NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 4—April 9, 2020

This exercise session follows up mainly on the topic of quiver of a (basic) finite-dimensional algebra over an algebraically closed field. That topic is discussed together with the topic of hereditary algebras, which has probably been the main focus of today's streamed lecture. Finally, there are some other nice exercises from chapter II in [1].

Hereafter, k is an algebraically closed field.

Quiver of a (hereditary) finite-dimensional algebra

Definition 1 (Right hereditary algebra; 1.1 in VII.1 in [1])**.** *An algebra A is called right hereditary if any right ideal of A is projective as an $A-$module.*

Theorem 2 (1.7 in VII.1 in [1])**.** *If A is a basic, connected, hereditary finite-dimensional algebra, then the quiver thereof, Q_A, is finite, connected, and acyclic quiver, and $A \cong kQ_A$. Any path algebra kQ where Q is a finite, connected and acyclic quiver is hereditary.*

Remark. Note that this theorem can be readily applied to (connected) path algebras $A = kQ$ as the quiver of such a path algebra is equal to the original quiver, $Q_A = Q$.

Exercise 1 (1 in VII.6 in [1]). Show that the following matrix algebras are hereditary:

(i) $$\begin{bmatrix} k & 0 & 0 & 0 \\ k & k & 0 & 0 \\ k & k & k & 0 \\ k & k & 0 & k \end{bmatrix}$$

(ii) $$\begin{bmatrix} k & 0 & 0 & 0 \\ k & k & 0 & 0 \\ k & 0 & k & 0 \\ k & 0 & 0 & k \end{bmatrix}$$

(iii) $$\begin{bmatrix} k & 0 & 0 & 0 & 0 & 0 \\ k & k & 0 & 0 & 0 & 0 \\ k & 0 & k & 0 & 0 & 0 \\ k & 0 & k & k & 0 & 0 \\ k & 0 & k & 0 & k & 0 \\ k & 0 & k & 0 & k & k \end{bmatrix}$$

(Hint: Express the matrix algebra as an admissible quotient of a path algebra of its quiver; note that the quotient is with respect to the zero ideal, and use the theorem above.)

Exercise 2 (partly 2 in VII.6 in [1]). Construct a hereditary matrix algebra such that its quiver is equal to the following:

(i)
$$\begin{tikzcd}
\circ & & \circ \arrow[dl] \\
& \circ \arrow[ul] \arrow[dl] & \\
\circ & & \circ \arrow[ul]
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
& & & \circ \arrow[dl] \\
\circ & \circ \arrow[l] & \circ \arrow[l] & \\
& & & \circ \arrow[ul]
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
& \circ \arrow[dl] \arrow[dr] & \\
\circ & & \circ \arrow[ll]
\end{tikzcd}$$

Other related exercises

Exercise 3 (18 in II.4 in [1]). Let A be a commutative finite-dimensional algebra. Show that A is a finite product of local algebras.

Exercise 4 (19 in II.4 in [1]). Let A be a basic and connected finite-dimensional algebra. Prove that $(Q_A)^{op} = Q_{A^{op}}$ and that there exists an admissible ideal I^{op} such that $A^{op} = kQ^{op}/I^{op}$.

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

Feel free to reach me at `jakub.kopriva@outlook.com`. Also, I am available for short Skype consultations after previous arrangment via e-mail.