

REPRESENTATION THEORY

1.

26/03/2020

- LAST TOPICS: JACOBSON RADICAL OF A MODULE
AND PROJECTIVE COVERS

- DEF: A RING, $M \in \text{Mod } A$. ~~THEN~~ THEN

$$\text{rad}(M) := \bigcap_{\substack{M' \subseteq M \\ \text{MAXIMAL}}} M' = \bigcap_{\substack{f: M \rightarrow S \\ S \text{ SIMPLE}}} \text{Ker } f$$

THE (JACOBSON) RADICAL OF M .

- P: A RING. THEN

$$(a) \text{rad}\left(\bigoplus_{i \in I} M_i\right) = \bigoplus_{i \in I} \text{rad}(M_i) \quad \left(\subseteq \bigoplus_{i \in I} M_i\right)$$

(b) $\forall f: M \rightarrow N$ A HOMO MORPHISM:

$$f(\text{rad } M) \subseteq \text{rad } N$$

$$(c) M \cdot \text{rad}(A) \subseteq \text{rad}(M).$$

IF A IS A FINITE DIM. ALGEBRA, THEN:

$$M \cdot \text{rad}(A) = \text{rad}(M)$$

- PF: (a) $f = (f_i)_{i \in I} : \bigoplus M_i \rightarrow S$

$$f\left(\left(m_i\right)_{i \in I}\right) = 0 \iff f_i(m_i) = 0 \quad \forall i \in I$$

$$(b) \quad m \in \text{rad}(M) \implies f(m) = 0 \quad \forall g: M \rightarrow S$$

$$\implies h f(m) = 0 \quad \forall h: N \rightarrow S$$

$$\implies f(m) \in \text{rad}(N)$$

$$(c) \quad m \in M \implies \begin{array}{ccc} f_m: & A & \longrightarrow M \\ & a & \longmapsto m \cdot a \end{array}$$

$$\stackrel{(b)}{\implies} m \cdot \text{rad}(A) \subseteq \text{rad}(M)$$

- SUPPOSE NOW A IS FIN. DIMENSIONAL

- THEN $\frac{A}{\text{rad}(A)}$ IS SEMISIMPLE AND $\frac{M}{M \cdot \text{rad}(A)}$ IS AN $\frac{A}{\text{rad}(A)}$ -MODULE \Rightarrow SEMISIMPLE

$$\Rightarrow 0 = \text{rad} \left(\frac{M}{M \cdot \text{rad}(A)} \right) = \frac{\text{rad}(M)}{M \cdot \text{rad}(A)}$$

- COROLLARY: $M \in \text{Mod } A$, A FIN. DIM. \Rightarrow

$$(\exists N > 0) (\text{rad}^N(M)) = 0$$

($\text{rad}^N(M) := \text{rad}(\text{rad}^{N-1}(M))$ INDUCTIVELY)

- DEF: $L_A \leq M_A$ IS SUPERFLUOUS SUBMODULE

IF $\forall N \leq M$:

$$\boxed{L + N = M \Rightarrow N = M}$$

- P: A FIN. DIM. ALGEBRA, $M \in \text{Mod } A$.

THEN $\text{rad}(M)$ IS SUPERFLUOUS IN M .

- PF: ~~LET~~ LET $\text{rad}(M) + N = M$, I.E.

$$M \cdot \text{rad}(A) + N = M$$

$$\Rightarrow (M \cdot \text{rad}(A) + N) \cdot \text{rad}(A) + N = M$$

$$M \cdot \text{rad}(A)^2 + N \cdot \text{rad}(A) + N$$

$$M \cdot \text{rad}(A)^2 + N$$

- GO ON BY INDUCTION:

$$M \cdot \text{rad}(A)^n + N = M \quad \forall n > 0$$

$$\Rightarrow N = M$$

- PROJECTIVE COVERS

- DEF: AN EPIMORPHISM $f: M \rightarrow N$ IS MINIMAL IF $\text{Ker} f$ IS SUPERFLUOUS IN M .

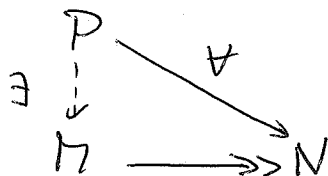
- L: f MINIMAL $\iff \left(\begin{array}{l} \forall g: L \rightarrow M, \\ f \circ g \text{ EPI} \implies g \text{ EPI} \end{array} \right)$

- DEF: A PROJECTIVE COVER OF M IS AN EPI.

$h: P \rightarrow M$ ~~WITH~~ SUCH THAT

- (1) P IS PROJECTIVE
- (2) h IS MINIMAL

- REM: PROJECTIVE MODULES:



EQV.: P_A PROJECTIVE $\iff P_A \leq \bigoplus A_A^{(\Gamma)}$ FOR SOME Γ

- P: LET $h: P_A \rightarrow M_A$ BE EPI WITH P PROJECTIVE
 TFAE:

(1) h IS MINIMAL (I.E. PROJ. COVER)

(2) ~~...~~ $\implies g: P \rightarrow P$ ISOMORPHISM

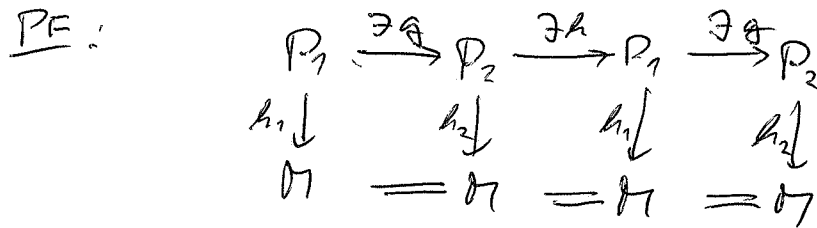
- PF: (1) \implies (2): g IS EPI BY MINIMALITY OF h
 $\implies g$ IS SPLIT EPI: $\implies g$ ISO BY MINIMALITY

~~(2) \implies (1): $N = \text{Im } g = P \implies g(N) = P$~~

(2) \implies (1) EXERCISE $\implies g$

- UNIQUENESS: $\mathcal{P} = \mathcal{M} \iff \mathcal{P} = \mathcal{Q} \oplus \mathcal{V}$ (where $\mathcal{V} = \mathcal{P} \cap \mathcal{Q}^\perp$)

- P: $h_1: P_1 \rightarrow M, h_2: P_2 \rightarrow M$ PROJ. COVERS
 $\implies \exists g: P_1 \xrightarrow{\cong} P_2$ SUCH THAT $h_2 g = h_1$.



- EXISTENCE:

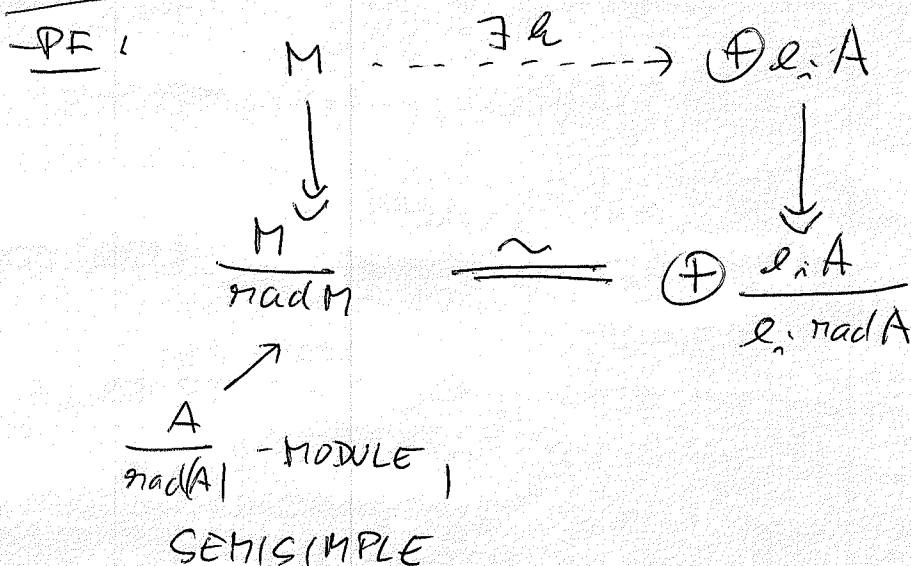
- L: A FIN. DIM. ALGEBRA, $e = e^2 \in A$ PRIMITIVE

\implies ~~$h: eA \rightarrow eA$~~ $h: eA \rightarrow \frac{eA}{e \text{ rad } A}$ IS PROJ. COVER.

- PF: - USE LIFTING OF IDEMPOTENTS

- T: A FIN. DIM. ALGEBRA, M_A MODULE

$\implies \exists$ PROJECTIVE COVER



DUALITY:

- A ALGEBRA OVER K

- $M_A \rightsquigarrow \text{Hom}_K(M, K)$ LEFT A-MODULE

$$f: M \rightarrow K, a \in A$$

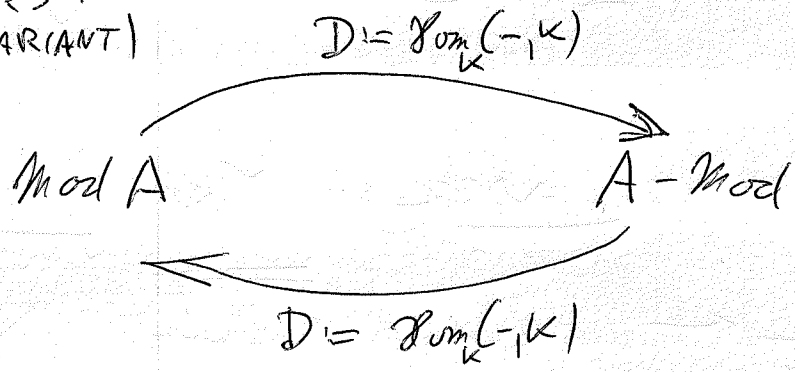
$$\rightsquigarrow (a \cdot f)(m) := f(m \cdot a)$$

- $A^N \rightsquigarrow \text{Hom}_K(N, K)$ RIGHT A-MODULE

$$\rightsquigarrow g: N \rightarrow K, a \in A$$

$$\rightsquigarrow (g \cdot a)(m) := g(a \cdot m)$$

- FUNCTORS:
(CONTRAVARIANT)



$$\rightsquigarrow \begin{matrix} \text{Mod } A & \xrightarrow{D} & D(D(M)) \\ m & \mapsto & ((\varphi: D(M) \rightarrow K) \mapsto \varphi(m)) \end{matrix}$$

$\rightsquigarrow M_M$ ISO IF M FIN. DIM.

- SO A IS A FIN. DIM. ALGEBRA

$$\rightsquigarrow (\text{Mod } A)^{\text{op}} \xrightarrow[\cong]{D} A\text{-mod}$$