NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 5—April 23, 2020

This exercise session is concerned mainly with integral quadratic forms, which capture some combinatorial and representation theoretic properties of quivers. The main reference is chapter VII in [1] since there are no exercises in [2]. There ia also an easy exercise on admissible orderings of vertices of quivers.

Integral quadratic forms (associated to quivers)

Exercise 1 (Based on 3.3 in VII.3 in [1]). Let q be a quadratic form on $\mathbb{Z}^n$. Express $\frac{\partial q}{\partial x_i}(\mathbf{x})$ using the associated symmetric bilinear form, $\mathbf{x}$ and $\mathbf{e}_i$ for all $i = 1, \dots, n$. Using this and some calculus, show that if q is positive semidefinite, then $q(\mathbf{x}) = 0$ if and only if $\frac{\partial q}{\partial x_i}(\mathbf{x}) = 0$ for all $i = 1, \dots, n$.

Exercise 2 (7 in VII.6 in [1]). Prove that the following quadratic forms are positive definite:

(i) $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - x_1x_3 - x_2x_4 - x_3x_4 + x_1x_4$.

(ii) $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4 - x_3x_4$

(Hint: You may construct the matrix A of the associated symmetric bilinear form form, and find the Cholesky decomposition by simultaneously performing Guassian elimination for rows and columns of A.)

Definition 1 (Weak positivity; 3.1 in VII.3 in [1])**.** *An integral quadratic form q on $\mathbb{Z}^n$ is weakly positive if for all $\mathbf{x} > 0$ we have that $q(\mathbf{x}) > 0$.*

Exercise 3 (8(a) in VII.6 in [1]). Show that the quadratic form $x_1^2 + x_2^2 + x_3^2 - x_1x_2 + x_1x_3 + x_2x_3$ is weakly positive, but not positive definite. (If you feel lost, you may see 3.2 in VII.3 in [1].)

Exercise 4 (10 in VII.6 in [1]). Find all positive $\mathbf{x} \in \mathbb{Z}^n$ such that $q_Q(\mathbf{x}) = 1$ for the following quiver Q with n vertices:

$$\circ \longrightarrow \dots \longrightarrow \circ$$

Exercise 5. Find all positive $\mathbf{x} \in \mathbb{Z}^2$ such that $q_Q(\mathbf{x}) = 1$ for the following quiver Q:

$$\circ \rightrightarrows \circ$$

Admissible orderings

Exercise 6. Label vertices of the following quivers, and find an admissible ordering thereof:

(i)
$$\begin{tikzcd}
& & & & & & \circ \arrow[dl] \\
\circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \\
& & & & & & \circ \arrow[ul]
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] \arrow[r] & \circ
\end{tikzcd}$$

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

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