Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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Table of contents

Forms associated with quivers and graphs—continued

Roots of Dynkin and Eucledian diagrams

Forms associated with quivers and graphs—continued

The yoga of forms [Kra, §§3.2 and 4.1]

- Let Q be a finite quiver with $Q_0 = \{1, \ldots, n\}$, Γ the underlying graph and $d_{i,j}$ be the number of edges $i — j$.
- Then we have the Euler form

$$\langle x, y\rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha\colon i \to j} x_i y_j.$$

- There is an associated quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$,

$$q(x) = \langle x, x\rangle = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha\colon i \to j} x_i x_j = \sum_{i \in \Gamma_0} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j.$$

- We cannot reconstruct $\langle -, -\rangle$ from Q, but we can reconstruct the symmetrized Euler form $(-, -)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$,

$$(x, y) = q(x + y) - q(x) - q(y) = \langle x, y\rangle + \langle y, x\rangle =$$
$$= \sum_{i \in \Gamma_0} (2 - 2d_{ii}) \cdot x_i y_i - \sum_{i \neq j} d_{ij} x_i y_j.$$

Some terminology related to forms

Definition
Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a quadratic form. Then

- q is positive definite if $q(x) > 0$ for all $x \in \mathbb{Z}^n \setminus \{0\}$.
- q is positive semi-definite if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$.

The radical of q is the subgroup

$$\operatorname{rad} q = \{x \in \mathbb{Z}^n \mid (x, -) \equiv 0\} \leq \mathbb{Z}^n,$$

where $(x, y) = q(x + y) - q(x) - q(y)$ is the associated bilinear form.

Definition
Given $x, y \in \mathbb{Z}^n$, we write $x \leq y$ if $x_i \leq y_i$ for all i, and $x < y$ if $x \leq y$ and $x \neq y$ (a partial order).

Finally, a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is sincere if $x_i \neq 0$ for all i.

A key lemma

Lemma ([Kra, Lemma 4.1.3])
Let Γ be a finite connected graph, $q\colon \mathbb{Z}^n \to \mathbb{Z}$ the associated quadratic form and $y \in \operatorname{rad} q$ such that $y > 0$. Then y is sincere and q is positive semidefinite. Moreover, for each $x \in \mathbb{Z}^n$ we have

$$q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q.$$

Proof.

- By the assumption on y, we have for each i:

$$0 = (e_i, y) = (2 - 2d_{ii})y_i - \sum_{i \neq j} d_{ij} y_j.$$

- If $y_i = 0$ for some i, then also $\sum_{i \neq j} d_{ij} y_j = 0$.
- Since $y_j \geq 0$ for each j, we have $y_j = 0$ whenever $d_{ij} > 0$, i.e. there is an edge $i — j$.
- As Γ is connected, this would imply $y = 0$, a contradiction!
- It follows that y is sincere.

Proof of the key lemma—continued

- We know that $(2-2d_{ii})y_i = \sum_{i\neq j} d_{ij}y_j$ for each i.
- Then $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is positive semi-definite, since

$$
\begin{aligned}
q(x) &= \sum_i (1-d_{ii})x_i^2 - \sum_{i<j} d_{ij}x_ix_j && (\text{def. of } q)\\
&= \sum_i (2-2d_{ii})y_i \cdot \frac{1}{2y_i} \cdot x_i^2 - \sum_{i<j} d_{ij}x_ix_j \\
&= \sum_{i\neq j} d_{ij} \cdot \frac{y_j}{2y_i} \cdot x_i^2 - \sum_{i<j} d_{ij}x_ix_j && (\text{by above})\\
&= \sum_{i<j} d_{ij} \cdot \frac{y_j}{2y_i} \cdot x_i^2 - \sum_{i<j} d_{ij}x_ix_j + \sum_{i<j} d_{ij} \cdot \frac{y_i}{2y_j} \cdot x_j^2 \\
&= \sum_{i<j} d_{ij} \cdot \frac{y_iy_j}{2} \cdot \left(\frac{x_i}{y_i} - \frac{x_j}{y_j}\right)^2 .
\end{aligned}
$$

Proof of the key lemma—still continued

- So far we know that $y \in \operatorname{rad} q$ is sincere and $\forall x \in \mathbb{Z}^n$:

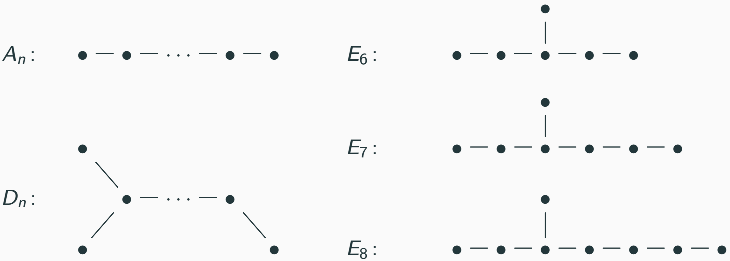
$$q(x) = \sum_{i<j} d_{ij} \cdot \frac{y_i y_j}{2} \cdot \left(\frac{x_i}{y_i} - \frac{x_j}{y_j} \right)^2 .$$

- It remains to prove that $q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q$.
- If $q(x) = 0$, then $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ whenever $i - j$ in Γ. Since Γ is connected, this implies that $x \in \mathbb{Q}y$.
- If $x \in \mathbb{Q}y$, then clearly $x \in \operatorname{rad} q$ (recall ‣ def. of rad q).
- Finally, $x \in \operatorname{rad} q$ always implies $q(x) = \frac{1}{2}(x, x) = 0$. □

When is q positive (semi-)definite? [Kra, §4.2]

Theorem ([Kra, Theorem 4.2.1])
Let Γ be a connected finite graph, $n = |\Gamma_0|$ and $q\colon \mathbb{Z}^n \to \mathbb{Z}$ the associated quadratic form.

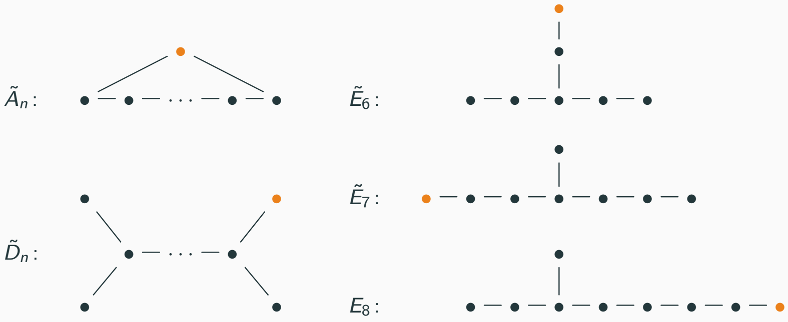
1. q is positive definite iff Γ is a Dynkin diagram:



When is q positive (semi-)definite? [Kra, §4.2]

Theorem ([Kra, Theorem 4.2.1])
Let Γ be a connected finite graph, $n = |\Gamma_0|$ and $q\colon \mathbb{Z}^n \to \mathbb{Z}$ the associated quadratic form.

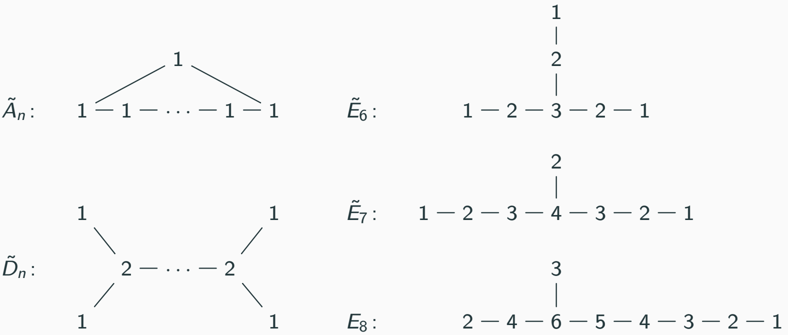
2. q is positive semi-definite but not positive definite iff Γ is a Euclidean diagram:



In that case, $\exists!\delta \in \mathbb{Z}^n$ such that $\delta > 0$ and $\operatorname{rad} q = \mathbb{Z}\delta$.

Proof of the theorem—the Euclidean case [Kra, §4.2]

- Suppose that Γ is a Euclidean diagram.
- Then we can explicitly find a positive radical vector $\delta \in \mathbb{Z}^n$:



- The quadratic form q is positive semi-definite and $\operatorname{rad} q = \mathbb{Z}\delta$ by the ‣ Key Lemma.

Proof of the theorem—the Dynkin case [Kra, §4.2]

- Suppose that Γ is a Dynkin diagram with $n = |\Gamma_0|$.
- Then there is an Euclidean diagram $\tilde{\Gamma}$ such that Γ is obtained from $\tilde{\Gamma}$ by deleting a vertex i.
- If q and $\tilde{q}$ are the corresponding quadratic forms and $x \in \mathbb{Z}^n$, then $q(x) = \tilde{q}(\tilde{x})$, where $\tilde{x} \in \mathbb{Z}^{n+1}$ is obtained from x by adding zero coordinate at vertex i. So q is positive semi-definite.
- The rest follows from the ▸ Key Lemma: If $x \neq 0$, then $x \notin \operatorname{rad} q$ (since $\tilde{x}$ is not sincere). Hence $q(x) \neq 0$.

Proof of the theorem—the remaining cases [Kra, §4.2]

- Suppose not that Γ is neither Dynkin nor Euclidean and $q\colon \mathbb{Z}^n \to \mathbb{Z}$ its quadratic form.
- Recall: $q(x) = \sum_{i\in\Gamma_0} x_i^2 - \sum_{i\leq j} d_{ij}x_ix_j$.
- We must prove that q is not positive semi-definite.
- Observation: There is a subgraph of $\Gamma' \subsetneq \Gamma$ which is Euclidean. Let $\delta \in \mathbb{Z}^{n'} \leq \mathbb{Z}^n$ its positive radical vector as above.
- Beware: We must also consider $\tilde{A}_0\colon$  with $q(x_1) \equiv 0$, and $\tilde{A}_1\colon$  with $q(x_1,x_2) = (x_1 - x_2)^2$!
- If $\Gamma_0' = \Gamma_0$, then $q(\delta) < 0$.
- If $\Gamma_0' \subsetneq \Gamma_0$, then there exists a vertex $j \in \Gamma_0 \setminus \Gamma_0'$ connected to Γ_0' by an edge. It follows that

$$q(2\delta+e_i) = 4q(\delta)+2(\delta,e_j)+q(e_j) \leq -2\sum_{i\in\Gamma_0'} d_{ij}\delta_i+(1-d_{ii}) < 0.$$

□

Back to quivers

Corollary
Let Q be a finite acyclic quiver and K a field. Then the form

$$\begin{aligned} q\colon \quad & \mathbb{Z}^n \to \mathbb{Z}, \\ & \underline{\dim} M \mapsto \dim_K \operatorname{Hom}(M, M) - \dim_K \operatorname{Ext}^1(M, M) \end{aligned}$$

is positive definite iff the underlying graph of Q is a Dynkin diagram.

Example
Let Q be an orientation of the $A_n = (1 - 2 - \cdots - n)$. Then

$$q(x) = \frac{1}{2} \cdot \left(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2\right).$$

Roots of Dynkin and Eucledian diagrams

Roots [Kra, §4.3]

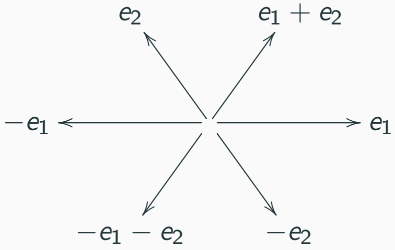
- Let Γ be a Dynkin or a Euclidean diagram.
- Let $q(x) = \sum_{i \in \Gamma_0} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j = \frac{1}{2}(x, x)$ and $(x, y) = q(x + y) - q(x) - q(y)$ be as before.
- Put $\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$.
- A root is a non-zero element of Δ.
- Motivation: Suppose Q is an acyclic quiver whose underlying diagram is Γ. We will show later that the roots of Γ are precisely the dimension vectors of indecomposable finitely generated KQ-modules.
- Observation: e_i is a root for each $i \in \Gamma_0$.
- Observation: $x \in \Delta \iff -x \in \Delta$.

Examples

- If q is positive def. ($= Q$ Dynkin), then $(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ extends to a scalar product $(-,-)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.
- The picture below is drawn in a basis of $\mathbb{R}^n$ where $(-,-)$ is the usual scalar product.

Example

- Let $\Gamma = A_2 = (2 — 1)$, so $q(x) = \frac{1}{2} \cdot \left(x_1^2 + (x_1 - x_2)^2 + x_2^2\right)$.
- The roots:



- Try to draw the roots of $A_3 = (3 — 2 — 1)$!

Properties of roots [Kra, §4.3]

Lemma (Proposition 4.3.1(3))
Let Γ be Dynkin or Euclidean. If $x \in \Delta$ is a root, then $x > 0$ or $x < 0$.

Proof.

- We can write $x = x^+ - x^-$, where $x^+, x^- \geq 0$ have disjoint support.
- $1 \geq q(x) = q(x^+) + q(x^-) - (x^+, x^-) \geq q(x^+) + q(x^-) \geq 0$.
- Thus, $q(x^+) = 0$ or $q(x^-) = 0$.
- If both x^+, x^- were non-zero, they would be sincere, a contradiction! □