INTRODUCTION TO AUSLANDER-REITEN THEORY

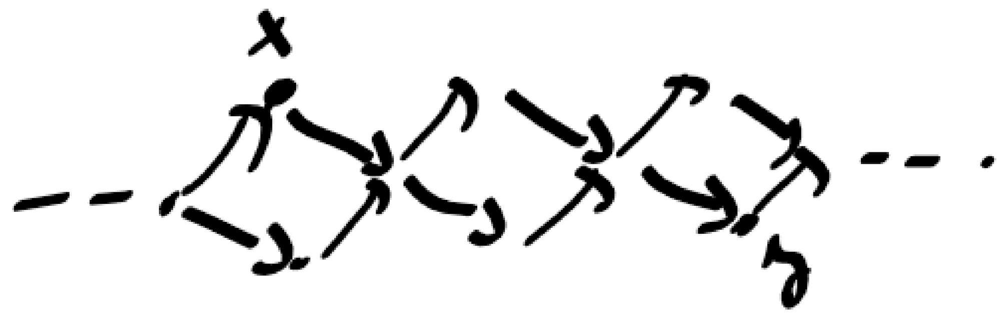
I. GOAL: DESCRIBE CATEGORIES LIKE $\underbrace{\mathrm{mod}\,A,\ D^b(A),\ K^b(\mathrm{proj}\,A)}_{A\text{ FIN. DIM. ALG OVER }k}$, $\underbrace{\mathrm{MCM}(R)}_{R\text{ COMM. LOCAL NOETHERIAN...}}$, $\underbrace{\mathrm{coh}\,X,\ D^b(\mathrm{coh}\,X)}_{X\text{ PROJECTIVE SCHEME OF FIN. TYPE OVER }k,\ldots}$

IN TERMS OF GENERATORS AND RELATION.

$\rightsquigarrow$ PICTURES OF THE CATEGORIES ...

– WHAT DOES IT MEAN: GENERATORS & RELATIONS

– LET Γ BE A QUIVER (NOT NECESSARILY FINITE), E.G.



– LET k BE A FIELD (OR A COMMUTATIVE RING)

$\rightsquigarrow$ PATH CATEGORY $k\Gamma$:

- OBJECTS: VERTICES OF Γ
- MORPHISMS: $\mathrm{Hom}(x,y)$ = VECTOR SPACE WHOSE BASIS IS SET OF ALL PATHS $x \rightsquigarrow y$ IN Γ
- COMPOSITION: INDUCED BY CONCATENATION OF PATHS
- UNITS: TRIVIAL PATHS

$\rightsquigarrow$ $k\Gamma$ IS A k-LINEAR CATEGORY

– IF $\mathcal{I} \subseteq k\Gamma$ IS A 2-SIDED IDEAL (I.E. $\forall x,y \in k\Gamma$: $\mathcal{I}(x,y) \underset{\text{SUBSP.}}{\subseteq} \mathrm{Hom}(x,y)$ + CLOSED UNDER COMPOSITION WITH ANY MAP)

$\rightsquigarrow$ QUOTIENT k-LINEAR CATEGORY $\mathcal{C}_{\Gamma,\mathcal{I}} := \dfrac{k\Gamma}{\mathcal{I}}$ .... OBJECTS: SAME AS IN $\mathcal{C}$
MORPHISMS: $\mathrm{Hom}_{\mathcal{C}_{\Gamma,\mathcal{I}}}(x,y) = \dfrac{\mathrm{Hom}_{k\Gamma}(x,y)}{\mathcal{I}(x,y)}$

II. EXAMPLES:

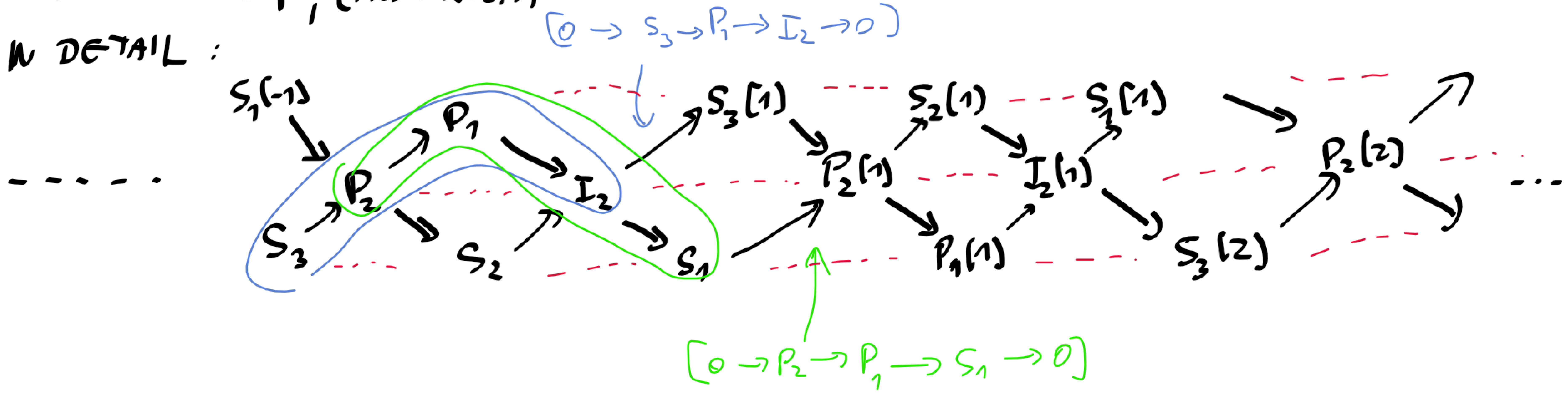
- $Q = (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$, CAN WE PRESENT $\operatorname{mod} kQ$ BY GENERATORS & RELATIONS IN THIS WAY?
- WE BETTER LOOK AT $\operatorname{ind} kQ$ = A SKELETON OF THE FULL SUBCATEGORY OF $\operatorname{mod} kQ$ SPANNED BY INDECOMPOSABLES

$$
\begin{tikzcd}[column sep=small]
 & & \underset{P_1 = I_3}{k \xrightarrow{1} k \xrightarrow{1} k} \arrow[dr] & & \\
 & \underset{P_2}{0 \to k \xrightarrow{1} k} \arrow[ur] \arrow[dr] & & \underset{I_2}{k \xrightarrow{1} k \to 0} \arrow[dr] & \\
\underset{S_3}{0 \to 0 \to k} \arrow[ur] & & \underset{S_2}{0 \to k \to 0} \arrow[ur] & & \underset{S_1}{k \to 0 \to 0}
\end{tikzcd}
$$

$\rightsquigarrow$ $\operatorname{ind} kQ \cong \mathcal{C}_{\Gamma, \Delta}$, $\Gamma =$
$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & \cdot \arrow[dr] & & \\
 & \cdot \arrow[ur] \arrow[dr] & & \cdot \arrow[dr] & \\
\cdot \arrow[ur] & & \cdot \arrow[ur] & & \cdot
\end{tikzcd}
$$
, $\Delta = (\text{MESH RELATIONS})$

- We can do a similar trick for $D^b(A)$ [Happel, '87]
- We look at ind $D^b(A)$
- Fact: • ind $D^b(A) \cong \mathcal{C}_{\Gamma}$, (mesh rel's) where Γ : ....

• more in detail:



- Comments on what we use here:

① A is **hereditary**, so

ⓐ every complex in $D^b(A)$ is iso in $D^b(A)$ to a bounded cpx of f.g. projectives ($D^b(A) \simeq \mathcal{K}^b(\mathrm{proj}\, A)$)

ⓑ every $X \in D^b(A)$ is iso to its homology, i.e. $X_\bullet \cong \bigoplus_{n \in \mathbb{Z}} H_n(X_\bullet)[n]$

$$\begin{array}{ccccccccccccc}
X_\bullet : & \cdots \to & P_{i+2} & \to & P_{i+1} & \xrightarrow{d} & P_i & \xrightarrow{d} & P_{i-1} & \to & \cdots \\
\uparrow & & & & \uparrow \oplus & & \uparrow\wr\!\!\!\uparrow & & & & \\
\text{complex of projectives} \to & \cdots \to & 0 & \to & B_i(X) & \rightarrowtail & Z_i(X) & \to & 0 & \to & \cdots \\
\downarrow\wr & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_i(X)[i] & \cdots \to & 0 & \to & 0 & \to & H_i(X) & \to & 0 & \to & \cdots
\end{array}$$

(c) EVERY $X \in D^b(A)$ IS A FINITE $\oplus$ OF OBJECTS WITH LOCAL ENDO. RINGS

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(2) KRULL-SCHMIDT THEOREM (A PART OF IT)

- I: SUPPOSE $\mathcal{C}$ IS AN ADDITIVE CATEGORY AND $X \in \mathcal{C}$ DECOMPOSES TO A FINITE $\oplus$ OF OBJECTS WITH LOCAL ENDO. RINGS. THEN EVERY OTHER SUCH DECOMPOSITION DIFFERS ONLY UP TO THE ORDER AND ISOMORPHISM OF SUMMANDS.

- SKETCH:

$$\mathrm{Hom}(X,-) : \mathrm{add}\, X \xrightarrow{\sim} \mathrm{proj}\, \underbrace{\mathrm{End}(X)}_{\text{SEMIPERFECT}}$$

III. THE RADICAL OF A PREADDITIVE ABELIAN CATEGORY

- RECALL: R RING $\rightsquigarrow$ $\mathrm{rad}(R) = \{ r \in R \mid \forall a \in R : 1_R - ar \text{ IS INVERTIBLE} \}$

(JACOBSON RADICAL) $= \{ r \in R \mid$ —"— $1_R - ra$ —"— $\}$

IS A 2-SIDED IDEAL

- DEF: LET $\mathcal{C}$ BE A PREADDITIVE CATEGORY. THE RADICAL $\mathrm{rad}_{\mathcal{C}}$ IS THE 2-SIDED IDEAL (NEEDS TO BE PROVED!) GIVEN AS FOLLOWS:

[ASSEM-SIMSON-SKOWRONSKI, APP. A3]

$\forall x, y \in \mathcal{C}$: $\mathrm{rad}_{\mathcal{C}}(x,y) = \{ h : x \to y \mid \forall g : y \to x : 1_x - gh \text{ IS INVERTIBLE} \}$

$(\subseteq \mathrm{Hom}_{\mathcal{C}}(x,y))$

- POWERS OF THE RADICAL:

- $\mathrm{rad}^n_{\mathcal{C}}(x,y) = \{ \sum_{\text{FINITE}} x \to r_1 \to r_2 \to \dots \to r_{n-1} \to y \}$

IN $\mathrm{rad}_{\mathcal{C}}$

- $\mathrm{rad}^\infty_{\mathcal{C}}(x,y) = \bigcap_{n \geq 1} \mathrm{rad}^n(x,y)$

ALSO 2-SIDED IDEALS

$\rightsquigarrow$ MOREOVER, $\mathrm{rad}^\infty\left(\dfrac{\mathcal{C}}{\mathrm{rad}^\infty_{\mathcal{C}}} \right) = \{0\}$

— PROPERTIES OF $\mathrm{rad}_{\mathcal{C}}$:

1. IF $f = (f_{ij}) : \bigoplus_{i=1}^{m} X_i \to \bigoplus_{j=1}^{n} Y_j$, THEN $f \in \mathrm{rad}_{\mathcal{C}} \iff \forall i,j : f_{ij} \in \mathrm{rad}_{\mathcal{C}}$

   $f_{ij} : X_i \to Y_j$

   SAME FOR $\mathrm{rad}_{\mathcal{C}}^{m}$, $m \in \mathbb{N} \cup \{\infty\}$, OR MORE GENERALLY, FOR ANY 2-SIDED IDEAL OF $\mathcal{C}$.

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2. $\mathrm{rad}_{\mathcal{C}}(R,R) = \mathrm{rad}(\mathrm{End}(R))$
   $R \in \mathcal{C}$

   BEWARE: DOES NOT WORK FOR rad^{m}, $m \geq 2$!!!

   $\mathrm{rad}_{\mathcal{C}}^{m}(R,R) \supsetneq \mathrm{rad}^{m}(\mathrm{End}(R))$

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3. SUPPOSE $x, y \in \mathcal{C}$ HAVE LOCAL ENDO. RINGS. THEN

   $$\mathrm{rad}_{\mathcal{C}}(x,y) = \{ h : x \to y \mid h \text{ NON-ISO} \}$$

   IN PARTICULAR IF $x \not\cong y$, THEN $\mathrm{rad}_{\mathcal{C}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(x,y)$

- SUPPOSE $\mathcal{C}$ IS SKELETAL (DIFFERENT OBJECTS ARE NON-ISO), k-LINEAR ($k = \bar{k}$ A FIELD), Hom-FINITE, ALL OBJECTS HAVE LOCAL ENDOMORPHISM RINGS

$\rightsquigarrow$ $\underline{\Gamma_{\mathcal{C}}, \text{ QUIVER OF } \mathcal{C}}$ ... VERTICES : OBJECTS OF $\mathcal{C}$

ARROWS : # ARROWS $x \to y$ $= \dim_k \dfrac{\mathrm{rad}_{\mathcal{C}}(x,y)}{\mathrm{rad}^2_{\mathcal{C}}(x,y)}$

- AS IN GABRIEL'S THEOREM, WE HAVE A FULL k-LINEAR FUNCTOR, BIJECTIVE ON OBJECTS:

$$k\Gamma_{\mathcal{C}} \twoheadrightarrow \frac{\mathcal{C}}{\mathrm{rad}^{\infty}_{\mathcal{C}}}$$

$$\Rightarrow \quad \frac{k\Gamma_{\mathcal{C}}}{I} \cong \frac{\mathcal{C}}{\mathrm{rad}^{\infty}_{\mathcal{C}}}$$

- DEF: IF A IS A FIN. DIM. ALGEBRA AND $\mathcal{C} = \mathrm{ind}\, A$, THEN $\Gamma_{\mathcal{C}} =: \Gamma_A$ IS CALLED THE AUSLANDER-REITEN QUIVER OF A.

IV. AUSLANDER-REITEN THEORY (WHY: TOOLS TO COMPUTE $\operatorname{rad}_{\mathcal{C}}$, $\operatorname{rad}^2_{\mathcal{C}}$)

- A FIN. DIM. ALG. / $k = \bar{k}$

$\rightsquigarrow$ AUSLANDER REITEN TRANSLATION: CERTAIN BIJECTION BETWEEN

$$\left\{\begin{matrix}\text{NON-PROJECTIVE}\\ \text{INDECOMPOSABLES}\end{matrix}\right\} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \{\text{NON-INJ. INDECOMPOSABLES}\}$$

- T ([ASS, THM IV.2.13]): $\forall M, N \in \operatorname{mod} A$ WE HAVE FUNCTORIAL ISO'S

$$\operatorname{Ext}^1_A(M,N) \cong D\,\overline{\operatorname{Hom}}_A(N, \tau M) \cong D\,\underline{\operatorname{Hom}}_A(\tau^- N, M)$$

FUNCTORIAL

$\rightsquigarrow$ IF $N = \tau M$, M IS INDECOMPOSABLE NON-PROJECTIVE:

$$\underbrace{\operatorname{Ext}^1_A(M, \tau M)} \cong D\,\underline{\operatorname{Hom}}_A(M,M) = D\left(\frac{\operatorname{End}_A(M)}{\{\text{MORPHISMS } M \to \text{proj} \to M\}}\right)$$

LOCAL k-alg, 1-dim top

$\operatorname{End}(M)$-BIMODULE WITH SIMPLE SOCLE FROM BOTH LEFT AND RIGHT, THE SOCLE IS 1-dim, REPRESENTED BY

$0 \to \tau M \to E \to M \to 0$ SEQ.

$\exists$ $\in \operatorname{rad}_{\operatorname{mod} A}$

$\rightsquigarrow$ GENERATORS FOR $\operatorname{rad}_{\operatorname{mod} A}$ AS A RIGHT IDEAL