Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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Finite representation type

The main theorem

Theorem (Gabriel) [Kra, Theorem 5.1.1 and Corollary 5.3.3]
Let K be a field and Q a finite connected quiver.

1. There are only finitely many indecomposable representations in $\mathrm{rep}_K(Q)$ iff Q is of Dynkin type.
2. If Q is of Dynkin type, then $M \mapsto \underline{\dim} M$ induces a bijection

$$\left\{ \begin{array}{c} \text{indecomposable} \\ \text{representations of } Q \end{array} \right\} / \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{positive} \\ \text{roots of } Q \end{array} \right\}$$

The Dynkin case

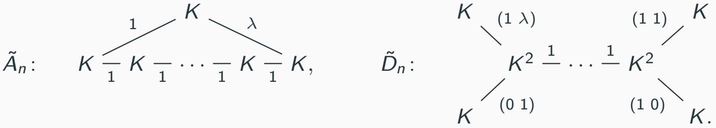
- Suppose that Q is of Dynkin type.
- If $M \in$ *ind*-Q, consider the shortest expression such that

$$\sigma_i \sigma_{i-1} \cdots \sigma_1 (\sigma_n \cdots \sigma_2 \sigma_1)^r (\underline{\dim} M) < 0$$

- Then $\sigma_{i-1} \cdots \sigma_1 c^r (\underline{\dim} M) = e_i$, so $S^+_{i-1} \cdots S^+_1 C^r(M) \cong S(i)$.
- Consequently $M \cong C^{-r} P(i)$ is preprojective and preprojective indecomposable representations are determined by their dimension vectors.
- On the other hand, any positive root $x \in \mathbb{Z}^n$ is of the form $c^{-r} \sigma_1 \cdots \sigma_{i-1}(e_i)$ for some $i \in Q_0$ and $r \geq 0$ such that all shorter expressions are also positive roots.
- It follows that $\underline{\dim} M = x$ for $M = C^{-r} P(i)$.

The general case

- Since any non-Dynkin quiver has a subquiver of Euclidean type, it suffices to prove that there are infinitely many indecomposable representations for Euclidean quivers.
- Method 1: Find ∞-many indecomposable regular representations.
- Suppose for simplicity that K is an infinite field. Then we have from instance

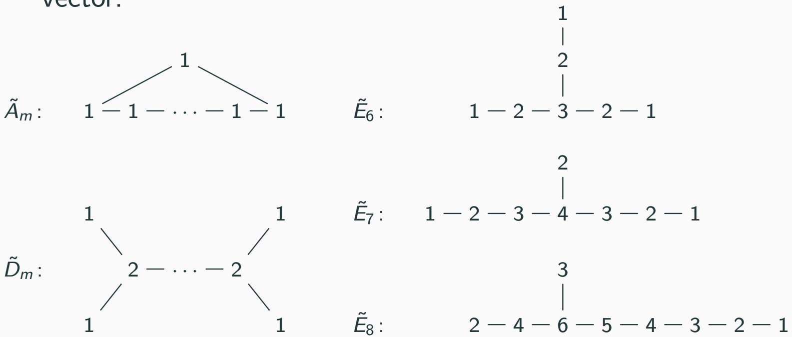


- Method 2: Prove that there are ∞-many preprojectives and preinjectives in the Euclidean type.

Representations of Euclidean quivers

The defect of a representation [Kra, §5.2]

- Let Q be an acyclic quiver of Euclidean type ($\tilde{A}_m$, $\tilde{D}_m$ or $\tilde{E}_{6,7,8}$), $n = m + 1 = |Q_0|$ and $\delta \in \mathbb{Z}^n$ the generating radical vector:



- Given $x \in \mathbb{Z}^n$, we have $0 = (\delta, x) = \langle \delta, x \rangle + \langle x, \delta \rangle$.
- The number $\partial x := \langle \delta, x \rangle = -\langle x, \delta \rangle$ if called the defect of x.
- If $M \in \text{rep}_K(Q)$, then the defect of M is $\partial M := \partial \underline{\dim} M$.

Indecomposable representations sorted out [Kra, §5.2]

Proposition ([Kra, Proposition 5.2.1])
Let Q be an acyclic quiver of Euclidean type, $n = |Q_0|$ and $M \in \textit{ind-}Q$.

1. M is preprojective iff $\partial M < 0$.
2. M is preinjective iff $\partial M > 0$.
3. M is regular iff $\partial M = 0$.

Proof.

- Since $(\delta, e_i) = 0$ for each $i \in Q_0$, we have $\sigma_i(\delta) = \delta - (\delta, e_i)e_i = \delta$. Hence $c^{\pm 1}(\delta) = \delta$.
- Say $M = C^r P(i)$ is preprojective ($r \leq 0$, $i \in Q_0$).
- Since the Euler form is invariant under $c\colon \mathbb{Z}^n \to \mathbb{Z}^n$, we have

$$\partial M = -\left\langle c^r(\underline{\dim} P(i)), c^r(\delta)\right\rangle = -\langle \underline{\dim} P(i), \delta\rangle = -\delta_i < 0.$$

- Similarly $\partial M > 0$ for M preinjective. It remains to prove that $\partial M = 0$ if M is regular.

Periodicity of c in the Euclidean case

- Recall that if Q is a quiver of Euclidean type and $\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$, then $\Delta/\mathbb{Z}\delta$ is finite.
- Since $e_i \in \Delta$ for each i, there exists $h > 0$ such that $c^h(x) - x \in \mathbb{Z}\delta$ for each $x \in \mathbb{Z}^n$.

Lemma ([Kra, Lemma 4.4.5])
Let Q be a quiver of Euclidean type and $x \in \mathbb{Z}^n$. Then

1. If $c^r(x) > 0$ for all $r \in \mathbb{Z}$, then $c^h(x) = x$.
2. If $c^h(x) = x$, then $\partial x = 0$.

Proof.

1. Suppose $c^h(x) = x + m\delta$ for $m \neq 0$. Then $c^{kh}(x) = x + km\delta$ by induction, so $c^{kh}(x) \not> 0$ for some $k \in \mathbb{Z}$ since δ is sincere.
2. Then $y = \sum_{r=0}^{h-1} c^r(x)$ is fixed by c, so $y \in \mathbb{Z}\delta$. Now $0 = \langle \delta, y \rangle = \sum_{r=0}^{h-1} \left\langle c^r(\delta), c^r(x) \right\rangle = h \cdot \langle \delta, x \rangle$. □

Preprojectives and preinjectives in the Euclidean case

Theorem ([Kra, Theorem 5.3.1])
Let Q be a finite acyclic quiver of Euclidean type. The assignment $M \mapsto \underline{\dim} M$ induces a bijections between

1. the isomorphism classes of indecomposable preprojective representations of Q and the positive roots of Q with negative defect and
2. the isomorphism classes of indecomposable preinjective representations of Q and the positive roots of Q with positive defect.

The preprojective and preinjective indecomposables form $2n$ countably infinite series $C^{-r}P(i)$ and $C^{r}I(i)$, $r \geq 0$, $i \in Q_0$ of pairwise non-isomorphic representations.

Preprojectives and preinjectives in the Euclidean case—proof

Proof.

- If $x \in \Delta$ has non-zero defect, then $c^r(x) < 0$ for some $r \in \mathbb{Z}$.
- Then $x = \underline{\dim} M$ for M indecomposable preprojective (if $r > 0$) or preinjective (if $r < 0$).
- Finally, $C^r(P(i))$ is non-injective (so non-zero) for each $r \leq 0$, since $\partial C^r(P(i)) = \partial P(i) < 0$. Hence the preprojectives form n countably infinite series of pairwise non-isomorphic representations.
- Similarly for preinjectives. $\square$