NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 3—April 16, 2021

Our goal today is to focus also on some exercises about idempotents, local algebras, and projective covers and on some exercises that deal with path algebras.

Idempotents, local algebras, and projective covers

Exercise 1 (4.9, I.4 in [1]). Let us have an algebra over k:

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \alpha_{21} & \lambda & 0 \\ \alpha_{31} & \alpha_{32} & \lambda \end{pmatrix} ; \lambda, \alpha_{21}, \alpha_{31}, \alpha_{32} \in k \right\}$$

Show that:

(i) B is indeed a well-defined algebra.

(ii) B is local.

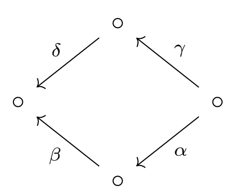
Exercise 2. Given a commutative finite-dimensional algebra over k, find a decomposition thereof.

Exercise 3. Exhibit a k-algebra with no non-trivivial idempotents that is not local.

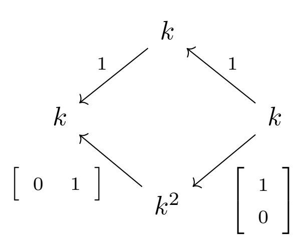
Exercise 4 (7 in I.6 in [1]). Show that $k[t]/(t^3)$ as a module over $k[t]$ (which is a path algebra of a quiver with a single vertex and a loop) has no projective cover.

Path algebras

Exercise 5 (7, III.4 in [1]). Let us have the following quiver:



bound by $\alpha\beta = 0$. Show irreducibility of the following representation:



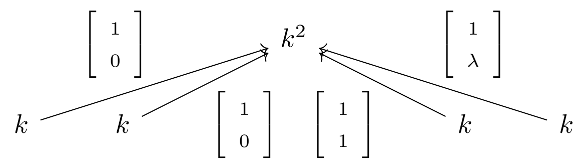
(Hint: Compute the endomorphism ring.)

Exercise 6 (Subspace quiver). Let Q be a quiver with vertices labelled $0, \dots, n$ and n arrows such that there is one arrow $i \circ \to \circ\, 0$ for each $1 \leq i \leq n$. Then, find:

(i) An embedding $kQ \to M_{n+1}(k)$ (1.3(d), II.1 and 3.5(c), II.3 in [1]);

(ii) Natural direct decomposition for every module over kQ (2.4.2 in [2]).

Exercise 7 (Subspace quiver continued). Express the result of (ii) in Exercise 6 with idempotents. In other words, explicitely describe (not necessarily primitive) idempotents in endomorphism rings of finite-dimensional representations of the subspace quiver. Furthermore, compute endomorphism rings of some non-trivial indecomposable representations of the subspace quiver. For instance:



for suitable values of $\lambda \in k$ (find those as well).

Exercise 8 (Kronecker quiver; 15, III.4 in [1]). Show that the endomorphism ring of the following representation of the Kronecker quiver $(1 \circ \rightrightarrows \circ\, 2)$:

$$k[t] \overset{1}{\underset{\cdot t}{\rightrightarrows}} k[t]$$

is not local; even though, the representation is indecomposable.

Exercise 9 (Kronecker quiver continued). For any $n \in \mathbb{N}$ find φ, ψ maps such that

$$k^n \overset{\varphi}{\underset{\psi}{\rightrightarrows}} k^{n+1}$$

is indecomposable. Compute the endomorphism ring of such a representation.

Exercise 10 (14, II.4 in [1]). Find a quiver Q and an admissible ideal $I \subseteq kQ$ such that $kQ/I \cong B$, where B is as in Exercise 1. (Hint: Q is a quiver with a single vertex and two loops.)

Exercise 11 (5, II.4 in [1]). Let Q be a finite and acyclic quiver. Prove that kQ is a connected $k-$algebra if and only if kQ/R_Q^2 is a connected $k-$algebra (R_Q is the ideal of kQ generated by all arrows of Q).

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

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