Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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May 7, 2020

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Preprojectives and preinjectives for Euclidean quivers

Preprojectives and preinjectives—reminder

- If Q is a finite acyclic quiver of Euclidean type, then the defect of $x \in \mathbb{Z}^{|Q_0|}$ is defined as

$$\partial x := \langle \delta, x \rangle = -\langle x, \delta \rangle.$$

Theorem ([Kra, Theorem 5.3.1])
The assignment $M \mapsto \underline{\dim} M$ induces a bijections between

1. the isomorphism classes of indecomposable preprojective representations of Q and the positive roots of Q with negative defect; and
2. the isomorphism classes of indecomposable preinjective representations of Q and the positive roots of Q with positive defect.

These form $2n$ countably infinite series $C^{-r}P(i)$ and $C^{r}I(i)$, $r \geq 0$, $i \in Q_0$ of pairwise non-isomorphic representations.

Proof.

- If $x \in \Delta$ has non-zero defect, then $c^r(x) < 0$ for some $r \in \mathbb{Z}$.
- Then $x = \underline{\dim} M$ for M indecomposable preprojective (if $r > 0$) or preinjective (if $r < 0$).
- Finally, $C^r(P(i))$ is non-injective (so non-zero) for each $r \leq 0$, since $\partial C^r(P(i)) = \partial P(i) < 0$. Hence the preprojectives form n countably infinite series of pairwise non-isomorphic representations.
- Similarly for preinjectives. □

Direction of morphisms for hereditary path algebras

Reflection functors and morphisms

Lemma (a baby version of [Kra, Lemma 7.3.1])
Let Q be any finite acyclic quiver with a sink i, and let $M, N \in \mathrm{rep}_K(Q)$ be indecomposable representations non-isomorphic to $S(i)$. Then S_i^+ induces an isomorphism

$$\mathrm{Hom}(M, N) \xrightarrow{\sim} \mathrm{Hom}(S_i^+ M, S_i^+ N).$$

Proof.

- For each $f\colon M \to N$ and for each $g\colon S_i^+ M \to S_i^- N$, we have commutative diagrams with isomorphisms in rows:

$$\begin{array}{ccc} S_i^- S_i^+ M & \xrightarrow[\sim]{\iota_M} & M \\ {\scriptstyle S_i^- S_i^+ f}\downarrow & & \downarrow{\scriptstyle f} \\ S_i^- S_i^+ N & \xrightarrow[\sim]{\iota_N} & N, \end{array} \qquad \begin{array}{ccc} S_i^+ M & \xrightarrow[\sim]{\pi_{S_i^+ M}} & S_i^+ S_i^- S_i^+ M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle S_i^+ S_i^- g} \\ S_i^+ N & \xrightarrow[\sim]{\pi_{S_i^+ N}} & S_i^+ S_i^- S_i^+ N. \end{array}$$

- Hence, the map from the statement is injective ($S_i^+ f_1 = S_i^+ f_2 \implies S_i^- S_i^+ f_1 = S_i^- S_i^+ f_2 \implies f_1 = f_2$) and surjective (given g, the map $\iota_N \circ S_i^- g \circ \iota_M^{-1}$ is a preimage). $\square$

Lemma ([Kra, Lemma 9.1.1])
Let Q be a finite acyclic quiver and M, N indecomposable representations.

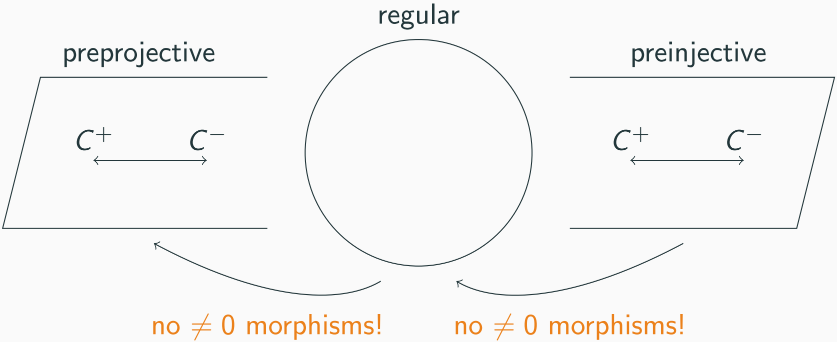
1. If N is preprojective and M is not, then $\mathrm{Hom}(M, N) = 0$.
2. If M is preinjective and N is not, then $\mathrm{Hom}(M, N) = 0$.

Proof.

- We prove part 1., the other is dual.
- Suppose we have $0 \neq f\colon M \to N$.
  Then $S^+_{i-1} \cdots S^+_1 C^r(N) \cong S(i) = P(i) \in \mathrm{rep}_K(\sigma_{i-1} \cdots \sigma_1 Q)$ for some $1 \leq i \leq n$ and $r \geq 0$, and
  $$0 \neq g := S^+_{i-1} \cdots S^+_1 C^r(f)\colon S^+_{i-1} \cdots S^+_1 C^r(M) \to S(i).$$
- Then g is a split epimorphism, and so is $f = C^{-r} S^-_1 \cdots S^-_{i-1}(g)$, $\lightning$!

Direction of morphisms—summary

$\mathrm{rep}_K(Q)$: (Q connected finite acyclic quiver, not Dynkin)



- Remark: The same argument shows that $\mathrm{Hom}(C^{-r}P(i), C^{-s}P(j)) = 0$ if $s < r$ or $s = r$ and there is no path $j \rightsquigarrow i$. A dual observation holds for preinjectives.

Closure properties of regular representations

Definition
A representation $M \in \mathrm{rep}_{\mathrm{K}}(Q)$ is preprojective/preinjective/regular if each indecomposable summand of M is such.

Lemma ([Kra, Lemma 9.1.3])
Let $f\colon M \to N$ be a morphism of regular representations of Q.

1. $\mathrm{Im}(f)$ is regular.
2. If Q is of Euclidean type, then $\mathrm{Ker}(f), \mathrm{Coker}(f)$ are regular.

Proof.

1. If L is an indecomposable summand of $\mathrm{Im}(f)$, there is a non-zero morphism $M \to L$ and $L \to N$.
2. We have $\partial M = \partial\,\mathrm{Ker}(f) + \partial\,\mathrm{Im}(f)$, so $\partial\,\mathrm{Ker}(f) = 0$ by part 1. If K is an indecomposable summand of $\mathrm{Ker}(f)$, then K is not preinjective, so $\partial K \leq 0$. It follows that $\partial K = 0$ and K is regular for each K.

Representations of the Kronecker quiver

The polynomial ring

- Here, let K be algebraically closed.
- We first consider finite dimensional representations of $\bullet \circlearrowleft x$ (Euclidean type $\tilde{A}_0$).
- Well known from linear algebra (Jordan normal form): Each such indecomposable representation is of the form

$$K^n \circlearrowleft J(n,\lambda)$$

where $n \geq 1$, $\lambda \in K$ and

$$J(n,\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

(if we identify KQ with $K[x]$, the representation corresponds to the $K[x]$-module $K[x]/\big((x-\lambda)^n\big)$.)

Jordan representations of the Kronecker quiver [Kra, §9.3]

- The Kronecker quiver: $\bullet \rightrightarrows \bullet$ (Euclidean type $\tilde{A}_1$).
- Let $\lambda_0, \lambda_1 \in K$ and $n \geq 1$. We define representations

$$R_{n,(\lambda_0:1)}\colon\ K^n \overset{J(n,\lambda_0)}{\underset{1_{K^n}}{\rightrightarrows}} K^n, \qquad R_{n,(1:\lambda_1)}\colon\ K^n \overset{1_{K^n}}{\underset{J(n,\lambda_1)}{\rightrightarrows}} K^n.$$

- Note that $R_{n,(1:\lambda_1)} \cong R_{n,(\lambda_1^{-1}:1)}$ if $\lambda_1 \neq 0$. Indeed, $R_{n,(1:\lambda_1)}$ is isomorphic to

$$R_{n,(\lambda_0:1)}\colon\ K^n \overset{J(n,\lambda_1)^{-1}}{\underset{1_{K^n}}{\rightrightarrows}} K^n,$$

  and the matrices $J(n,\lambda_1)^{-1}$ and $J(n,\lambda_1^{-1})$ are similar (both induce indec. $K[x]$-modules and have the same eigenvalue).
- Upshot: If $\lambda = (\lambda_0 : \lambda_1) \in \mathbb{P}^1_K$ and $n \geq 1$, we have an indecomposable regular representation $R_{n,\lambda}$.

Theorem (Kronecker, [Kra, Theorem 9.3.1])
Let K be algebraically closed. The following is a complete list of pairwise non-isomorphic indecomposable representations of $\bullet \rightrightarrows \bullet$:

1. $P_n \colon K^n \underset{\binom{0}{I}}{\overset{\binom{I}{0}}{\rightrightarrows}} K^{n+1}$, $n \geq 0$ (preprojectives),
2. $I_n \colon K^{n+1} \underset{(0\ I)}{\overset{(I\ 0)}{\rightrightarrows}} K^n$, $n \geq 0$ (preinjectives),
3. $R_{n,\lambda}$, $n \geq 1$, $\lambda \in \mathbb{P}^1_K$ (regular).

Regulars at different points of $\mathbb{P}^1_K$ do not see each other

Lemma ([Kra, Lemma 9.3.2(2)])
If $m, n \geq 1$ and $\lambda \neq \mu \in \mathbb{P}^1_K$, then

$$\mathrm{Hom}(R_{n,\lambda}, R_{m,\mu}) = 0 = \mathrm{Ext}^1(R_{n,\lambda}, R_{m,\mu}).$$

Proof.

- Note that for each $0 < \ell < n$, we have a short exact sequence

$$0 \to R_{\ell,\lambda} \to R_{n,\lambda} \to R_{n-\ell,\lambda} \to 0.$$

- Using the long exact sequence of Hom's and Ext's, it suffices to prove the lemma for $n = 1$, and by symmetry also for $m = 1$.
- It is a straightforward computation that

$$\mathrm{Hom}(R_{1,\lambda}, R_{1,\mu}) = 0 = \mathrm{Ext}^1(R_{1,\lambda}, R_{1,\mu}). \qquad \square$$

Each regular representation contains some $R_{1,\lambda}$

Lemma ([Kra, Lemma 9.3.3])
Any indecomposable regular representation M contains a subrepresentation isomorphic to $R_{1,\lambda}$ for some $\lambda \in \mathbb{P}^1_K$.

Proof.

- M is of the form $(V \underset{g}{\overset{f}{\rightrightarrows}} W)$ with $0 = \partial M = \dim V - \dim W$.
- If g is an isomorphism, then $M \cong R_{n,(\lambda_0:1)}$ for some $\lambda_0 \in K$ and we are done.
- Note also that $\mathrm{Ker}(f) \cap \mathrm{Ker}(g) = 0$, or else M would have a simple injective subrepresentation (so a summand) isomorphic to $K \rightrightarrows 0$.
- So if g is not an isomorphism, then $\exists x \in \mathrm{Ker}\, g \setminus \mathrm{Ker}\, f$, and M has a subrepresentation $K \underset{0}{\overset{1}{\rightrightarrows}} K$, which is $R_{n,(1:0)}$. □

Classification of indecomposable regular representations

Proposition ([Kra, Proposition 9.3.4])
The indecomposable regular representations M are up to isomorphism precisely $R_{n,\lambda}$, where $n \geq 1$ and $\lambda \in \mathbb{P}^1_K$.

Proof.

- Again $M \cong (V \overset{f}{\underset{g}{\rightrightarrows}} W)$ with $n := \dim V = \dim W$.
- We prove the proposition by induction on n; $n = 1$ is clear.
- If $n > 1$, we have a monomorphism $\iota \colon R_{1,\lambda} \rightarrowtail M$ (the previous lemma) and $\operatorname{Coker} \iota$ is regular.
- By inductive hypothesis, we have a short exact sequence
$$0 \to R_{1,\lambda} \to M \to \bigoplus_{i=1}^{s} R_{n_i,\lambda_i} \to 0.$$
- We must have $\lambda_i = \lambda$ for all i (or else $\operatorname{Ext}^1(R_{n_i,\lambda_i}, R_\lambda) = 0$ and $M \twoheadrightarrow R_{n_i,\lambda_i}$ would split).
- Thus, f or g is an isomorphism and we use linear algebra. □