NMAG442 Representation Theory of Finite-Dimensional Algebras

Excercise session 1—February 27, 2020

Wedderburn-Artin theorem (I.3 in [1])

Definition 1 (Semisimple modules and rings)**.** *A module M in $\mathsf{Mod}-R$ is called simple if it has no proper submodules (other than zero submodule and itself). It is called semisimple (or completely reducible) if it is a direct sum of simple $R-$modules. Finally, a ring S is semisimple if it is semisimple as a module over itself.*

Definition 2 (Socle of a module)**.** *Let M in $\mathsf{Mod}-R$ be a module. Then,* $\mathrm{soc}(M)$ *is the submodule of M generated by all simple submodules of M. It is referred to as the socle of M.*

Exercise 1. Prove that, for M and N right modules over R:

(i) M is semisimple if and only if $\mathrm{soc}(M) = M$. (Hint: Use Zorn lemma.)

(ii) Let $f : M \to N$ be an $R-$module homomorphism. Then, $f(\mathrm{soc}(M)) \subseteq \mathrm{soc}(N)$.

(iii) Epimorphic image of a semisimple module is semisimple.

(iv) R is semisimple if and only if all right modules over R are semisimple.

Exercise 2 (Schur lemma). Let $S_1, S_2 \in \mathsf{Mod}-R$, and $f : S_1 \to S_2$ be a non-zero homomorphism between them. Then, prove the following:

(i) If S_1 is simple, f is a monomorphism.

(ii) If S_2 is simple, f is an epimorphims.

(iii) If both are simple, f is an isomorphism.

Exercise 3. Find a simple example of a ring R (preferrably a finite-dimensional algebra over a field k) and an $R-$module M such that M is not simple, yet $\mathrm{End}_R(M)$ is a division ring.

Exercise 4. Let R be a finite-dimensional algebra over an *algebraically closed* field k. Then, for every S, a simple module over R, prove that $\mathrm{End}_R(S) \cong k$.

Exercise 5 (Wedderburn-Artin theorem). Let R be a ring. Then, prove that the following propositions are equivalent:

(i) R is semisimple.

(ii) R is isomorphic to $M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$ for $m_1, \ldots, m_n \in \mathbb{N}$ and division rings $D_1, \ldots, D_n$.

Introductory examples of finite-dimensional algebras

Exercise 6 (Kronecker algebra, [3]). Let $K(2) = 1\bullet \rightrightarrows \bullet 2$ be a quiver. Its path algebra $kK(2)$ is called Kronecker algebra. There two tasks regarding this algebra:

(i) Find an embedding $kK(2) \to M_3(k)$.

(ii) Describe all indecomposable representations of $kK(2)$ of dimension vector $(1,1)$ (meaning that the vector spaces corresponding to vertices of $K(2)$ are both of dimension 1), and paramterize them using a well-known object from algebraic geometry. Also, describe homomorphisms between these representations.

Exercise 7 (1.4(c), II.1 in [1]). Show that, given a quiver Q, kQ is finite-dimensional if and only if Q is acyclic and Q_0, Q_1 are finite.

Exercise 8 (1.3(b), II.1 in [1]). Find an isomorphic algebra to kQ where Q has a single vertex with n loops.

Exercise 9 (Subspace quiver). Let Q be a quiver with vertices labelled $0, \dots, n$ and n arrows such that there is one arrow $i\bullet \to \bullet 0$ for each $1 \leq i \leq n$. Then, find:

(i) An embedding $kQ \to M_{n+1}(k)$ (1.3(d), II.1 and 3.5(c), II.3 in [1]);

(ii) Natural direct decomposition for every module over kQ (2.4.2 in [2]).

Exercise 10 (4.9, I.4 in [1]). Let us have an algebra over k:

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ \alpha_{21} & \lambda & 0 \\ \alpha_{31} & \alpha_{32} & \lambda \end{pmatrix} ; \lambda, \alpha_{21}, \alpha_{31}, \alpha_{32} \in k \right\}$$

Show that:

(i) B is indeed a well-defined algebra.

(ii) B is local.

References

[1] Assem, I., Skowronski, A., and Simson, D. *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, vol. 65. Cambridge University Press, 2006.

[2] Krause, H. Representations of quivers via reflection functors. *arXiv preprint arXiv:0804.1428* (2008).

[3] Kronecker, L. Algebraische reduction der schaaren bilinearer formen. *Sitzungsber. Akad. Berlin.* (1890), 1225–1237.

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