Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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April 16, 2020

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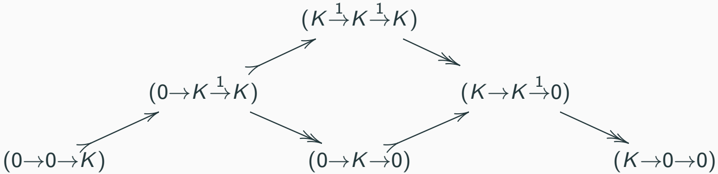
Table of contents

Motivation

Longer-term goal

- Let K be a field and Q a finite acyclic quiver and $A = KQ$ the finite dimensional hereditary algebra.
- Let *ind*-A be a skeleton of the full subcategory of *mod*-A formed by indecomposable modules.
- Aim: to understand the category *ind*-A, or at least for which quivers there are only finitely many isomorphism classes of finitely generated indecomposable A-modules.

Example
If K any field, $Q = (1 \to 2 \to 3)$, then *ind*-KQ looks like:



The Euler form

The group K_0

- Aim: Given an abelian category $\mathcal{A}$, understand functions $\delta\colon \operatorname{obj}\mathcal{A} \to G$, G abelian group, such that

$$\delta(L) = \delta(K) + \delta(M) \qquad \forall (0 \to K \to L \to M \to 0).$$

- A prototype: $\dim_K\colon \text{vect-}K \to \mathbb{Z}$, $V \mapsto \dim_K(V)$.
- Key observation: There is a universal such function $\mathcal{A} \to K_0(\mathcal{A})$!
  - Generators of $K_0(\mathcal{A})$: Isomorphism classes $[M]$, $M \in \operatorname{obj}\mathcal{A}$.
  - Relations in $K_0(\mathcal{A})$: $[L] = [K] + [M] \quad \forall (0 \to K \to L \to M \to 0)$.
- How can we compute $K_0(\mathit{mod}\text{-}A)$ for finite dimensional algebras?

K_0 for a path algebra

Definition
Given $M = (M_i, f_\alpha) \in \mathrm{rep}_\mathsf{K}(Q, I)$, we define the dimension vector $\underline{\dim} M := \big(\dim_K(M_i)\big)_{i \in Q_0}$.

Proposition
$K_0(\textit{mod-}KQ/I) \cong K_0(\mathrm{rep}_\mathsf{K}(Q, I)) \cong \mathbb{Z}^{Q_0}$.

Proof.

- We have a group homomorphism $\varphi\colon K_0(\mathrm{rep}_\mathsf{K}(Q, I)) \to \mathbb{Z}^{Q_0}$ which sends $[M] \mapsto \underline{\dim}(M)$.
- φ is surjective since it maps simples $[S_i]$ to a basis of $\mathbb{Z}^{Q_0}$.
- Each $M \in \mathrm{rep}_\mathsf{K}(Q, I)$ has a filtration
$$0 = M_0 \leq M_1 \leq M_2 \leq \cdots \leq M_{\ell-1} \leq M_\ell = M$$
such that each M_i/M_{i-1} is simple, so $[M] = \sum_{i=1}^{\ell} [M_i/M_{i-1}]$.
- The classes of simples $[S_i]$, $i \in Q_0$, are linearly independent since their images over φ are such. Hence φ is injective.

An attempt to compute dimensions of Hom-groups

- Let $A = KQ$ for Q finite acyclic and $M, N \in \textit{mod-}A$.
- Can we compute $\dim_K \operatorname{Hom}_A(M, N)$ from $\underline{\dim}(M)$ and $\underline{\dim}(N)$?
- Not really in general. However we actually can compute

$$\dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N)!$$

- Idea: If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact in *mod*-A, we have

$$0 \to \operatorname{Hom}_A(M_3, N) \to \operatorname{Hom}_A(M_2, N) \to \operatorname{Hom}_A(M_1, N) \to$$
$$\to \operatorname{Ext}^1_A(M_3, N) \to \operatorname{Ext}^1_A(M_2, N) \to \operatorname{Ext}^1_A(M_1, N) \to 0.$$

- So $(\dim_K \operatorname{Hom}_A(M_2, N) - \dim_K \operatorname{Ext}^1_A(M_2, N)) = (\dim_K \operatorname{Hom}_A(M_3, N) - \dim_K \operatorname{Ext}^1_A(M_3, N)) + (\dim_K \operatorname{Hom}_A(M_1, N) - \dim_K \operatorname{Ext}^1_A(M_1, N))$.
- One can also do similar considerations for N.

The Euler form

- By the previous slide, we obtain a bilinear form

$$\langle -,- \rangle \colon K_0(mod\text{-}A) \times K_0(mod\text{-}A) \to \mathbb{Z},$$
$$([M],[N]) \mapsto \dim_K \mathrm{Hom}_A(M,N) - \dim_K \mathrm{Ext}^1_A(M,N).$$

- This is the Euler form.
- More generally, if A has finite global dimension, we have

$$([M],[N]) \mapsto \sum_{n=0}^{\mathrm{gldim}(A)} (-1)^n \dim_K \mathrm{Ext}^n_A(M,N).$$

- Since $[M] \to \underline{\dim}(M)$ induces $K_0(mod\text{-}A) \cong \mathbb{Z}^{Q_0}$, the Euler form can be computed just from the dimension vectors.

A formula for the Euler form

- Given a finite acyclic quiver Q with $Q_0 = \{1, \dots, n\}$ vertices and a field K, we have the Euler bilinear form such that for each $M, N \in \text{rep}_K\, Q$,

$$\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N),$$

- The dimension vectors $e_i := \underline{\dim} S_i$ of simples S_i, $i \in Q_0$, form the standard basis for $\mathbb{Z}^n$.
- Then

$$\begin{aligned} \langle e_i, e_j \rangle &= \dim_K \operatorname{Hom}(S_i, S_j) - \dim_K \operatorname{Ext}^1(S_i, S_j) \\ &= \delta_{ij} - |\{\alpha \colon i \to j \text{ in } Q_1\}|. \end{aligned}$$

- Hence, if $x = (x_1, \dots x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$, then

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \colon i \to j} x_i y_j.$$

Forms associated with quivers and graphs

The yoga of forms [Kra, §§3.2 and 4.1]

- Let Q be a finite quiver with $Q_0 = \{1, \dots, n\}$, Γ the underlying graph and $d_{i,j}$ be the number of edges $i — j$.
- Then we have the Euler form

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha\colon i \to j} x_i y_j .$$

- There is an associated quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$,

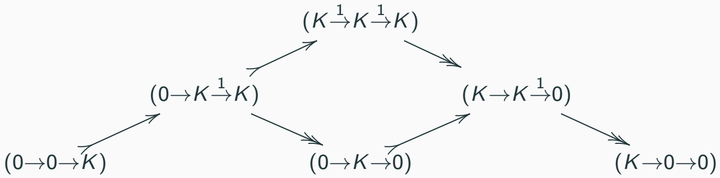
$$q(x) = \langle x, x \rangle = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha\colon i \to j} x_i x_j = \sum_{i \in \Gamma_0} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j .$$

- We cannot reconstruct $\langle -, - \rangle$ from Q, but we can reconstruct the symmetrized Euler form $(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$,

$$(x, y) = q(x + y) - q(x) - q(y) = \langle x, y \rangle + \langle y, x \rangle =$$
$$= \sum_{i \in \Gamma_0} (2 - 2d_{ii}) \cdot x_i y_i - \sum_{i \neq j} d_{ij} x_i y_j .$$

Example—continued

- Let $Q = (1 \to 2 \to 3)$, so that $\Gamma = (1 — 2 — 3)$.
- $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_1 y_2 - x_2 y_3$.
- $$q(x) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 \\ = \frac{1}{2} \cdot \left(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 \right).$$
- This is a positive definite quadratic form, i.e. $q(x) > 0$ $(\forall x \in \mathbb{Z}^3 \setminus \{0\})$, which will turn out to be closely related to the finite representation type of Q!



Some terminology related to forms

Definition
Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a quadratic form. Then

- q is positive definite if $q(x) > 0$ for all $x \in \mathbb{Z}^n \setminus \{0\}$.
- q is positive semi-definite if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$.

The radical of q is the subgroup

$$\operatorname{rad} q = \{x \in \mathbb{Z}^n \mid (x, -) \equiv 0\} \leq \mathbb{Z}^n,$$

where $(x, y) = q(x + y) - q(x) - q(y)$ is the associated bilinear form.

Definition
Given $x, y \in \mathbb{Z}^n$, we write $x \leq y$ if $x_i \leq y_i$ for all i, and $x < y$ if $x \leq y$ and $x \neq y$ (a partial order).

Finally, a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is sincere if $x_i \neq 0$ for all i.