NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 4—April 30, 2021

Our goal today is to explore path algebras and their representations and to employ some previously discussed concepts in doing so.

We work over an algebraically closed k and with finite-dimensional modules.

Path algebras and their representations

Exercise 1 (7, III.4 in [1]). Let us have the following quiver:

$$\begin{tikzcd}
 & \circ \arrow[dl, "\delta"'] & \\
\circ & & \circ \arrow[ul, "\gamma"'] \arrow[dl, "\alpha"] \\
 & \circ \arrow[ul, "\beta"] &
\end{tikzcd}$$

bound by $\alpha\beta = 0$. Show indecomposability of the following representation:

$$\begin{tikzcd}
 & k \arrow[dl, "1"'] & \\
k & & k \arrow[ul, "1"'] \arrow[dl, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \\
 & k^2 \arrow[ul, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] &
\end{tikzcd}$$

(Hint: Compute the endomorphism ring.)

Exercise 2 (Subspace quiver). Let Q be a quiver with vertices labelled $0, \ldots, n$ and n arrows such that there is one arrow $i \circ \to \circ\, 0$ for each $1 \leq i \leq n$. Then, find:

(i) An embedding $kQ \to M_{n+1}(k)$ (1.3(d), II.1 and 3.5(c), II.3 in [1]);

(ii) Natural direct decomposition for every module over kQ (2.4.2 in [2]).

Exercise 3 (Subspace quiver continued). Express the result of (ii) in Exercise 2 with idempotents. In other words, explicitely describe (not necessarily primitive) idempotents in endomorphism rings of finite-dimensional representations of the subspace quiver. Furthermore, compute endomorphism rings of some non-trivial indecomposable representations of the subspace quiver. For instance:

$$\begin{tikzcd}
 & & k^2 & & \\
k \arrow[urr, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & k \arrow[ur, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"'] & & k \arrow[ul, "\begin{bmatrix} 1 \\ 1 \end{bmatrix}"] & k \arrow[ull, "\begin{bmatrix} 1 \\ \lambda \end{bmatrix}"']
\end{tikzcd}$$

for suitable values of $\lambda \in k$ (find those as well).

Exercise 4 (Kronecker quiver; 15, III.4 in [1]). Show that the endomorphism ring of the following representation of the Kronecker quiver ($1 \circ \rightrightarrows \circ\, 2$):

$$k[t] \underset{\cdot t}{\overset{1}{\rightrightarrows}} k[t]$$

is not local; even though, the representation is indecomposable.

Exercise 5 (Kronecker quiver continued). For any $n \in \mathbb{N}$ find φ, ψ maps such that

$$k^n \underset{\psi}{\overset{\varphi}{\rightrightarrows}} k^{n+1}$$

is indecomposable. Compute the endomorphism ring of such a representation.

Exercise 6 (5, II.4 in [1]). Let Q be a finite and acyclic quiver. Prove that kQ is a connected $k-$algebra if and only if kQ/R_Q^2 is a connected $k-$algebra (R_Q is the ideal of kQ generated by all arrows of Q).

Exercise 7 (5(d) and 7(d) in III.2 in [1]). Given a quiver on four vertices:

$$\begin{tikzcd}
 & \circ \arrow[dl, "\delta"'] & \\
\circ \arrow[loop left, "\lambda"] & & \circ \arrow[ul, "\gamma"'] \arrow[dl, "\alpha"] \\
 & \circ \arrow[ul, "\beta"] &
\end{tikzcd}$$

bound by the following relations $\alpha\beta = \gamma\delta$ and $\lambda^3 = 0$ (reason that the ideal generated by these realitions is admissible), compute all simple, indecomposable projective (with radicals) and injective modules (with factors by socles) over this quiver.

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

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