REPRESENTATION THEORY

26/03/2020

- LAST TOPICS: <u>JACOBSON RADICAL OF A MODULE AND PROJECTIVE COVERS</u>

- <u>DEF</u>: <u>A</u> RING, $M \in \operatorname{mod} A$. ~~[illegible]~~ THEN

$$\operatorname{rad}(M) := \bigcap_{\substack{M' \subseteq M \\ \text{MAXIMAL}}} M' = \bigcap_{\substack{f: M \to S \\ S \text{ SIMPLE}}} \operatorname{Ker} f \, ,$$

THE (<u>JACOBSON</u>) <u>RADICAL OF M</u>.

- <u>P</u>: <u>A</u> A RING. THEN

(a) $\operatorname{rad}\left(\bigoplus_{i \in I} M_i\right) = \bigoplus_{i \in I} \operatorname{rad}(M_i) \quad \left(\subseteq \bigoplus_{i \in I} M_i\right)$

(b) $\forall f: M \to N$ A HOMOMORPHISM:

$$f(\operatorname{rad} M) \subseteq \operatorname{rad} N$$

(c) $M \cdot \operatorname{rad}(A) \subseteq \operatorname{rad}(M)$.

IF A IS A FINITE DIM. ALGEBRA, THEN:

$M \cdot \operatorname{rad}(A) = \operatorname{rad}(M)$

- <u>PF</u>: (a) $f = (f_i)_{i \in I} : \bigoplus M_i \longrightarrow S$

$$f((m_i)_{i \in I}) = 0 \iff f_i(m_i) = 0 \;\; \forall i \in I$$

(b) $m \in \operatorname{rad}(M) \implies g(m) = 0 \;\; \forall g: M \to S$

$\implies hf(m) = 0 \;\; \forall h: N \to S$

$\implies f(m) \in \operatorname{rad}(N)$

(c) $m \in M \implies f_m : A \longrightarrow M$, $a \longmapsto m \cdot a$

$\overset{(b)}{\implies} m \cdot \operatorname{rad}(A) \subseteq \operatorname{rad}(M)$

- SUPPOSE NOW A IS FIN. DIMENSIONAL
- THEN $\frac{A}{rad(A)}$ IS SEMISIMPLE AND $\frac{M}{M \cdot rad(A)}$ IS AN $\frac{A}{rad(A)}$ - MODULE $\Rightarrow$ SEMISIMPLE

$$\Rightarrow 0 = rad\left(\frac{M}{M \cdot rad A}\right) = \frac{rad(M)}{M \cdot rad(A)}$$

- <u>COROLLARY</u>: $M \in \mathrm{Mod}\,A$, A FIN. DIM. $\Rightarrow$

$$(\exists N > 0)\left(rad^N(M)\right) = 0$$

$$\left(rad^N(M) := rad(rad^{N-1}(M)) \text{ INDUCTIVELY}\right)$$

- <u>DEF</u>: $L_A \leq M_A$ IS <u>SUPERFLUOUS SUBMODULE</u> IF $\forall N \leq M$:

$$\boxed{L + N = M \Rightarrow N = M}$$

- <u>P</u>: A FIN. DIM. ALGEBRA, $M \in \mathrm{Mod}\,A$. THEN $rad(M)$ IS SUPERFLUOUS IN M.

- <u>PF</u>: LET $rad(M) + N = M$, I.E.

$$M \cdot rad(A) + N = M$$

$$\Rightarrow (M \cdot rad(A) + N) \cdot rad(A) + N = M$$

$$\| $$

$$M \cdot rad(A)^2 + N \cdot rad A + N$$

$$\|$$

$$M \cdot rad(A)^2 + N$$

— GO ON BY INDUCTION:

$$M \cdot rad(A)^m + N = M \quad \forall m > 0$$

$$\Rightarrow N = M$$

– PROJECTIVE COVERS

– **DEF**: AN EPIMORPHISM $f : M \twoheadrightarrow N$ IS MINIMAL IF $\operatorname{Ker} f$ IS SUPERFLUOUS IN M.

– **L**: f MINIMAL $\iff \left(\forall g : L \to M,\ fg \text{ EPI} \Rightarrow g \text{ EPI} \right)$

– **DEF**: A PROJECTIVE COVER OF M IS AN EPI. $h : P \to M$ ~~WITH~~ SUCH THAT

(1) P IS PROJECTIVE

(2) h IS MINIMAL

– **REM**: PROJECTIVE MODULES:

$$\begin{tikzcd} P \arrow[d, dashed, "\exists"'] \arrow[dr, "\forall"] & \\ M \arrow[r, two heads] & N \end{tikzcd}$$

EQUIV.: P_A PROJECTIVE $\iff P_A \leq_\oplus A_A^{(I)}$ FOR SOME I

– **P**: LET $h : P_A \to M_A$ BE EPI WITH P PROJECTIVE

TFAE:

(1) h IS MINIMAL (I.E. PROJ. COVER)

(2) $$\begin{tikzcd} P \arrow[d, "g"'] \arrow[dr, two heads, "h"] \arrow[d, phantom, shift left=4, "/\!/"] & \\ P \arrow[r, "h"'] & M \end{tikzcd} \Longrightarrow g : P \to P \text{ ISOMORPHISM}$$

– **PF**: (1) ⟹ (2): • g IS EPI BY MINIMALITY OF h

⟹ g IS SPLIT EPI: $$\begin{tikzcd} P \arrow[d, "\exists"'] \arrow[dr, "id"] & \\ g : P \arrow[r, two heads] & P \end{tikzcd}$$

⟹ g ISO BY MINIMALITY

~~(2) ⟹ (1): $N + \operatorname{Ker} g = P$ ⟹ $g(N) = P$~~

(2) ⟹ (1) EXERCISE

- UNIQUENESS:

- P: $h_1 : P_1 \to M$, $h_2 : P_2 \to M$ PROJ. COVERS
  $\Longrightarrow \exists g : P_1 \xrightarrow{\cong} P_2$ SUCH THAT $h_2 g = h_1$.

PF:

$$\begin{tikzcd} P_1 \arrow[r, "\exists g"] \arrow[d, "h_1"] & P_2 \arrow[r, "\exists h"] \arrow[d, "h_2"] & P_1 \arrow[r, "\exists g"] \arrow[d, "h_1"] & P_2 \arrow[d, "h_2"] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd}$$

- EXISTENCE:

- L: A FIN. DIM. ALGEBRA, $e = e^2 \in A$ PRIMITIVE
  $\Longrightarrow$ $h : eA \longrightarrow \dfrac{eA}{e\, \mathrm{rad} A}$ IS PROJ. COVER

- PF: - USE LIFTING OF IDEMPOTENTS

- T: A FIN. DIM. ALGEBRA, M_A MODULE
  $\Longrightarrow$ $\exists$ PROJECTIVE COVER

- PF:

$$\begin{tikzcd} M \arrow[r, dashed, "\exists h"] \arrow[d, two heads] & \bigoplus e_i A \arrow[d, two heads] \\ \dfrac{M}{\mathrm{rad} M} \arrow[r, "\sim"] & \bigoplus \dfrac{e_i A}{e_i \, \mathrm{rad} A} \end{tikzcd}$$

$\dfrac{A}{\mathrm{rad}(A)}$ - MODULE,
SEMISIMPLE

DUALITY:

- $\underline{A}$ ALGEBRA OVER $\mathbb{K}$

- $M_A \rightsquigarrow \mathcal{H}om_{\mathbb{K}}(M, \mathbb{K})$ LEFT A-MODULE

$$f : M \to \mathbb{K}, \quad a \in A$$
$$\rightsquigarrow (a \cdot f)(m) := f(m \cdot a)$$

- ${}_A N \rightsquigarrow \mathcal{H}om_{\mathbb{K}}(N, \mathbb{K})$ RIGHT A-MODULE

$$\rightsquigarrow g : N \to \mathbb{K}, \quad a \in A$$
$$\rightsquigarrow (g a)(m) := g(a \cdot m)$$

- FUNCTORS: (CONTRAVARIANT)

$$\begin{tikzcd} \mathrm{Mod}\, A \arrow[rr, bend left, "D := \mathcal{H}om_{\mathbb{K}}(-, \mathbb{K})"] & & A\text{-}\mathrm{Mod} \arrow[ll, bend left, "D := \mathcal{H}om_{\mathbb{K}}(-, \mathbb{K})"] \end{tikzcd}$$

$$\rightsquigarrow \quad \eta_M : M_A \longrightarrow D(D(M))$$
$$m \longmapsto \big((\varphi : DM \to \mathbb{K}) \longmapsto \varphi(m)\big)$$

$\rightsquigarrow$ η_M ISO IF M FIN. DIM.

- SO $\underline{A}$ IS A FIN. DIM. ALGEBRA

$$\rightsquigarrow \quad (\mathrm{mod}\, A)^{op} \xrightarrow[\simeq]{D} A\text{-}\mathrm{mod}$$
