Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

Jan Šťovíček

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Department of Algebra, Charles University, Prague

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Definition ([ASS, Definition II.3.1])
Let A be a finite dimensional algebra over K with a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$. Suppose that $A/\operatorname{rad}(A) \cong K \times K \times \cdots \times K$ (n copies of K, this happens e.g. if K is algebraically closed and A is basic). Then the quiver Q_A of A, is defined as follows:

vertices: $(Q_A)_0 = \{1, 2, \ldots, n\}$,

arrows: The arrows $i \to j$ are chosen to form a basis $\{\alpha_1, \ldots, \alpha_{m_{ij}}\}$ of $e_i(\operatorname{rad}(A)/\operatorname{rad}^2(A))e_j \cong e_i \operatorname{rad}(A)e_j/e_i \operatorname{rad}^2(A)e_j$.

The quiver of a finite dimensional algebra, continued

Lemma ([ASS, Lemma II.3.2])
The quiver Q_A does not depend on the choice of the complete set of primitive orthogonal idempotents.

Proof.

- Recall that $e_i A e_j \cong \mathrm{Hom}_A(e_j A, e_i A)$ via $a \mapsto a \cdot -$.
- The latter bijection restricts to bijections

$$e_i \operatorname{rad}(A) e_j \cong \mathrm{Hom}_A(e_j A, e_i \operatorname{rad}(A)),$$
$$e_i \operatorname{rad}^2(A) e_j \cong \mathrm{Hom}_A(e_j A, e_i \operatorname{rad}^2(A)),$$

  hence

$$e_i \operatorname{rad}(A) e_j / e_i \operatorname{rad}^2(A) e_j \cong \mathrm{Hom}_A(e_j A, e_i \operatorname{rad}(A) / e_i \operatorname{rad}^2(A)). \quad \square$$

Lemma ([ASS, Lemma II.3.6], exercise)
Let K be a field, Q a finite quiver, $I \leq KQ$ an admissible ideal and $A = KQ/I$. Then $Q \cong Q_A$ (an isomorphism of quivers).

Comparing A and KQ_A

- Recall: If we have a complete set of primitive orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ in A, then Q_A is as follows:

  vertices: $(Q_A)_0 = \{1, 2, \dots, n\}$,

  arrows: The arrows $i \to j$ are chosen to form a basis $\{\alpha_1, \dots, \alpha_{m_{ij}}\}$ of $e_i \operatorname{rad}(A) e_j / e_i \operatorname{rad}^2(A) e_j$.

- We can lift $\{\alpha_1, \dots, \alpha_{m_{ij}}\}$ to elements $\{a_1, \dots, a_{m_{ij}}\}$ in $e_i \operatorname{rad}(A) e_j$.
- This allows us to define and algebra homomorphism (cf. [ASS, Theorem II.1.8])

$$\begin{aligned} \varphi_A \colon KQ_A &\to A, \\ e_i &\mapsto e_i, \\ (\alpha_\ell \colon i \to j) &\mapsto a_\ell \in e_i \operatorname{rad}(A) e_j. \end{aligned}$$

A as a factor algebra of KQ_A

Lemma (essentially [ASS, Lemma II.3.3])
The algebra homomorphism $\varphi_A\colon KQ_A \to A$ is surjective.

Proof.

- We will prove by induction on $m \geq 2$ that $R_{Q_A} \to \operatorname{rad}(A)/\operatorname{rad}^m(A)$ is surjective.
- For $m = 2$, this follows from the fact that $R_{Q_A}/R_{Q_A}^2 \xrightarrow{\sim} \operatorname{rad}(A)/\operatorname{rad}^2(A)$ by the construction.
- Suppose now $m \geq 2$ and $a \in A$. By inductive hypothesis, $a = \varphi_A(q) + c$ for some $q \in R_{Q_A}$ and $c \in \operatorname{rad}^m(A)$.
- Therefore, $c = \sum_i a_i b_i$ for some $a_i \in \operatorname{rad}(A)$ and $b_i \in \operatorname{rad}^{m-1}(A)$. Again induction, $a_i = \varphi_A(q_i) + c_i$ and $b_i = \varphi_A(r_i) + d_i$, where $q_i, r_i \in R_{Q_A}$ and $c_i, d_i \in \operatorname{rad}^m(A)$.
- $a = \varphi(q) + \sum_i a_i b_i = \varphi(q) + \sum_i \left(\varphi_A(q_i) + c_i\right)\left(\varphi_A(r_i) + d_i\right) = \varphi_A\left(q + \sum_i q_i r_i\right) + \dots$ with omitted terms in $\operatorname{rad}^{m+1}(A)$. $\square$

Theorem (Gabriel, [ASS, Theorem II.3.7])
Let A be a finite dimensional algebra over K such that $A/rad(A) \cong K \times K \times \cdots \times K$. Then $A \cong KQ_A/I$, where I is an admissible ideal.

Proof.

- Consider the surjective homomorphism $\varphi_A \colon KQ_A \to A$ (last slide) and put $I = \operatorname{Ker} \varphi_A$.
- Then $KQ_A/I \cong A$; we must show that I is admissible.
- Since φ_A induces $KQ_A/R^2_{Q_A} \xrightarrow{\sim} A/\operatorname{rad}^2(A)$, we have $I \leq R^2_{Q_A}$.
- On the other hand, $\varphi_A(R^m_{Q_A}) \leq \operatorname{rad}^m(A)$ for each $m \geq 0$, so $R^m_{Q_A} \leq I$ if we choose $m \gg 2$ so that $\operatorname{rad}^m(A) = 0$. □

Corollary
Any finite dimensional algebra over an algebraically closed field is Morita equivalent to KQ/I, where Q is a finite quiver and an admissible ideal $I \leq KQ$.

Crash course in homological algebra

Resolutions

- Let A be a ring and M_A a right module.
- Then we can construct a so-called projective resolution of M, i.e. an exact sequence

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$$

with $P_0, P_1, P_2, \dots$ projective.
- This is in general not unique (unless we take projective covers at all steps), but it is unique (when formulated rigorously) up to adding or splitting off sequences

$$\cdots \to 0 \to P \xrightarrow{1_P} P \to 0 \to \cdots .$$

- Dual considerations apply to an injective coresolution of N_A,

$$0 \to N \xrightarrow{\iota} E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \xrightarrow{d^3} \cdots$$

Ext groups

- If $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence, then only
$$0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3)$$
is exact, the last map need not be surjective.
- There is a natural way to complete the left exact sequence to a long exact sequence
$$\begin{gathered} 0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3) \to \\ \to \mathrm{Ext}_A^1(M, N_1) \to \mathrm{Ext}_A^1(M, N_2) \to \mathrm{Ext}_A^1(M, N_3) \to \\ \to \mathrm{Ext}_A^2(M, N_1) \to \mathrm{Ext}_A^2(M, N_2) \to \mathrm{Ext}_A^2(M, N_3) \to \cdots \end{gathered}$$
- If $\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$ is a projective resolution of M, we get a complex
$$\cdots \xleftarrow{d_3^*} \mathrm{Hom}_A(P_2, N) \xleftarrow{d_2^*} \mathrm{Hom}_A(P_1, N) \xleftarrow{d_1^*} \mathrm{Hom}_A(P_0, N) \xleftarrow{d_0^*} 0$$
and put $\mathrm{Ext}^n(M, N) = H^i\big(\mathrm{Hom}_A(P_\bullet, N)\big) = \mathrm{Ker}\, d_{n+1}^* / \mathrm{Im}\, d_n^*$.

Properties of Ext groups

- If $0 \to N \xrightarrow{\iota} E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \xrightarrow{d^3} \cdots$ is an injective coresolution, we obtain $\mathrm{Ext}^n_A(M,N)$ also as the comohology of the complex

$$0 \xrightarrow{d^0_*} \mathrm{Hom}_A(M,E^0) \xrightarrow{d^1_*} \mathrm{Hom}_A(M,E^1) \xrightarrow{d^2_*} \mathrm{Hom}_A(M,E^2) \xrightarrow{d^3_*} \cdots$$

and for each exact $0 \to M_1 \to M_2 \to M_3 \to 0$ we have exact

$$\begin{gathered} 0 \to \mathrm{Hom}_A(M_3,N) \to \mathrm{Hom}_A(M_2,N) \to \mathrm{Hom}_A(M_1,N) \to \\ \to \mathrm{Ext}^1_A(M_3,N) \to \mathrm{Ext}^1_A(M_2,N) \to \mathrm{Ext}^1_A(M_1,N) \to \\ \to \mathrm{Ext}^2_A(M_3,N) \to \mathrm{Ext}^2_A(M_2,N) \to \mathrm{Ext}^2_A(M_1,N) \to \cdots \end{gathered}$$

- The construction is functorial, we have functors

$$\mathrm{Ext}^n_A(-,-)\colon (\mathit{Mod}\text{-}A)^{\mathrm{op}} \times \mathit{Mod}\text{-}A \to \mathrm{Ab}.$$

- There is a natural equivalence $\mathrm{Ext}^0_A(-,-) \cong \mathrm{Hom}_A(-,-)$.

Ext **groups and extensions**

- Let $\mathcal{E}(M,N) = \{\varepsilon\colon 0 \to N \to E \to M \to 0\}$ be the class of all short exact sequences with N and M.
- Equivalence relation on $\mathcal{E}(M,N)$: $\varepsilon_1 \sim \varepsilon_2$ if

$$\begin{array}{llllllllll}
\varepsilon_1\colon & 0 & \longrightarrow & N & \xrightarrow{\iota_1} & E_1 & \xrightarrow{\pi_1} & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow \exists & & \| & & \\
\varepsilon_2\colon & 0 & \longrightarrow & N & \xrightarrow{\iota_2} & E_2 & \xrightarrow{\pi_2} & M & \longrightarrow & 0.
\end{array}$$

- There are constructive bijections $\operatorname{Ext}^1_A(M,N) \cong \mathcal{E}(M,N)/\sim$.
- The zero element of $\operatorname{Ext}^1_A(M,N)$ corresponds to the equivalence class of

$$0 \to N \overset{\oplus}{\rightarrowtail} N \oplus M \overset{\oplus}{\twoheadrightarrow} M \to 0.$$