NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 2—March 12, 2020

Examples of finite-dimensional algebras (from the last session)

Exercise 1 (Kronecker algebra, [3]). Let $K(2) = 1 \circ \rightrightarrows \circ\, 2$ be a quiver. Its path algebra $kK(2)$ is called Kronecker algebra. There two tasks regarding this algebra:

(i) Find an embedding $kK(2) \to M_3(k)$.

(ii) Describe all indecomposable representations of $kK(2)$ of dimension vector $(1, 1)$ (meaning that the vector spaces corresponding to vertices of $K(2)$ are both of dimension 1), and paramterize them using a well-known object from algebraic geometry. Also, describe homomorphisms between these representations.

Exercise 2 (1.4(c), II.1 in [1]). Show that, given a quiver Q, kQ is finite-dimensional if and only if Q is acyclic and Q_0, Q_1 are finite.

Exercise 3 (1.3(b), II.1 in [1]). Find an isomorphic algebra to kQ where Q has a single vertex with n loops.

Exercise 4 (Subspace quiver). Let Q be a quiver with vertices labelled $0, \dots, n$ and n arrows such that there is one arrow $i \circ \to \circ\, 0$ for each $1 \leq i \leq n$. Then, find:

(i) An embedding $kQ \to M_{n+1}(k)$ (1.3(d), II.1 and 3.5(c), II.3 in [1]);

(ii) Natural direct decomposition for every module over kQ (2.4.2 in [2]).

Exercise 5 (4.9, I.4 in [1]). Let us have an algebra over k:

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \alpha_{21} & \lambda & 0 \\ \alpha_{31} & \alpha_{32} & \lambda \end{pmatrix} ; \lambda, \alpha_{21}, \alpha_{31}, \alpha_{32} \in k \right\}$$

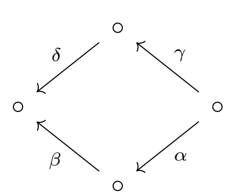
Show that:

(i) B is indeed a well-defined algebra.

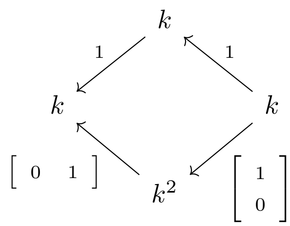
(ii) B is local.

Idempotents and direct decomposition of modules

Exercise 6 (7, III.4 in [1]). Let us have the following quiver:

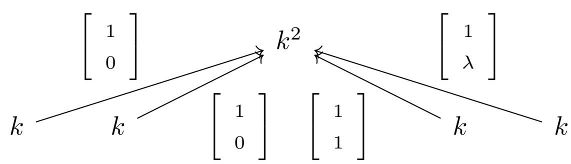


bound by $\alpha\beta = 0$. Show irreducibility of the following representation:



(Hint: Compute the endomorphism ring.)

Exercise 7 (Subspace quiver continued). Express the result of (ii) in Exercise 4 with idempotents. In other words, explicitely describe (not necessarily primitive) idempotents in endomorphism rings of finite-dimensional representations of the subspace quiver. Furthermore, compute endomorphism rings of some non-trivial indecomposable representations of the subspace quiver. For instance:



for suitable values of $\lambda \in k$ (find those as well).

Exercise 8 (Kronecker quiver continued; 15, III.4 in [1]). Show that the endomorphism ring of the following representation of the Kronecker quiver (see Exercise 1):

$$k[t] \overset{1}{\underset{\cdot t}{\rightrightarrows}} k[t]$$

is not local; even though, the representation is indecomposable.

Exercise 9 (Kronecker quiver continued). For any $n \in \mathbb{N}$ find φ, ψ maps such that

$$k^n \overset{\varphi}{\underset{\psi}{\rightrightarrows}} k^{n+1}$$

is indecomposable. Compute the endomorphism ring of such a representation.

Exercise 10 (14, II.4 in [1]). Find a quiver Q and an admissible ideal $I \subseteq kQ$ such that $kQ/I \cong B$, where B is as in Exercise 5. (Hint: Q is a quiver with a single vertex and two loops.)

Exercise 11 (5, II.4 in [1]). Let Q be a finite and acyclic quiver. Prove that kQ is a connected $k-$algebra if and only if kQ/R_Q^2 is a connected $k-$algebra (R_Q is the ideal of kQ generated by all arrows of Q).

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

[3] Kronecker, L. Algebraische reduction der schaaren bilinearer formen. *Sitzungsber. Akad. Berlin.* (1890), 1225–1237.

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