NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 1—March 12, 2021

Our goal today is to review the material in Section I.3 in [1]. There are two main topics which we cover: Wedderburn-Artin theorem and (Jacobson) radical of a module over a finite-dimensional algebra.

Wedderburn-Artin theorem

Definition 1 (Semisimple modules and rings). *A module M in $\text{Mod} - R$ is called simple if it has no proper submodules (other than zero submodule and itself). It is called semisimple (or completely reducible) if it is a direct sum of simple $R-$modules. Finally, a ring S is semisimple if it is semisimple as a module over itself.*

Definition 2 (Socle of a module). *Let M in $\text{Mod} - R$ be a module. Then,* $\text{soc}(M)$ *is the submodule of M generated by all simple submodules of M. It is referred to as the socle of M.*

Exercise 1. Prove that, for M and N right modules over R:

(i) M is semisimple if and only if $\text{soc}(M) = M$. (Hint: Use Zorn lemma.)

(ii) Let $f : M \to N$ be an $R-$module homomorphism. Then, $f(\text{soc}(M)) \subseteq \text{soc}(N)$.

(iii) Epimorphic image of a semisimple module is semisimple.

(iv) R is semisimple if and only if all right modules over R are semisimple.

Exercise 2 (Schur lemma; 3.1 in chapter I in [1]). Let $S_1, S_2 \in \text{Mod} - R$, and $f : S_1 \to S_2$ be a non-zero homomorphism between them. Then, prove the following:

(i) If S_1 is simple, f is a monomorphism.

(ii) If S_2 is simple, f is an epimorphims.

(iii) If both are simple, f is an isomorphism.

Exercise 3. Find a simple example of a ring R (preferrably a finite-dimensional algebra over a field k) and an $R-$module M such that M is not simple, yet $\text{End}_R(M)$ is a division ring.

Exercise 4 (Corollary 3.2 in chapter I in [1]). Let R be a finite-dimensional algebra over an *algebraically closed* field k. Then, for every S, a simple module over R, prove that $\text{End}_R(S) \cong k$.

Exercise 5 (Wedderburn-Artin theorem; 3.4 in chapter I in [1]). Let R be a ring. Then, prove that the following propositions are equivalent:

(i) R is semisimple.

(ii) R is isomorphic to $M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$ for $m_1, \ldots, m_n \in \mathbb{N}$ and division rings $D_1, \ldots, D_n$.

Exercise 6 (Wedderburn-Artin theorem continued; 3.4 in chapter I in [1]). Let A be a finite-dimensional algebra. Then, prove that the following propositions are equivalent:

(i) A is semisimple.

(ii) $\operatorname{rad} A = 0$.

(Hint: Show that every right ideal in A splits.)

Radical of a module over a finite-dimensional algebra

Definition 3 (Radical of a module; 3.6 in chapter I in [1])**.** *Let $M \in \mathrm{Mod} - R$. The (Jacobson) radical of M,* $\operatorname{rad} M$ *is the intersection of all maximal submodules of M.*

Definition 4 (Superflous submodule; 5.5(a) in chapter I in [1])**.** *Let $M \in \mathrm{Mod} - R$, and let $L \leq M$ be a submodule. Say that L is superflous if for any $N \leq M$ such that $L + N = M$, N needs to equal M.*

Exercise 7 (3.7 in chapter I in [1]). Prove that, for M and N right modules over R:

(i) An element $m \in M$ is in the radical of M if it is in the kernel for every module homomorphism $f : M \to S$ where S is simple.

(ii) $\mathrm{rad}(M \oplus N) = \mathrm{rad}M \oplus \mathrm{rad}N$.

(iii) Let $f : M \to N$ be an $R-$module homomorphism. Then, $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$.

(iv) If R is a finite dimensional algebra, then $M \cdot \operatorname{rad} R = \operatorname{rad} M$. (Hint: Key is that $R/\operatorname{rad} R$ is semisimple.)

(v) $\operatorname{rad} M$ is a superflous submodule of M.

Definition 5 (Top of a module)**.** *Let A be a finite-dimensional algebra, and $M \in$* $\mathrm{Mod} - A$*. Its top is defined as* $\operatorname{top} M = M/\operatorname{rad} M$*.*

Exercise 8 (Corollary 3.9(a) in chapter I in [1]). Let A be a finite-dimensional algebra, and $M, N \in \mathrm{Mod} - A$. Then $f : M \to N$, a module homomorphism, is surjective if and only if the induced map $\operatorname{top} f : \operatorname{top} M \to \operatorname{top} N$ is surjective.
(Hint: Determine what $\operatorname{top} f$ is, and use properties of the module radical.)

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

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