Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

Jan Šťovíček

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Department of Algebra, Charles University, Prague

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The Harada-Sai lemma and consequences—continued

The Harada-Sai lemma [Kra, §6.3]

Lemma (Harada-Sai, [Kra, Lemma 6.3.1])
Let $n \geq 1$ and suppose we have in ind-A a chain of non-isomorphisms $X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{2^n-2}} X_{2^n-1} \xrightarrow{\varphi_{2^n-1}} X_{2^n}$ *between modules of dimension $\leq n$. Then* $\varphi_{2^n-1} \cdots \varphi_1 = 0$.

Corollary
Suppose that A is a finite dimensional algebra which is of finite representation type. Then $\exists N > 0$ such that $\mathrm{Rad}^N(X, Y) = 0$ for all $X, Y \in$ *ind-A*. In particular, each non-isomorphism in *ind-A* is a sum of compositions of irreducible morphisms by [Kra, Proposition 6.2.4].

Preprojective (and preinjective) representations

- Suppose that $A = KQ$, where Q is a finite acyclic quiver.
- The for each pair of preprojectives $X, Y \in \textit{ind-}KQ$, there exists $N > 0$ such that $\mathrm{Rad}^N(X, Y) = 0$
- Indeed, suppose that $X = C^{-r}P(i)$ and $Y = C^{-s}P(j)$ and we have a chain of non-isomorphisms

$$X = X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{N-2}} X_{N-1} \xrightarrow{\varphi_{N-1}} X_N = Y.$$

- If all the $\varphi_i \neq 0$, then each X_i is isomorphic to $C^{-t}P(k)$ for $k \in Q_0$ and t is between r and s (last time).
- So $\varphi_{N-1} \cdots \varphi_1 = 0$ if $N \gg 0$.
- In particular, each non-isomorphism between preprojective modules in *ind*-KQ is a sum of compositions of irreducible morphisms by [Kra, Proposition 6.2.4].

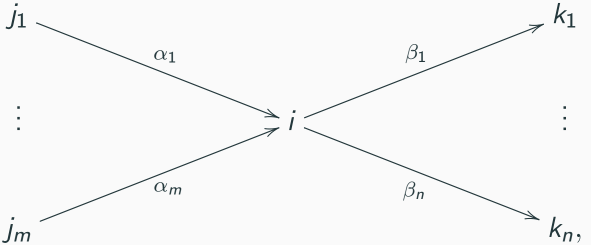
Irredicible morphisms between preprojectives

Morphisms induced by arrows

- If Q is a finite acyclic quiver and $\alpha\colon i \to j$ is an arrow, then we have a monomorphism

$$\alpha^* = \alpha \cdot -\colon P(j) = e_j \cdot KQ \rightarrowtail e_i \cdot KQ = P(i).$$

- We can collect there together:



$$\bigoplus_{\alpha\colon j\to i} P(j) \xleftarrow{\tau(i)} P(i) \overset{\sigma(i)}{\leftarrowtail} \bigoplus_{\alpha\colon i\to k} P(k) = \operatorname{rad} P(i).$$

Radical morphisms to projectives

Lemma ([Kra, Lemma 7.1.2])
TFAE for a morphism $\varphi\colon X \to P(i)$, $X \in$ *mod-KQ:*

1. $\varphi \in \operatorname{Rad}(X, P(i))$.
2. φ *is not an epimorphism.*
3. φ *admits a factorization* $\varphi = \sigma(i)\varphi'$.

Proof.

- 1. $\iff$ 2. follows from [Kra, Lemma 6.2.2] (see here) and the fact that each epimorphism to $P(i)$ is a split epimorphism.
- 2. $\iff$ 3. follow from the fact that $\sigma(i)\colon \operatorname{rad} P(i) \rightarrowtail P(i)$ and $\operatorname{rad} P(i)$ is the unique maximal submodule of $P(i)$. $\square$

Irreducible morphims to projectives

Lemma ([Kra, Lemma 7.1.3(1)])
Let $i \in Q_0$ be a vertex and X an indecomposable representation. If $\varphi\colon X \to P(i)$ is an irreducible morphism then there is an arrow $\beta\colon i \to k$ such that $X \cong P(k)$.

Proof.

- We have a factorization $X \xrightarrow{\varphi'} \bigoplus_{\alpha\colon i \to k} P(k) \overset{\sigma(i)}{\rightarrowtail} P(i)$.
- Since $\sigma(i)$ is not a split epi, φ' is a split mono.
- So $X \cong P(k)$ for some arrow $\beta\colon i \to k$. □

Irreducible morphims to projectives—continued

Notation

- If $X, Y \in$ *ind-KQ*, $\operatorname{Irr}(X, Y) := \operatorname{Rad}(X, Y)/\operatorname{Rad}^2(X, Y)$.
- If $i, j \in Q_0$, then $Q_1(i, j) = \{\alpha \in Q_1 \mid \alpha\colon i \to j\}$.

Lemma ([Kra, Lemma 7.1.4])
Let $i, j \in Q_0$. Then the map sending an arrow $\alpha\colon i \to j$ to the coset $\alpha^* + \operatorname{Rad}^2(P(j), P(i))$ induces a K-linear isomorphism

$$KQ_1(i, j) \xrightarrow{\sim} \operatorname{Irr}(P(j), P(i)).$$

Proof.
This follow from the facts that

1. $\operatorname{Hom}(P(j), P(i))$ has a basis formed by all paths $i \rightsquigarrow j$ and
2. If $X \in$ *ind-KQ* and there are non-zero morphisms $P(j) \to X \to P(i)$, then $X \cong P(k)$ for some $k \in Q_0$. $\square$

Irreducible morphims from a simple projective

Lemma ([Kra, Lemma 7.1.3(1)])
Let $i \in Q_0$ be a sink and $X = (X_i, f_\alpha)$ an indecomposable representation. If $\varphi\colon P(i) = S(i) \to X$ is an irreducible morphism then there is an arrow $\alpha\colon j \to i$ such that $X \cong P(j)$.

Proof.

- It suffices to prove that there is a factorization $\varphi = \varphi' \tau(i)$, where $\tau(i)\colon P(i) \to \bigoplus_{\alpha\colon j\to i} P(j)$ is as before.
- Indeed, since $\tau(i)$ is not a split mono, φ' is a split epi and consequently $P(j) \cong X$ for some arrow $\alpha\colon j \to i$.
- To finish the proof, note that we have

$$\begin{array}{ccc} \operatorname{Hom}\left(\bigoplus_{\alpha\colon j\to i} P(j), X\right) & \xrightarrow{\operatorname{Hom}(\tau(i),X)} & \operatorname{Hom}\left(P(i), X\right) \\ \sim\downarrow & & \downarrow\sim \\ \bigoplus_{\alpha\colon j\to i} X_j & \xrightarrow[(f_\alpha)]{} & X_i. \end{array}$$

- The lower map is surjective by the prop. of S_i^+, since $X \not\cong S(i)$. $\square$

Reflection functors and the radical

Lemma ([Kra, Lemma 7.3.1])
Let $i \in Q_0$ be a sink and $X, Y \in$ ind-KQ not isomorphic to $S(i)$. Then S_i^+ induces isomorphisms $\mathrm{Rad}^n(X, Y) \xrightarrow{\sim} \mathrm{Rad}^n(S_i^+ X, S_i^+ Y)$ *for all $n \geq 0$. In particular, S_i^+ induces* $\mathrm{Irr}(X, Y) \xrightarrow{\sim} \mathrm{Irr}(S_i^+ X, S_i^+ Y)$.

Proof.

- We know the result for $n = 0$ already and $n = 1$ is easy.
- For $n \geq 2$, it suffices to prove that if we have non-zero morphisms $X \to Z \to Y$ in *ind-KQ*, then $Z \ncong S(i)$. This is clear as well, since otherwise $X \to Z$ would be a split mono, so an isomorphism. □

Irreducible morphisms from indecomposable projectives

- Suppose $Q_0 = \{1, \ldots, n\}$ is admissibly ordered and we have an arrow $\alpha \colon i \to j$ $(i > j)$.
- We construct an irreducible morphism $\alpha_* \colon P(i) \to C^- P(j)$ as follows:
- Put $Q' = \sigma_{i-1} \cdots \sigma_1 Q$ (so i is a sink of Q').
- Then we have an irreducible morphism in *ind*-KQ' (since we have an arrow $\alpha' \colon j \to i$ in Q'),

$$\alpha'^* \colon P'(i) = S'(i) \to P'(j) = S_i^- \cdots S_n^- S_1^- \cdots S_{j-1}^- S'(j)$$

- Then we just take $\alpha_* := S_1^- \cdots S_{i-1}^- \alpha'^*$ and use the last lemma to prove that it is irreducible.

Irreducible morphisms between indecomposable preprojectives

Proposition [Kra, Proposition 7.3.4]
Let Q be a finite acyclic quiver, $X = C^{-r}P(i)$ and $Y = C^{-s}P(j)$ ($i, j \in Q_0$ and $r, s \geq 0$). Then

$$\mathrm{Irr}(X, Y) = \frac{\mathrm{Rad}(X, Y)}{\mathrm{Rad}^2(X, Y)} = \begin{cases} KQ_1(j, i) & \text{if } s = r, \\ KQ_1(i, j) & \text{if } s = r + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

- If $s \leq r$, then
$$\mathrm{Irr}(X, Y) \cong \mathrm{Irr}(C^{s-r}P(i), P(j)) \cong \begin{cases} KQ_1(j, i) & \text{if } s = r, \\ 0 & \text{if } s < r. \end{cases}$$
- If $s \geq r + 1$, $\mathrm{Irr}(X, Y) \cong \mathrm{Irr}(P(i), C^{r-s}P(j)) \cong$
$$\mathrm{Irr}(S(i), S_i^- \cdots S_n^- C^{r-s+1}P(j)) \cong \begin{cases} KQ_1(i, j) & \text{if } s = r + 1, \\ 0 & \text{if } s > r + 1. \end{cases} \quad \square$$

Summary [Kra, §§7.4 and 7.5]

- Any morphism between preprojective representations is a K-linear combination of the identity morphisms and compositions of morphisms (where $\alpha\colon i \to j$, $r \geq 0$):
  1. $C^{-r}\alpha^*\colon C^{-r}P(j) \to C^{-r}P(i)$ and
  2. $C^{-r}\alpha_*\colon C^{-r}P(i) \to C^{-r-1}P(j)$.
- The relations between these compositions are generated by the relations

$$\sum_{\alpha\colon j\to i} C^{-r}\alpha_* C^{-r}\alpha^* + \sum_{\alpha\colon i\to k} C^{-r-1}\alpha^* C^{-r}\alpha_* = 0,$$

where $i \in Q_0$ and $r \geq 0$ such that $C^{-r-1}P(i) \neq 0$.

- These relations all come from short exact sequences in *mod-KQ*,

$$0 \to C^{-r}P(i) \to \bigoplus_{\alpha\colon j\to i} C^{-r}P(j) \oplus \bigoplus_{\alpha\colon i\to k} C^{-r-1}P(k) \to C^{-r-1}P(i) \to 0.$$