NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 3—March 26, 2020

This exercise session follows up mainly on the self-study from the last week, which was focused on projective, injective, and simple modules over finite-dimensional algebras. At first, there are excercises whose solutions are available in the main reference [1]; they are followed by exercise without solutions provided in the main reference to allow independent appliacation of known principles. Finally, there are some exercises regarding the Morita equivalence and projective covers.

Hereafter, k is an algebraically closed field.

Simple, projective, and injective modules over finite dimensional algebras (with solutions in [1])

Exercise 1 (Kronecker quiver continued; 3(a) in III.2 in [1]). For any $n \in \mathbb{N}$ there is an indecomposable representation of the Kronecker quiver $K(2)$ of the form:

$$k^{n+1} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} k^n$$

Where $\varphi = [I_n, \mathbf{0}]$ and $\psi = [\mathbf{0}, I_n]$ as block matrices. Recall what simple modules over the $kK(2)$ are (you may also do this for indecomposable projectives and injectives), and compute socle, radical, and top of such indecomposable representations.

Exercise 2 (Subspace quiver continued; 5(a) and 7(a) in III.2 in [1]). Given a subspace quiver on three vertices (*i.e.* configurations of two subspaces within a single space) or $\circ \to \circ \leftarrow \circ$, compute all simple, indecomposable projective (with radicals) and injective modules (with factors by socles) over this quiver. (Hint)

Exercise 3 (5(d) and 7(d) in III.2 in [1]). Given a quiver on four vertices:

$$\begin{tikzcd}
& \circ \arrow[dl, "\delta"'] & \\
\circ \arrow[loop left, "\lambda"] & & \circ \arrow[ul, "\gamma"'] \arrow[dl, "\alpha"] \\
& \circ \arrow[ul, "\beta"] &
\end{tikzcd}$$

bound by the following relations $\alpha\beta = \gamma\delta$ and $\lambda^3 = 0$ (reason that the ideal generated by these realitions is admissible), compute all simple, indecomposable projective (with radicals) and injective modules (with factors by socles) over this quiver.

Simple, projective, and injective modules over finite dimensional algebras (without solutions in [1])

Exercise 4 (4.(b) in III.4 in [1]). Given a quiver:

$$\begin{tikzcd} & \circ \arrow[d] & \\ \circ \arrow[r, "\alpha"] & \circ \arrow[r, "\beta"] & \circ \end{tikzcd}$$

bound by $\alpha\beta = 0$ (*i.e.* comnposition of any two arrows is zero), compute all simple, indecomposable projective (with radicals) and injective modules (with factors by socles) over this quiver.

Exercise 5 (4.(c) in III.4 in [1]). Given a quiver $\circ \to \circ \to \circ \to \circ$ bound by the second power of its arrow ideal (*i.e.* composition of any two arrows is zero), compute all simple, indecomposable projective (with radicals) and injective modules (with factors by socles) over this quiver.

Morita equivalence

Exercise 6 (14 in I.6 in [1]). Let A be a basic finite-dimensional algebra over k, and let M be a finite-dimensional module over A. Show that $\ell(M)$, the length of composition series of M, is equal to the dimension of M over k.

Exercise 7 (15 in I.6 in [1]). Let A be a finite-dimensional algebra over k. Show that the following propositions are equivalent:

(i) A is basic.

(ii) Every simple right $A-$module is one-dimensional over k.

(iii) The length of composition series of any finite-dimensional module over A is equal to its dimension over k.

Projective covers

Exercise 8 (7 in I.6 in [1]). Show that $k[t]/(t^3)$ as a module over $k[t]$ (which is a path algebra of a quiver with a single vertex and a loop) has no projective cover.

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

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