightarrow{\partial_\varepsilon} \mathrm{Ext}^1_{\mathcal{A}}(M,X) \xrightarrow{i_*} \mathrm{Ext}^1_{\mathcal{A}}(M,Y) \xrightarrow{\pi_*} \mathrm{Ext}^1_{\mathcal{A}}(M,Z) \xrightarrow{\partial_\varepsilon} \mathrm{Ext}^2_{\mathcal{A}}(M,X)$$

$$\varepsilon : 0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$$

- $i_* : \mathrm{Ext}^1_{\mathcal{A}}(M,X) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(M,Y)$

$$[\eta : 0 \to X \to E \to M \to 0]_\sim \longmapsto [\mu : 0 \to Y \to F \to M \to 0], \text{ WHERE}$$

$$\begin{array}{l} \eta : 0 \to X \to E \to M \to 0 \\ \qquad\quad i \downarrow \quad \ulcorner \downarrow \quad \parallel \\ \mu : 0 \to Y \to F \to M \to 0 \end{array}$$

- $\partial_\varepsilon : \mathrm{Ext}^1_{\mathcal{A}}(M,Z) \longrightarrow \mathrm{Ext}^2_{\mathcal{A}}(M,X)$

$$[\eta : 0 \to Z \xrightarrow{j} E \xrightarrow{r} M \to 0] \longmapsto [0 \to X \xrightarrow{i} Y \xrightarrow{j \circ \pi} E \xrightarrow{r} M \to 0]_\sim$$

– COMPLEXES:

- $\mathcal{A}$ ABELIAN CATEGORY
- A (COCHAIN) COMPLEX IN $\mathcal{A}$ IS A DIAGRAM OF THE FORM

$$X^\bullet: \quad \cdots \longrightarrow X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} X^2 \longrightarrow \cdots$$

SUCH THAT $d^i d^{i-1} = 0 \quad \forall i \in \mathbb{Z}$

- A MORPHISM OF CHAIN COMPLEXES:

$$\begin{tikzcd}
X^\bullet: \arrow[d, "f=(f^i \mid i\in\mathbb{Z})"] & \cdots \arrow[r] & X^{-1} \arrow[r, "d_X"] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "="] & X^0 \arrow[r, "d_X"] \arrow[d, "f^0"] \arrow[dr, phantom, "="] & X^1 \arrow[r, "d_X"] \arrow[d, "f^1"] \arrow[dr, phantom, "="] & X^2 \arrow[r] \arrow[d, "f^2"] & \cdots \\
Y^\bullet: & \cdots \arrow[r] & Y^{-1} \arrow[r, "d_Y"'] & Y^0 \arrow[r, "d_Y"'] & Y^1 \arrow[r, "d_Y"'] & Y^2 \arrow[r] & \cdots
\end{tikzcd}$$

$\rightsquigarrow$ HAVE ABELIAN CATEGORY OF COCHAIN COMPLEXES IN $\mathcal{A}$, DENOTE IT BY $\mathcal{C}(\mathcal{A})$.

– COHOMOLOGY: $X^\bullet \in \mathcal{C}(\mathcal{A})$, $n \in \mathbb{Z}$ $\rightsquigarrow$ $H^n(X^\bullet) = \dfrac{\ker d_X^n}{\operatorname{Im} d_X^{n-1}}$,

$$\begin{tikzcd}
& X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d] & X^n \arrow[r, "d_X^n"] & X^{n+1} & \\
0 \arrow[r] & \operatorname{Im} d_X^{n-1} \arrow[ur] \arrow[r, dotted, "\exists!"] & \ker d_X^n \arrow[u] \arrow[r] & H^n(X^\bullet) \arrow[d] & \\
& & & 0 &
\end{tikzcd}$$

– ACTUALLY, WE OBTAIN A FUNCTOR $H^n: \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{A}$

- **KEY FACT:**

LET $\mathcal{A}$ BE AN ABELIAN CATEGORY AND

$$\begin{tikzcd}
X^\bullet \arrow[d, "i^\bullet"'] & \cdots \arrow[r] & X^{-1} \arrow[r] \arrow[d, "i^{-1}"] & X^0 \arrow[r] \arrow[d, "i^0"] & X^1 \arrow[r] \arrow[d, "i^1"] & \cdots \\
Y^\bullet \arrow[d, "\lambda^\bullet"'] & \cdots \arrow[r] & Y^{-1} \arrow[r] \arrow[d, "\lambda^{-1}"] & Y^0 \arrow[r] \arrow[d, "\lambda^0"] & Y^1 \arrow[r] \arrow[d, "\lambda^1"] & \cdots \\
Z^\bullet & \cdots \arrow[r] & Z^{-1} \arrow[r] & Z^0 \arrow[r] & Z^1 \arrow[r] & \cdots
\end{tikzcd}$$

SNAKE LEMMA SITUATION

IS A S.E.S. IN $\mathcal{C}(\mathcal{A})$.

THEN WE HAVE A LONG EXACT SEQUENCE OF COHOMOLOGIES IN $\mathcal{A}$

$$\cdots \to H^{-1}(Y^\bullet) \xrightarrow{H^{-1}(\lambda^\bullet)} H^{-1}(Z^\bullet) \xrightarrow{\partial^{-1}} H^0(X^\bullet) \xrightarrow{H^0(i^\bullet)} H^0(Y^\bullet) \xrightarrow{H^0(\lambda^\bullet)} H^0(Z^\bullet) \xrightarrow{\partial^0} H^1(X^\bullet) \xrightarrow{H^1(i^\bullet)} H^1(Y^\bullet) \to \cdots$$

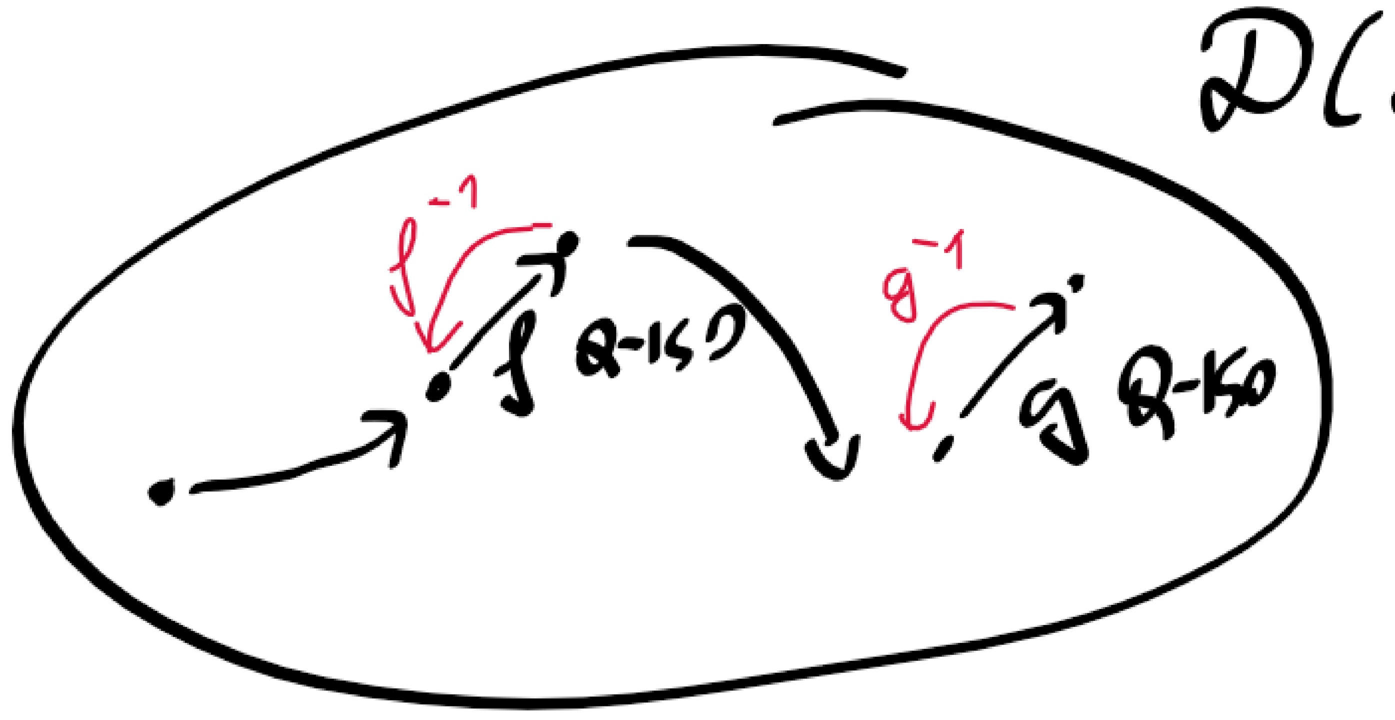
- **DEF:** A MAP $f^\bullet : X^\bullet \to Y^\bullet$ IN $\mathcal{C}(\mathcal{A})$ IS CALLED A QUASI-ISOMORPHISM IF $H^n(f^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$ IS ISO IN $\mathcal{A}$ $\forall n \in \mathbb{Z}$.

- DENOTE $\mathcal{W} = \{ f^\bullet : X^\bullet \to Y^\bullet \mid f^\bullet \text{ IS QUASI-ISO} \}$

- THEN THE DERIVED CATEGORY $\mathcal{D}(\mathcal{A})$ IS CONSTRUCTED AS

$$\mathcal{C}(\mathcal{A}) \xrightarrow{Q} \mathcal{D}(\mathcal{A}) = \mathcal{C}(\mathcal{A})[\mathcal{W}^{-1}]$$

$\mathcal{D}(\mathcal{A})$



$\rightsquigarrow$ $\mathcal{D}(\mathcal{A})$ HAS MORPHISMS AS

$$\bullet \xrightarrow{a} \bullet \xrightarrow{f^{-1}} \bullet \xrightarrow{b} \bullet \xrightarrow{g^{-1}} \bullet \xrightarrow{c} \bullet \rightsquigarrow c\, g^{-1}\, b\, f^{-1}\, a$$