Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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Table of contents

Hereditary algebras

Reminder

- If A is a ring, and $M, N \in \mathit{Mod}\text{-}A$ modules, we have constructed groups $\operatorname{Ext}^i_A(M, N)$, $i \geq 0$, using either a projective resolution of M or an injective coresolution of N.
- The construction is functorial, we have functors

$$\operatorname{Ext}^n_A(-,-) \colon (\mathit{Mod}\text{-}A)^{\mathrm{op}} \times \mathit{Mod}\text{-}A \to \mathrm{Ab},$$

  and $\operatorname{Ext}^0_A(-,-) \cong \operatorname{Hom}_A(-,-)$.
- $\operatorname{Ext}^1_A(M, N) \cong \{0 \to N \to E \to M \to 0\}/\!\sim$ and the zero element is

$$[\, 0 \to N \overset{\oplus}{\rightarrowtail} N \oplus M \overset{\oplus}{\twoheadrightarrow} M \to 0 \,]_{\sim}.$$

Homological dimensions

Definition
The projective dimension of $M \in Mod\text{-}A$ is the smallest integer n such that M has a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0,$$

or the projective dimension is infinite if a finite projective resolution does not exist.

Dually, the injective dimension of N is the smallest n such that

$$0 \to N \to E^0 \to \cdots \to E^n \to 0,$$

or infinite if a finite injective coresolution does not exist.

Ext-groups and homological dimensions

Proposition
The following are equivalent:

1. The projective dimension of M is at most n;
2. $\mathrm{Ext}_A^i(M, -) \equiv 0$ for all $i > n$;
3. $\mathrm{Ext}_A^{n+1}(M, -) \equiv 0$.

Proposition
The following are equivalent for $N \in \mathit{Mod}\text{-}A$:

1. The injective dimension of M is at most n;
2. $\mathrm{Ext}_A^i(-, N) \equiv 0$ for all $i > n$;
3. $\mathrm{Ext}_A^{n+1}(-, N) \equiv 0$;
4. $\mathrm{Ext}_A^{n+1}(A/I, N) = 0$ for each $I_A \leq A$ (Baer lemma);

If A is a finite dimensional algebra, these are equivalent to

5. $\mathrm{Ext}_A^{n+1}(S, N) \equiv 0$ for each S_A simple.

Homological characterization of semisimple rings

Theorem
Let A be a ring. Then the following are equivalent:

1. *The ring is semisimple.*
2. *Each short exact sequence of A-modules* $0 \to N \to E \to M \to 0$ *splits.*
3. $\mathrm{Ext}^1_A(-,-) \equiv 0$.
4. *Each A-module is projective.*
5. *Each A-module is injective.*

The global dimension

Definition
The right global dimension of a ring A is defined as

$$\operatorname{gldim}(A) = \sup\{\text{proj. dim.}\, M \mid M \in \mathit{Mod}\text{-}A\}.$$

Remark

1. $\operatorname{gldim}(A) \leq n$ if and only if $\operatorname{Ext}_A^{n+1}(-,-) \equiv 0$.
2. A is semisimple if and only if $\operatorname{gldim}(A) = 0$.
3. $\operatorname{gldim}(A) = \sup\{\text{proj. dim.}\, A/I \mid I_A \leq A\}$ (Baer lemma).
4. If A is a finite dimensional algebra, then even $\operatorname{gldim}(A) = \sup\{\text{proj. dim.}\, S \mid S_A \text{ simple}\}$.
5. One can define a left global dimension. If A is left and right noetherian, then the dimensions agree (non-trivial!).

Hereditary rings ([ASS, §VII.1])

Definition
A ring A is (right) hereditary if it satisfies the equivalent conditions in the

Theorem ([ASS, Theorem VII.1.4])
The following are equivalent for a ring A:

1. $\operatorname{gldim}(A) \leq 1$,
2. $\operatorname{proj.dim.}(A/I) \leq 1$ *for each* $I_A \leq A$,
3. *each right ideal* $I_A \leq A$ *is projective,*
4. *each submodule of a projective right module is projective,*
5. *each factor of an injective right module is injective.*

If A is a finite-dimensional algebra, these are further equivalent to

6. $\operatorname{proj.dim.}(S) \leq 1$ *for each* S_A *simple.*

Examples of hereditary rings

Example
Most well-known: $\mathbb{Z}$!

Lemma ([ASS, Theorem VII.1.7(a)])
If K is a field and Q is a finite acyclic quiver, then KQ is hereditary.

Proof.

- S_A simple $\implies$ $S \cong (e_i A)/(e_i \operatorname{rad}(A))$ for $i \in Q_0$.
- So we have $0 \to e_i \operatorname{rad}(A) \to e_i A \to S \to 0$.
- Now $e_i \operatorname{rad}(A) = \oplus_{(\alpha\colon i \to j)} \alpha A \cong \oplus_{(\alpha\colon i \to j)} e_j A$ is projective. □

Remark
KQ is hereditary even if Q has oriented cycles (only need: $|Q_0| < \infty$). The proof is much harder, uses non-commutative Gröbner bases.

Characterization of hereditary finite dimensional algebras

Theorem ([ASS, Theorem VII.1.7(b)])
Let A be a hereditary finite dimensional algebra over K such that $A/rad(A) \cong K \times K \times \cdots \times K$ (e.g. if K is alg. closed and A is basic). Then $A \cong KQ_A$, so Q_A is acyclic.

Corollary
Any hereditary finite dimensional algebra over an algebraically closed field is Morita equivalent to KQ, where Q is a finite acyclic quiver.

Proof of the characterization

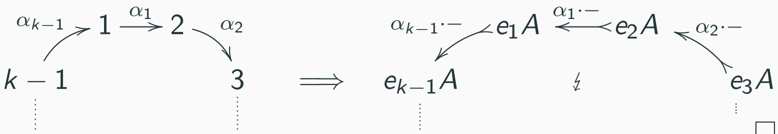
Lemma ([ASS, Corollary VII.1.5(a)])
Let A be hereditary and $f\colon Q \to P$ a map between indecomposable projectives. If f is non-zero, then it is a monomorphism.

Proof.

- $\operatorname{Im} f \leq P$ is projective, so $Q \cong \operatorname{Ker} f \oplus \operatorname{Im} f$.
- If $f \neq 0$, then $\operatorname{Im} f \cong Q$ and, hence, $\operatorname{Ker} f = 0$. □

Corollary
Suppose that A is hereditary and $A/rad(A) \cong K \times K \times \cdots \times K$. Then Q_A is acyclic.

Proof.



□

Proof of the characterization—continued

Theorem ([ASS, Theorem VII.1.7(b)])
Let A be a hereditary finite dimensional algebra over K such that $A/rad(A) \cong K \times K \times \cdots \times K$. Then $A \cong KQ_A$, so Q_A is acyclic.

Proof.

- WLOG $A = KQ_A/I$ with Q_A acyclic and I admissible.
- By [ARS, Lemma III.1.11]: $I \neq 0 \implies A$ not hereditary.
- To this end, we have an exact sequence of A-modules

$$0 \to \frac{I}{R_Q \cdot I} \to \frac{R_Q}{R_Q \cdot I} \xrightarrow{p} \frac{R_Q}{I} \to 0.$$

- As R_Q is projective over KQ, then R_Q/R_QI is proj. over A.
- Further, $0 \neq I/R_QI \leq R_Q^2/R_QI = \mathrm{rad}(R_Q/R_QI)$.
- So p is a projective cover in *mod*-A and $\mathrm{rad}(A)_A = R_Q/I$ is non-projective A-module. □