

ALGEBRAIC TRIANGULATED CATEGORIES

- A TRIANGULATED CATEGORY CONSISTS OF

• AN ADDITIVE CATEGORY $\mathcal{T}$

• AN AUTOEQUIVALENCE $\mathcal{T} \xrightarrow{\quad} \mathcal{T}$

$$X \longmapsto X[1] \quad (\text{ALT.: } \Sigma X)$$

(SHIFT / SUSPENSION
FUNCTOR)

• A CLASS OF (DISTINGUISHED) TRIANGLES:

$$\Delta = \{ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \}$$

- MORPHISM OF TRIANGLES:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow \cong & & g \downarrow \cong & & h \downarrow \cong & & \downarrow f[1] \cong \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

$\in \Delta$

$\Downarrow (\text{TR})$

$\in \Delta$

- THIS STRUCTURE IS SUBJECT TO AXIOMS! [H288]

(TR1) • Δ IS CLOSED UNDER TAKING ISOMORPHIC DIAGRAMMS

• $\forall m: X \rightarrow Y \quad \exists \quad (X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]) \in \Delta$

• $\forall X \in \mathcal{T} : \quad (X \xrightarrow{1_X} X \rightarrow 0 \rightarrow X[1]) \in \Delta$

(TR2) IF $(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]) \in \Delta$, THEN $(Y \xrightarrow{-u} Z \xrightarrow{-w} X[1] \xrightarrow{-u[1]} Y[1])$
 $\triangle$

(TR3) EACH DIAGRAM

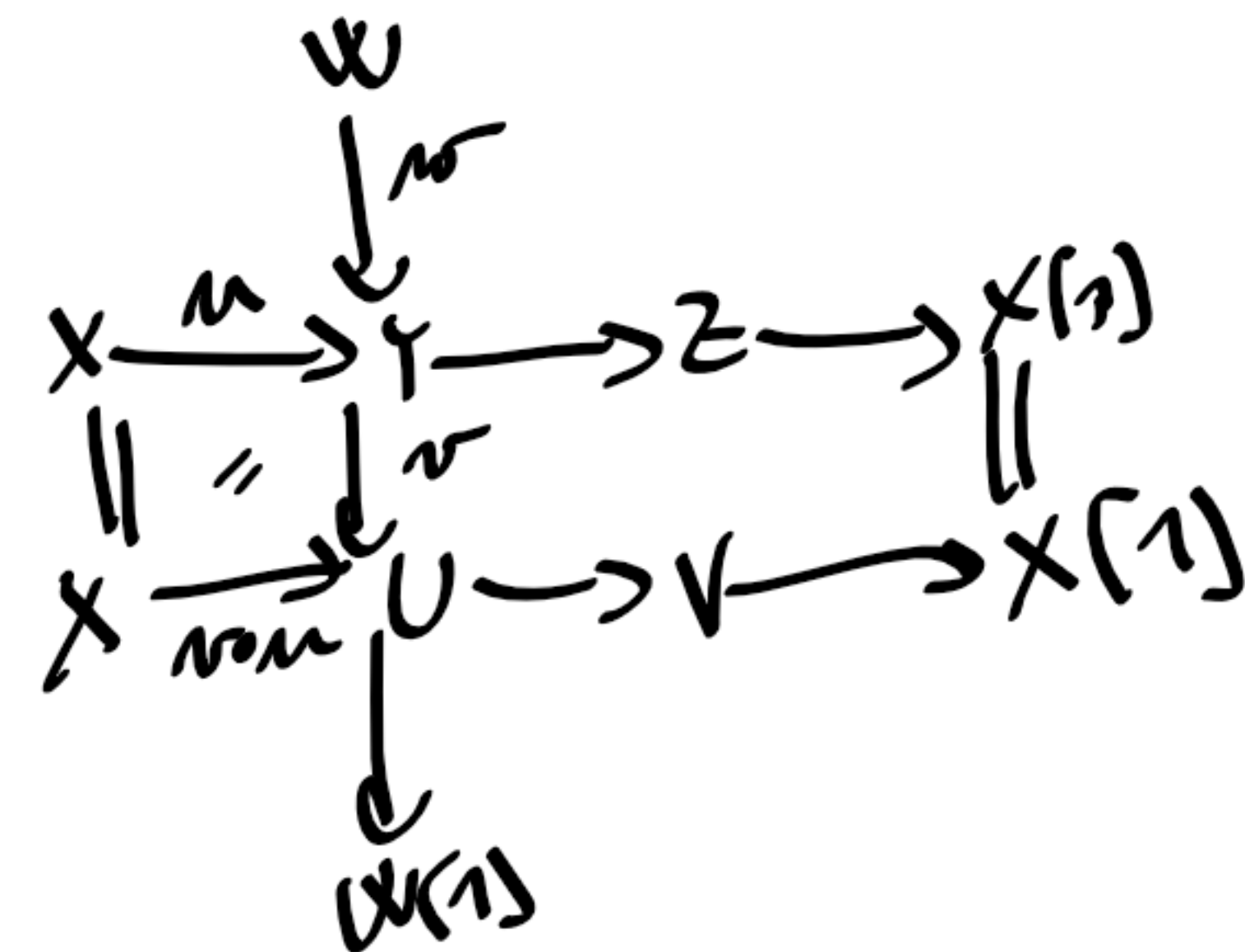
$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ f \downarrow & \cong & \downarrow g & \cong & \downarrow \exists h & \cong & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

CAN BE COMPLETED TO A MORPHISM OF TRIANGLES.

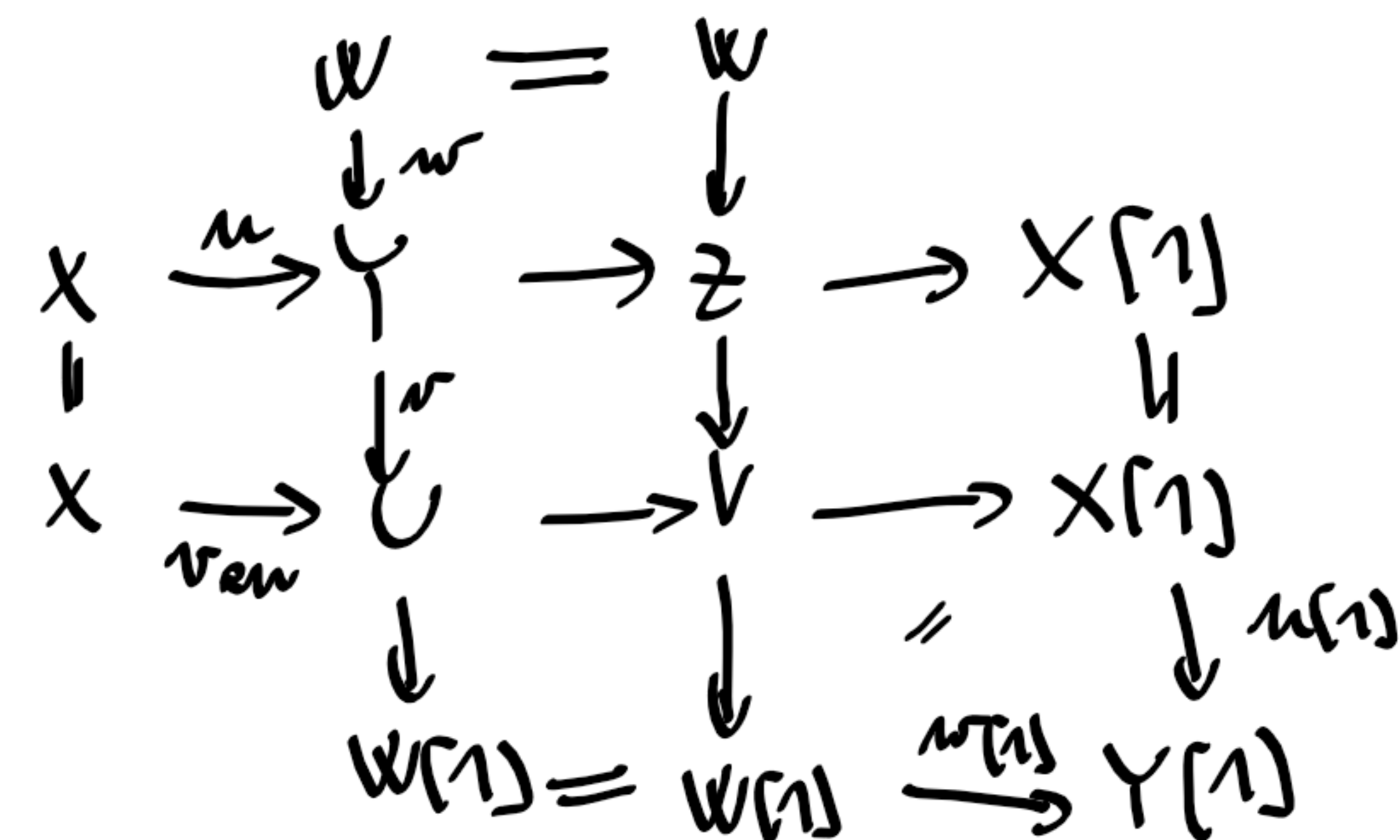
! • NON-UNIQUE
 • TRIANGLE COMPLETION NON-FUNCTORIAL
 IN GENERAL [Ve97]

(TR4) (OCTAHEDRAL AXIOMS)

- SAY WE HAVE 3 TRIANGLES IN Δ OF THE FORM



- THEN WE CAN COMPLETE THIS DIAGRAM TO :



↑ SO THAT THIS IS ALSO A TRIANGLE

- REMARKS TO (TR4):

① IF WE APPLY (TR4), WE EVEN GET A TRIANGLE

$$Y \longrightarrow V \oplus Z \longrightarrow V \longrightarrow Y[1]$$

② NEEMAN'S EQUIVALENT REFORMULATION OF (TR4):

(NEEM) EACH DIAGRAM

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \xrightarrow{-u[1]} Y[1] \\ f \downarrow & = & \downarrow g & = & \downarrow \exists h & = & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

CAN BE COMPLETED TO A MORPHISM OF TRIANGLES IN SUCH A WAY THAT THE MAPPING CONE

$$X' \oplus Y \longrightarrow Y' \oplus Z \longrightarrow Z' \oplus X[1] \longrightarrow \overset{(X' \oplus Y)[1]}{X'[1] \oplus Y[1]} \underset{\text{or can}}{\longrightarrow}$$

IS A TRIANGLE.

(3)

A CONSEQUENCE OF (TR1) — (TRY1):

VERDIER'S 3x3 LEMMA: EACH DIAGRAM

$$\begin{array}{ccccccc} X & \xrightarrow{\mu} & Y & \xrightarrow{\nu} & Z & \xrightarrow{\omega} & X(1) \\ f \downarrow & = & \downarrow g & & & & \downarrow f(1) \\ X' & \xrightarrow{\mu'} & Y' & \xrightarrow{\nu'} & Z' & \xrightarrow{\omega'} & X'(1) \end{array}$$

CAN BE COMPLETED TO A DIAGRAM WITH TRIANGLES
IN ROWS & COLUMNS:

$$\begin{array}{ccccccc} X & \xrightarrow{\mu} & Y & \xrightarrow{\nu} & Z & \xrightarrow{\omega} & X(1) \\ f \downarrow & = & g \downarrow & = & h \downarrow & = & \downarrow f(1) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'(1) \\ t \downarrow & = & \downarrow & = & \downarrow & = & \downarrow t(1) \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''(1) \\ s \downarrow & = & \downarrow & = & \downarrow & \text{ANTI-COMMUTES} & \downarrow s(1) \\ X(1) & \xrightarrow{\mu(1)} & Y(1) & \xrightarrow{\nu(1)} & Z(1) & \xrightarrow{-\omega(1)} & X(2) \end{array}$$

- ALGEBRAIC TRIANGULATED CATEGORIES

- AXIOMS (TR3) & (TR4) SPEAK ONLY OF EXISTENCE OF CERTAIN MAPS & TRIANGLES, **NO UNIQUENESS!!!**
- SOMETIMES THIS IS TOO WEAK TO PROVE THINGS!
- USUAL FIX: DO NOT LOOK JUST ON A TRIANGULATED CATEGORY ALONE, BUT CONSIDER ALSO HOW IT WAS CONSTRUCTED!

• DEF: A TRIANGULATED CATEGORY $(\mathcal{T}, \mathcal{M}, \Delta)$ IS ALGEBRAIC, [K094] IF IT IS TRIANGLE EQUIVALENT TO [K07] Σ , WHERE Σ IS A FROBENIUS EXACT CATEGORY.

• A ^(EXACT) TRIA. FUNCTOR $(\mathcal{T}, \mathcal{M}, \Delta_{\mathcal{T}}) \longrightarrow (\mathcal{U}, \mathcal{L}_n, \Delta_n)$ IS GIVEN BY:

- AN ADDITIVE FUNCTOR $F: \mathcal{T} \longrightarrow \mathcal{U}$
- AN NATURAL EQUIVALENCE $(1)_n \circ F \stackrel{\cong}{=} F \circ (1)_{\mathcal{T}}$ } STRUCTURE
- $\forall (x \xrightarrow{\sim} y \xrightarrow{\sim} z \xrightarrow{\sim} x(1)_{\mathcal{T}}) \in \Delta_{\mathcal{T}} : (Fx \xrightarrow{F_{\sim}} Fy \xrightarrow{F_{\sim}} Fz \xrightarrow{F_{\sim}} Fx(1)_n) \in \Delta_n \leftarrow \text{AXIOM}$

SOME WORK $\leadsto$ TRIANGLE EQUIVALENCE

- FROBENIUS EXACT CATEGORIES AND THEIR STABLE CATEGORIES

- EXACT CATEGORY (B.10)
 - Σ ADDITIVE CATEGORY
 - $\mathcal{E} = \{ 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \}$
 - $\nearrow$ INFLATION (ADH. MONO)
 - $\nwarrow$ DEFLECTION (ADH. EPI)
 } + AXIOMS
- QUILLEN/GABRIEL: Σ SMALL EXACT CATEGORY
 - $\Rightarrow$ UP TO EQUIVALENCE, $\Sigma \subseteq \mathcal{A}$ EXTENSION CLOSED IN $\mathcal{A}$ ABELIAN, $\mathcal{E} = \{ 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \text{ EXACT IN } \mathcal{A} \mid X, Y, Z \in \Sigma \}$
- Σ IS FROBENIUS IF
 - (1) Σ HAS ENOUGH PROJECTIVES ($\forall X \in \Sigma \exists P \in \Sigma$ PROJECTIVE & $P \twoheadrightarrow X$)
 - (2) Σ HAS ENOUGH INJECTIVES
 - (3) $\forall X \in \Sigma : X \text{ PROJECTIVE} \Leftrightarrow X \text{ INJECTIVE}$

-
- EXPL: (1) $\Sigma = \text{Mod } R$, $\mathcal{E} = \{ \text{ALL S. E. S.} \}$ IS FROBENIUS IFF R IS QUASI-FROBENIUS
 E.G.: $R = kG$, k FIELD, G FINITE GROUP.
 - (2) $\mathcal{B}$ ADDITIVE CAT. $\leadsto \Sigma = \mathcal{C}(\mathcal{B})$, $\mathcal{E} = \{ 0 \rightarrow X' \rightarrow Y' \rightarrow Z' \rightarrow 0 \mid \begin{matrix} \text{in } \mathcal{B} \\ \text{is split exact} \end{matrix} \}$
 - $0 \rightarrow \begin{smallmatrix} \uparrow \downarrow \\ 1 \ 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} \uparrow \downarrow & \uparrow \downarrow \\ 1 \ 0 & 1 \ 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} \uparrow \downarrow \\ 1 \ 0 \end{smallmatrix} \rightarrow 0$

- THE STABLE CATEGORY

- Σ FROBENIUS EXACT

$\rightsquigarrow$

STABLE CATEGORY

$\underline{\Sigma}$

= ADDITIVE QUOTIENT
MODULO ALL MAPS WHICH
FACTOR THROUGH A PROJ. OBJECT,
I.E.

SO $\text{ob } \underline{\Sigma} = \text{ob } \Sigma$, BUT

$$\text{Hom}_{\underline{\Sigma}}(X, Y) := \frac{\text{Hom}_{\Sigma}(X, Y)}{\{f: X \rightarrow Y \mid \exists X \xrightarrow{f} Y \text{ factoring through a proj. obj.}\}}$$

- POINT: THERE IS A NICE TRIANGULATED STRUCTURE ON $\underline{\Sigma}$!

- SHIFT FUNCTOR:
"COSTRYCY FUNCTOR"

$$(1): \begin{array}{ccc} \Sigma & \xrightarrow{\quad} & \Sigma \\ X & \xrightarrow{\quad} & \Sigma X \end{array}$$

$$0 \rightarrow X \rightarrow \overset{\uparrow \text{INJECTIVE}}{E} \rightarrow \Sigma X \rightarrow 0$$

WELL-DEF. (SKETCH)

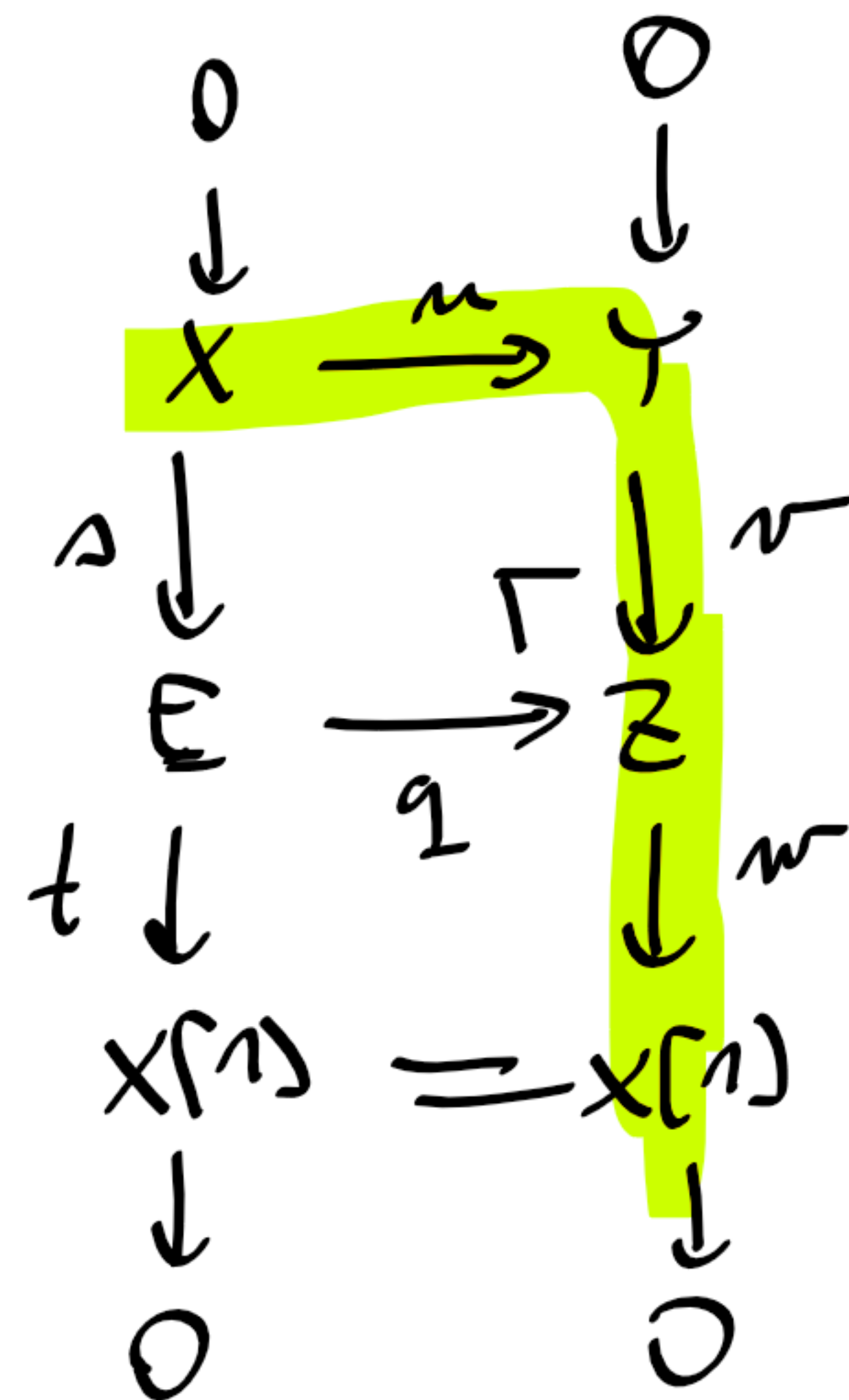
$$\begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & E_X & \rightarrow & \Sigma X \rightarrow 0 \\ & & \downarrow f & & \downarrow \exists & & \downarrow f(1) \\ 0 & \rightarrow & Y & \rightarrow & E_Y & \rightarrow & \Sigma Y \rightarrow 0 \end{array}$$

$$\begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & E_X & \rightarrow & \Sigma X \rightarrow 0 \\ & & \downarrow 0 & & \downarrow \exists' & & \downarrow \text{FACTORS THROUGH } E_Y \\ 0 & \rightarrow & Y & \rightarrow & E_Y & \rightarrow & \Sigma Y \rightarrow 0 \end{array}$$

• TRIANGLES:

$$X \xrightarrow{u \in \Sigma} Y$$

$\leadsto$ EXTEND TO A TRIANGLE
IN $\underline{\Sigma}$



$\leadsto$ BY DEFINITION, $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X(1)$ IS A TRIANGLE IN $\underline{\Sigma}$,
UP TO ISO IN $\underline{\Sigma}$, ALL TRIANGLES ARE LIKE THAT

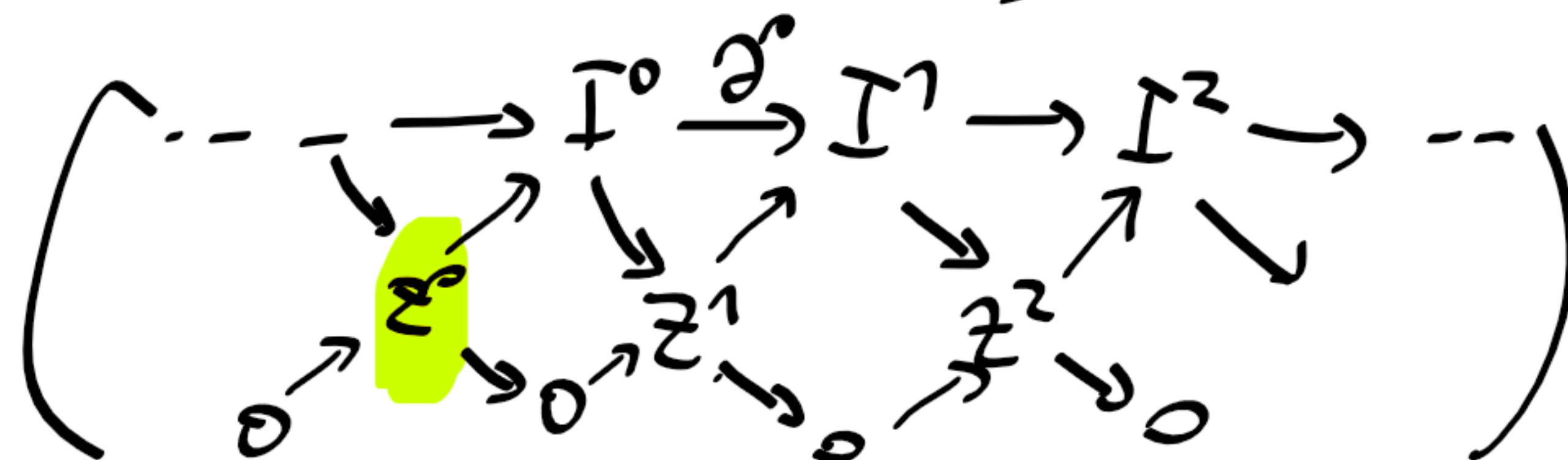
- NOTE: WE HAVE COMBINATIONS IN $\underline{\Sigma}$: $0 \rightarrow X \xrightarrow{\binom{\Delta}{u}} E \oplus Y \xrightarrow{\binom{-q}{v}} Z \rightarrow 0$
 $0 \rightarrow Y \xrightarrow{u} Z \xrightarrow{w} X(1) \rightarrow 0$

- PROVE (TR1)-(TR4) FOR $(\underline{\Sigma}, \cap, \Delta)$ -- EXERCISE!

VISTA: TILTING THEORY (Ke94), [Kro7]

• LET $\mathcal{E}$ BE FROBENIUS EXACT CATEGORY

• THEN $\mathcal{E} \xleftarrow[\text{TRIA.}]{\cong} \mathcal{K}_{ac}(\mathcal{E}_{inj}) \subseteq \mathcal{K}(\mathcal{E}_{inj})$
 $\mathcal{E}_{inj} = \{X \mid X \text{ INJECTIVE}\} \subseteq \mathcal{E}$
 $\uparrow$
 ACYCLIC CPLX

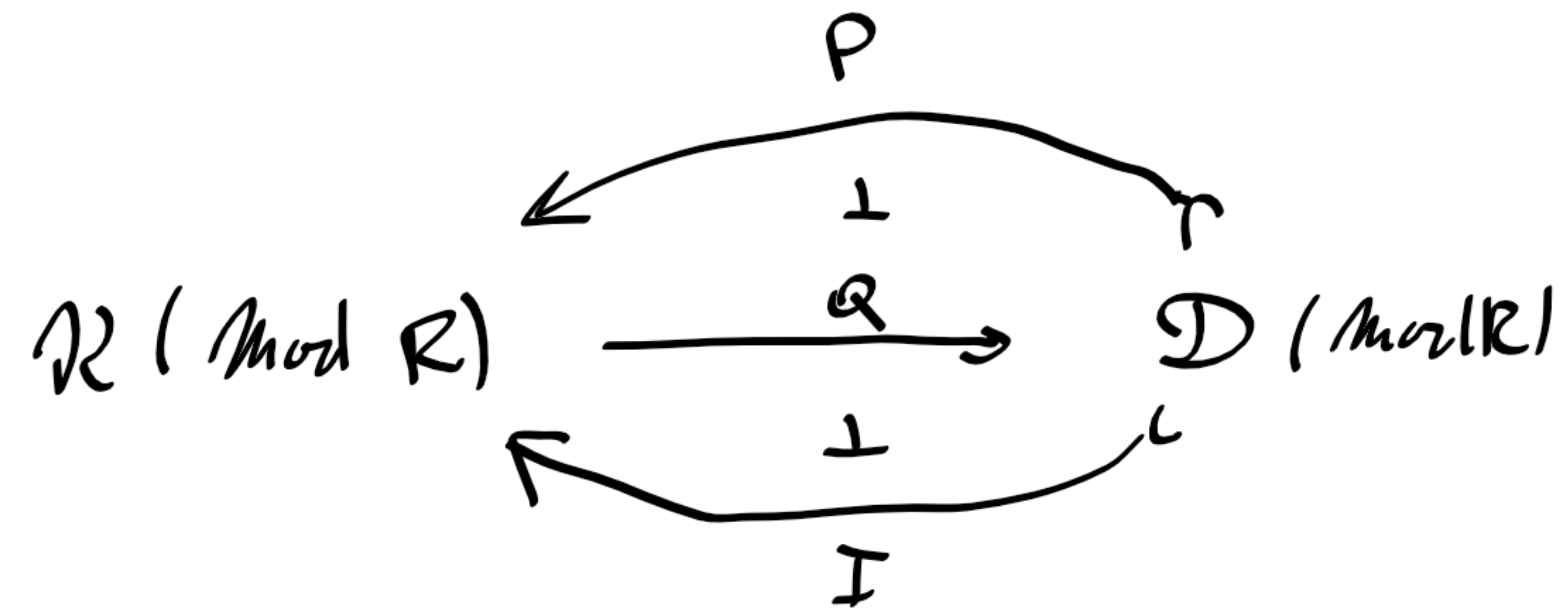


$$\ker d^0 = Z^0 \longleftarrow I^0$$

$X \longrightarrow \text{FULL RESOLUTION}$

• IF $T \in \mathcal{E}$ IS TILTING $\rightsquigarrow T \mapsto \hat{T} \in \mathcal{K}_{ac}(\mathcal{E}_{inj})$, SO $\hat{T}$ IS A COMPLEX,
 $\rightsquigarrow A := \text{End}^*(\hat{T})$ IS A' DGA $\rightsquigarrow \mathcal{E} \xrightarrow{\text{FULL}} \mathcal{D}(A)$

- DERIVED FUNCTORS :



$\Rightarrow D(\text{Mod } R) \cong \underbrace{\text{FULL SUBCAT.}}_{\text{TRIA.}} \text{ OF } R(\text{Mod } R) \Rightarrow D(\text{Mod } R) \text{ ALGEBRAIC}$