DERIVED CATEGORIES & DERIVED EQUIVALENCES

- LITERATURE

- WEIBEL, AN INTRODUCTION TO HOMOLOGICAL ALGEBRA, 1994
- ROTMAN, ——— // ———, 2009
- MAC LANE, HOMOLOGY
- CARTAN & EILENBERG

- "CLASSICAL HOMOLOGICAL ALGEBRA": Ext, Tor
  - SETUP: $\mathcal{A}$ ABELIAN CATEGORY $\underbrace{\left(\text{Mod-}R,\ R \text{ RING},\ \text{Rep}(Q,k),\ Q \text{ QUIVER},\ k \text{ FIELD}\right)}_{\text{EXAMPLES}}$

- A CATEGORY $\mathcal{A}$ IS ABELIAN IF
  1. IT IS ADDITIVE $\left(\text{Hom}_{\mathcal{A}}(X,Y) \in \mathcal{A}b,\ \underbrace{\text{Hom}_{\mathcal{A}}(X,Y) \times \text{Hom}_{\mathcal{A}}(Y,Z) \xrightarrow{\text{comp}} \text{Hom}_{\mathcal{A}}(X,Z)}_{\text{BILINEAR}} \right)$

     & $\mathcal{A}$ HAS FINITE DIRECT SUM
  2. $\mathcal{A}$ HAS ALL KERNELS $\left(\forall X \xrightarrow{f} Y \ \exists K \xrightarrow{g} X : \right.$ $\begin{tikzcd} K \arrow[r, "g"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] & Y \\ L \arrow[u, dashed, "\exists!"] \arrow[ur, "\ell\ \forall"'] \arrow[urr, bend right, "=0"'] & & \end{tikzcd}$ $\left.\right)$

     AND COKERNELS.

⋮

- A CATEGORY $\mathcal{A}$ IS ABELIAN IF

(1) IT IS ADDITIVE ($\mathrm{Hom}_{\mathcal{A}}(X,Y) \in \mathcal{A}b$, $\mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(Y,Z) \xrightarrow[\text{BILINEAR}]{\text{comp}} \mathrm{Hom}_{\mathcal{A}}(X,Z)$)

& $\mathcal{A}$ HAS FINITE DIRECT SUM

(2) $\mathcal{A}$ HAS ALL KERNELS AND COKERNELS. ($\forall X \xrightarrow{f} Y$ $\exists K \xrightarrow{k} X$:

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] & Y \\ L \arrow[u, dashed, "\exists!"] \arrow[ur, "\ell\ \forall"'] \arrow[urr, bend right, "=0"'] & & \end{tikzcd}$$

)

(3) $\mathcal{A}$ HAS IMAGES :

$$\begin{tikzcd} & X \arrow[rr, "f"] \arrow[dr, "\pi"'] \arrow[drr, dashed, "\exists!"] \arrow[rr, bend left, "0"] & & Y \arrow[dr, "c"] \arrow[dr, bend left, "=0"] & \\ K \arrow[ur, "k"] \arrow[ur, phantom, "\text{KERNEL}"'] & & I \arrow[r, dashed, "\sim"] \arrow[r, phantom, "\exists!"'] & L \arrow[u, "\ell"] & C \arrow[l, phantom, "\text{COKERNEL}"] \end{tikzcd}$$

REQUIRE THIS TO BE ISO

$I :=$ THE IMAGE OF f ,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & I \arrow[ur] & \end{tikzcd}$$

- NOW IN ANY ABELIAN CATEGORY, WE CAN DEFINE Ext - FUNCTORS.

- IDEA OF Ext   (Ext COMES FROM EXTENSIONS)

  - RECALL: A SHORT EXACT SEQUENCE (S.E.S.) IN $\mathcal{A}$
    IS A DIAGRAM OF THE FORM
    $$0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow 0$$
    SUCH THAT $i = \mathrm{Ker}(p)$ & $p = \mathrm{Coker}(i)$.

  - IF WE HAVE A S.E.S $0 \to X \to Y \to Z \to 0$, THEN WE SAY THAT Y IS AN EXTENSION OF X BY Z.

- THM (MITCHELL) LET $\mathcal{A}$ BE A SMALL ABELIAN CATEGORY.
  THE THERE EXISTS A RING R AND A FULLY-FAITHFUL & EXACT & ADDITIVE FUNCTOR
  $$\mathcal{A} \overset{F}{\hookrightarrow} \mathrm{Mod}\,R$$

  $\forall X,Y \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \mathrm{Hom}_R(FX,FY)$
  $f \longmapsto F(f)$

  F SENDS S.E.S. TO S.E.S.

IF $\mathcal{A}$ HAS ENOUGH PROJECTIVE OR INJECTIVE OBJECTS, WE CAN CONSTRUCT RESOLUTIONS

FOR GENERAL $\mathcal{A}$: YONEDA Ext's
(MAC LANE, HOMOLOGY, SECTION Ext W/O PROJECTIVES)

AXIOMATIC APPROACH

- IDEA: IF $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ IS EXACT IN $\mathcal{A}$ & $W \in \mathcal{A}$, THEN

$$0 \to \mathrm{Hom}_{\mathcal{A}}(W,X) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(W,Y) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathcal{A}}(W,Z) \quad \text{EXACT}$$

UNIVERSAL

$\ast$

$$\xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{A}}(W,X) \xrightarrow{i_*} \mathrm{Ext}^1_{\mathcal{A}}(W,Y) \xrightarrow{\pi_*} \mathrm{Ext}^1_{\mathcal{A}}(W,Z)$$
$$\to \mathrm{Ext}^2_{\mathcal{A}}(W,X) \xrightarrow{i_*} \mathrm{Ext}^2_{\mathcal{A}}(W,Y) \xrightarrow{\pi_*} \mathrm{Ext}^2_{\mathcal{A}}(W,Z)$$
$$\to \mathrm{Ext}^3_{\mathcal{A}}(W,X) \to \cdots$$

- PROVISIONAL NOTATION:

• IF $X, Y \in \mathcal{A}$, $n \geq 1$, DENOTE $\mathcal{E}^n(X,Y) = \{ \varepsilon : 0 \to Y \to E_1 \to \dots \to E_n \to X \to 0 \text{ EXACT IN } \mathcal{A} \}$

• WE DECLARE ELEMENTS $\varepsilon, \varepsilon'$ OF $\mathcal{E}^n(X,Y)$ EQUIVALENT PROVIDED THERE EXISTS A DIAGRAM AS FOLLOWS:

$$\begin{tikzcd}
\varepsilon: & 0 \arrow[r] & Y \arrow[r, "e_0"] \arrow[d, equal] & E_1 \arrow[r, "e_1"] \arrow[d, "f_1"] & E_2 \arrow[r, "e_2"] \arrow[d, "f_2"] & \dots \arrow[r] & E_n \arrow[r, "e_n"] \arrow[d, "f_n"] & X \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon': & 0 \arrow[r] & Y \arrow[r, "e'_0"'] & E'_1 \arrow[r, "e'_1"'] & E'_2 \arrow[r] & \dots \arrow[r] & E'_n \arrow[r, "e'_n"'] & X \arrow[r] & 0
\end{tikzcd}$$

• REM. IF $n = 1$, THEN f_1 AS ABOVE IS ISO:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "f_1"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & E'_1 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

NO ANALOGY OF THAT IF $n \geq 2$!

$$\begin{tikzcd}
\varepsilon: & 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, hook] & E_2 \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d, equal] & 0 \\
\varepsilon': & 0 \arrow[r] & Y \arrow[r] & E_1 \oplus W \arrow[r, "{\oplus\, 1_x}"'] & E_2 \oplus W \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

• DENOTE BY $\sim$ THE SMALLEST EQUIV. RELATION ON $\mathcal{E}^n(X,Y)$ SUCH THAT PAIR $\varepsilon, \varepsilon'$ AS ABOVE ARE EQUIV.

- THE YONEDA Ext: $X, Y \in \mathcal{A}$, $n \geq 1$, THEN PUT

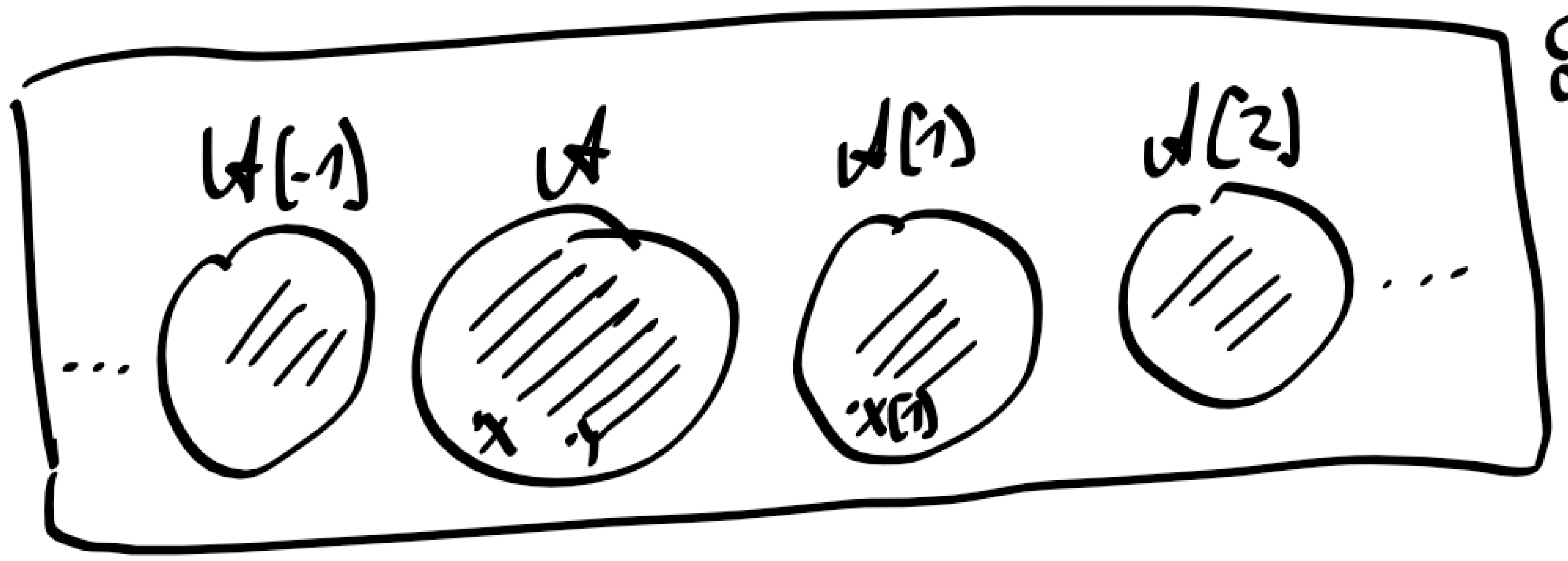
$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = \mathcal{E}^n(X,Y) / \sim$$

- FACTS:
  - $\mathrm{Ext}^n_{\mathcal{A}}(X,Y)$ IS NATURALLY AN ABELIAN GROUP (BAER SUM)
  - $\mathrm{Ext}^n_{\mathcal{A}} : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{A}b$ IS NATURALLY AN ADDITIVE FUNCTOR
  - FROM ANY S.E.S. $0 \to X \to Y \to Z \to 0$ AND $W \in \mathcal{A}$ WE GET (UNIVERSAL) LONG EXACT SEQ.'S AS IN $\circledast$

- BY DEF: $\mathrm{Ext}^0_{\mathcal{A}}(X,Y) := \mathrm{Hom}_{\mathcal{A}}(X,Y)$

$\mathrm{Ext}^{-n}(X,Y) = 0 \quad \forall\, n > 0$

- DERIVED CATEGORY



$\underline{D\mathcal{A}} \circlearrowleft_{\simeq} [1]$

DERIVED CATEGORY OF $\mathcal{A}$

$D\mathcal{A}$ WILL BE CONSTRUCTED SO THAT:

- COUNTABLY MANY COPIES OF $\mathcal{A}$ SIT INSIDE $D\mathcal{A}$
- $\forall X, Y \in \mathcal{A} \ \forall n \in \mathbb{Z}$: $\quad \mathrm{Ext}^n_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{D\mathcal{A}}(X, Y[n])$
- $D\mathcal{A}$ IS A TRIANGULATED CATEGORY, I.E. IT COMES WITH A DISTINGUISHED CLASS OF DIAGRAMS OF THE FORM

$$X \xrightarrow{\tilde{i}} Y \xrightarrow{\tilde{p}} Z \xrightarrow{\varepsilon} X[1]$$

$\rightsquigarrow$ "EXTENDS THE CLASS OF S.E.S. $\varepsilon: 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ IN $\mathcal{A}$."

$[\varepsilon]_{\sim} \in \mathrm{Ext}(Z, X) \cong \mathrm{Hom}_{D\mathcal{A}}(Z, X[1])$