Representation theory of finite dimensional algebras (NMAG 442)

Notes for the streamed lecture

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Table of contents

Injective envelopes

The socle of a module is essential

Definition
Let A be a ring. A submodule $L \leq M$ is essential if

$$(\forall N \leq M)(L \cap N = 0 \implies N = 0).$$

Lemma
If A is a finite-dimensional algebra and M_A a module, then $\operatorname{soc}(M)$ *is essential in M.*

Proof.

- Suppose that $0 \neq N \leq M$. Then N has a non-zero finite-dimensional $0 \neq N' \leq N$.
- Since N' is finite-dimensional, it has a simple submodule $S \leq N'(\leq N)$.
- Hence $\operatorname{soc}(N) \cap N \neq 0$. $\square$

Lemma
Let A be a finite-dimension algebra and $e \in A$ a primitive idempotent. Then

$$\iota\colon \operatorname{soc} D(Ae) \rightarrowtail D(Ae)$$

is an injective envelope

Proof.

- Use the equivalence $D\colon (A\text{-}mod)^{\mathrm{op}} \xrightarrow{\sim} mod\text{-}A$.
- ι is up to isomorphism the dual of $\pi\colon Ae \twoheadrightarrow Ae/\operatorname{rad}(Ae)$, which is a projective cover by [ASS, Prop. I.4.5(c)].
- As explained last time, D sends projective covers to injective envelopes (use the Baer lemma). □

Existence of injective envelopes

Theorem
If A is a finite dimensional algebra, then each $M \in \text{Mod-}A$ has an injective envelope.

Proof.

$$\begin{array}{ccc} \operatorname{soc}(M) & \xrightarrow{\sim} & \bigoplus_{i \in I} \operatorname{soc} D(Ae_i) \\ {\scriptstyle \text{ess}} \downarrow & & \downarrow {\scriptstyle \text{ess}} \\ M & \overset{\iota}{\dashrightarrow} & \bigoplus_{i \in I} D(Ae_i). \end{array}$$

- $\bigoplus_{i \in I} \operatorname{soc} D(Ae_i)$ is injective (non-trivial, use the Baer lemma and the fact that A is right noetherian),
- the right vertical arrow is an essential submodule (it is an embedding of the socle),
- ι is an embedding (if $\operatorname{Ker} \iota \neq 0$, then $\operatorname{Ker} \iota|_{\operatorname{soc}(M)} \neq 0$, a contradiction) and easily ι is essential. □

Structure of injective modules

Corollary
Let A be a finite dimensional algebra and $E \in Mod\text{-}A$ be injective. Then

$$E \cong \bigoplus_{i \in I} D(Ae_i).$$

Proof.

- E has an injective envelope of the form

$$\iota \colon E \to \bigoplus_{i \in I} D(Ae_i).$$

- Since E is injective, ι is an isomorphism (uniqueness of injective envelopes). □

Basic algebras and Morita equivalence

Basic algebras

Definition
Let A be a finite dimensional algebra and $\{e_1, e_2, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents (so $A_A \cong \bigoplus_{i=1}^{n} e_i A$). Then A is basic if $e_i A \ncong e_j A$ for $i \neq j$.

Lemma
A is basic if and only if $B := A/\operatorname{rad}(A)$ is basic.

Proof.

- Suppose that $\{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents.
- Then $\pi_i : e_i A \to e_i B$ is a projective cover.
- By the uniqueness of projective covers [ASS, Theorem I.5.8(b)], $e_i B \cong e_j B$ if and only if $e_i A \cong e_j A$. □

Remark
For the Wedderburn-Artin theorem, $B = A/\operatorname{rad}(A)$ is basic if and only if $B \cong D_1 \times D_2 \times \cdots \times D_n$, where all the D_i are division rings.

Division ring extensions of algebraically closed fields

Lemma ([ASS, Corollary I.3.2])
Let K be an algebraically closed field and D a finite-dimensional division ring over K. Then $D = K$.

Proof.

- Let $0 \neq d \in D$. Since $D \cong K^m$ as K-vector spaces and $K = \overline{K}$, the K-linear endomorphism

$$d \cdot - \colon D \to D$$

  has an eigenvector $v \in D$ associated with an eigenvalue $\lambda \in k$. I.e. $d \cdot v = \lambda \cdot v$.
- Since D is a division ring, we have $d = \lambda \in K$. □

Basic algebras over algebraically closed fields

Proposition ([ASS, Proposition I.6.2])
Let A be a finite dimensional algebra over an algebraically closed field. Then A is basic if and only if $A/\operatorname{rad}(A) \cong K \times K \times \cdots \times K$ as K-algebras.

In this case, all simple A-modules are one-dimensional over K.

Proof.
Just combine the Remark and the Lemma from the last 2 slides. □

Corollary
Let Q be a finite quiver and $I \leq KQ$ an admissible ideal. Then $A = KQ/I$ is a basic algebra.

A basic algebra associated to A

Definition
Assume that A is a finite dimensional algebra with a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$. A basic algebra associated to A is defined as

$$A^b = eAe,$$

where $e = e_{i_1} + \cdots + e_{i_k}$, and $e_{i_1}, \ldots, e_{i_k}$ are chosen so that

1. $e_{i_k}A \not\cong e_{i_\ell}A$ whenever $k \neq \ell$ and
2. each e_iA is isomorphic to one of $e_{i_1}A, \ldots, e_{i_k}A$.

Lemma ([ASS, Lemma I.6.5])
Up to isomorphism of K-algebras, A^b does not depend on any choices made above.

Proof.
- We have $eAe \cong \mathrm{End}(eA)$ via $a \mapsto a \cdot -$.
- Then use the Krull-Schmidt theorem. $\square$

Intermezzo—tensor products

- Let A be a ring and M_A and ${}_AN$ modules.
- Then $M \otimes_A N$ is an abelian group generated by symbols $m \otimes n$ (elementary tensors) with $m \in M$ and $n \in N$ subject to relations
  1. $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$,
  2. $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$,
  3. $m \cdot a \otimes n = m \otimes a \cdot n$.
- We in fact have a functor $\otimes_A \colon \mathit{Mod}\text{-}A \times A\text{-}\mathit{Mod} \to \mathrm{Ab}$.
- Computational rules:
  1. $\mathrm{Coker}(M \otimes_A f) \cong M \otimes_A \mathrm{Coker}\, f$ for each $f \colon N_1 \to N_2$ and similarly if we swap the coordinates.
  2. $\left(\bigoplus_{i \in I} M_i\right) \otimes_A \left(\bigoplus_{j \in J} N_j\right) \cong \bigoplus_{i \in I, j \in J} (M_i \otimes_A N_j)$.
  3. $A \otimes_A N \cong N$ via $a \otimes n \mapsto a \cdot n$ and $M \otimes_A A \cong M$ via $m \otimes a \mapsto m \cdot a$.

Intermezzo—tensor products, continued

- If B is another ring and ${}_B M_A$ is a bimodule (i.e. $(b \cdot m) \cdot a = b \cdot (m \cdot a)$), then $M \otimes_A N$ is a left B-module via

$$b \cdot (m \otimes n) := (b \cdot m) \otimes_A n$$

and we obtain a functor $\otimes_A \colon \textit{B-Mod-A} \times \textit{A-Mod} \to \textit{B-Mod}$.

- If K is a commutative ring, then we can view any $M \in \textit{Mod-K}$ as an K-K-bimodule and so we have

$$\otimes_K \colon \textit{Mod-K} \times \textit{Mod-K} \to \textit{Mod-K}.$$

- If K is a field and V, W vector spaces, then $V \otimes_K W$ is a basis-free way to construct a vector space with dimension $\dim_K V \cdot \dim_K W$.

Morita equivalence

Theorem (Morita, see also [ASS, §I.6])
Let A be a ring, P_A a finitely generated projective module which is a generator (i.e. A_A is a summand in some P_A^n), and $B = \mathrm{End}(P_A)$ (so ${}_BP_A$ is a B-A-bimodule).

Then we have inverse equivalences

$$\mathrm{Hom}_A(P,-)\colon \mathit{Mod}\text{-}A \rightleftarrows \mathit{Mod}\text{-}B : - \otimes_B P.$$

Moreover, each equivalence between module categories arises in this way up to natural equivalence.

Example
If $P_A = A_A^n$, then $B \cong M_n(A)$ and $\mathit{Mod}\text{-}A \simeq \mathit{Mod}\text{-}M_n(A)$.

Associated basic algebra is indeed basic

Proposition ([ASS, Corollary I.6.10])
Let A be a finite dimensional algebra and $A^b = eAe$ the associated basic algebra to A. Then A^b is a basic algebra and we have an equivalence

$$F\colon \mathit{Mod}\text{-}A \to \mathit{Mod}\text{-}A^b,$$
$$M \mapsto Me.$$

Proof.

- We just use the Morita equivalence for $P = eA$, so that $\operatorname{End}(P_A) \cong eAe = A^b$.
- Note also that $\operatorname{Hom}_A(eA, M) \cong Me$ via $f \mapsto f(e) = f(e \cdot e) = f(e) \cdot e \in Me$.
- Finally, $F(eA) = A^b$. Since eA decomposes to pairwise non-iso summands in $\mathit{Mod}\text{-}A$, the same property holds for A^b in $\mathit{Mod}\text{-}A^b$, so A^b is basic.