Embedding algebras into semimodules

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Why linear/affine representations?

	a	b	c	d	e	f	g
a	a	e	b	f	c	g	d
b	c	g	d	a	e	b	f
c	e	b	f	c	g	d	a
d	g	d	a	e	b	f	c
e	b	f	c	g	d	a	e
f	d	a	e	b	f	c	g
g	f	c	g	d	a	e	b

… a mess

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f	d	a	e	b	f	c	g
g	f	c	g	d	a	e	b

... a mess

This is just $\underbrace{(\mathbb{Z}_7, *) \text{ with } x * y = 2x + 4y}_{\textit{reduct of an abelian group (linear representation)}}$.

Subreducts of semimodules

- **A** is called a *reduct* of **B**, if all operations of **A** are terms of **B**.
- **A** is called a *subreduct* of **B**, if it is a subalgebra of a reduct of **B**.

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Let $\mathbf{B} = (B, +, \alpha\cdot : \alpha \in R)$ be a semimodule over a semiring $\mathbf{R}$:

Terms of **B** = expressions

$$t(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{R}$.

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- *Semiring* = "ring without subtraction"
- *Semimodule* = "module without subtraction"

Embedding into semimodules

Problem

Determine subreducts of semimodules.

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Every algebra is a subreduct of a semimodule.

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- Commutativity of the semiring yields entropic algebras.
- Idempotent + entropic = modes.

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Anna Romanowska (1990's): Which modes are subreducts of semimodules over commutative semirings?

Szendrei modes

Idempotency:

$$f(x, x, \dots, x) = x$$

Entropy (= all operations commute each other)

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) = g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn}))$$

Szendrei identities (= replace only one pair)

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) = f(f(x_{\pi(11)}, \dots, x_{\pi(1n)}), \dots, f(x_{\pi(n1)}, \dots, x_{\pi(nn)}))$$

where $\pi : ij \leftrightarrow ji$ for a single fixed ij.

Subreducts are Szendrei modes

Consider a subreduct (A, f) of a semimodule over a commutative semiring $\mathbf{R}$. Hence

$$f(a,b,c) = \alpha a + \beta b + \gamma c$$

for some $\alpha, \beta, \gamma \in R$.

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$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) = \alpha^2 x_1 + \alpha\beta x_2 + \alpha\gamma x_3 + \beta\alpha y_1 + \beta^2 y_2 + \dots,$$

$$f(f(x_1, y_1, x_3), f(x_2, y_2, y_3), f(z_1, z_2, z_3)) = \alpha^2 x_1 + \alpha\beta y_1 + \alpha\gamma x_3 + \beta\alpha x_2 + \beta^2 y_2 + \dots,$$

Since $\alpha\beta = \beta\alpha$, Szendrei identity follows.

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More general theorem (Michał Stronkowski)

A Szendrei entropic algebra with onto operations is a subreduct of a semimodule over a commutative semiring.

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An idempotent algebra is a subreduct of a semimodule over a commutative semiring if and only if it is a Szendrei mode.

More general theorem (Michał Stronkowski)

A Szendrei entropic algebra with onto operations is a subreduct of a semimodule over a commutative semiring.

J. Ježek and T. Kepka proved these results for binary algebras in early 1980's.

The embedding I.

Fix a Szendrei mode $\mathbf{A} = (A, f_\sigma : \sigma \in \Sigma)$.

- Let $\Omega = \{\alpha_{\sigma,i} : \sigma \in \Sigma \ n\text{-ary},\ i = 1, \dots, n\}$.
- Let $\mathbf{R}_\Sigma$ denote the polynomial semiring $\mathbb{N}[\Omega]/\theta$, where the congruence θ is generated by all pairs

$$(\alpha_{\sigma,1} + \cdots + \alpha_{\sigma,n}, 1).$$

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- On an $\mathbf{R}_\Sigma$-semimodule $\mathbf{M}$ define operations

$$g_\sigma(a_1, \dots, a_n) = \alpha_{\sigma,1} a_1 + \cdots + \alpha_{\sigma,n} a_n.$$

Then $(M, g_\sigma : \sigma \in \Sigma)$ is a Szendrei mode.

Where do we get a suitable $\mathbf{R}_\Sigma$-semimodule $\mathbf{M}$?

The embedding II.

- Let $\mathbf{F}(A)$ denote the free $\mathbf{R}_\Sigma$-semimodule over A.
- *Á. Szendrei:* $\langle A \rangle_{(F(A),\ g_\sigma : \sigma \in \Sigma)}$ is a free Szendrei mode.

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- *Á. Szendrei:* $\langle A\rangle_{(F(A),\ g_\sigma:\sigma\in\Sigma)}$ is a free Szendrei mode.
- Let ρ be the relation on $F(A)$ consisting of all pairs

  $$(u+\omega b, \quad u+\omega\alpha_{\sigma,1}a_1+\omega\alpha_{\sigma,2}a_2+\ldots+\omega\alpha_{\sigma,n}a_n),$$

  where $b=f_\sigma(a_1,\ldots,a_n)$ (in $\mathbf{A}$)
  and $u\in F(A)$, $\omega\in\Omega^*$ arbitrary.
- Let $\bar\rho$ be the congruence of $\mathbf{F}(A)$ generated by ρ.

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Theorem

$\mathbf{A}$ *embeds into* $(F(A)/\bar\rho,\ g_\sigma:\sigma\in\Sigma)$*, by* $a\mapsto a/\bar\rho$*.*

Non-embeddable modes

- Free modes (a syntactical proof by M. Stronkowski)
- A small example: ternary algebra $(\{0,1,2\}, f)$ with

$$f(x,y,z) = \begin{cases} 2-x & \text{if } y=z=1, \\ x & \text{otherwise.} \end{cases}$$

- Differential modes (many of them)

Affine representation

Linear representation = (semi)module term

$$f(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

Affine representation = (semi)module polynomial

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Theorem (Ježek-Kepka, Stronkowski)
There is a (non-idempotent) binary algebra, which is affine but non-linear over a semimodule over commutative semiring.

Module representations

Quasi-linear = subreduct of a module
Quasi-affine = polynomial subreduct of a module

- They are *abelian*, i.e. for every term t

$$t(x,u_1,\dots,u_k)=t(x,v_1,\dots,v_k)$$
$$\Rightarrow t(y,u_1,\dots,u_k)=t(y,v_1,\dots,v_k).$$

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- Under various additional assumptions, *abelian* algebras are *quasi-affine* (e.g., congruence modularity, C. Herrman; more by Á. Szendrei and K. Kearnes)
- *R. Quackenbush:* Not all *abelian* algebras are *quasi-affine* + an infinite scheme of quasiidentities for quasi-affineness

Quasi-affine = quasi-linear ?

???Theorem??? (S.+Stronkowski)
Quasi-affine algebras are quasi-linear.

1. use Ježek's embedding of **A** into a semimodule **M**
2. take the smallest congruence α of **M** such that the factor is $+$-cancellative (thus $\mathbf{M}/\alpha$ is a subreduct of a module)
3. quasiidentities describing that $\alpha \cap A^2$ is trivial
4. check that quasi-affine algebras satisfy them

Racapitulation

Semimodules:

	affine		linear
general	ALL	$\Leftrightarrow$	ALL
commutative semiring	?	$\not\Leftrightarrow$	?
commutative, idempotent	Szendrei modes	$\Leftrightarrow$	Szendrei modes

Modules:

	affine		linear
general	Q. axioms	$\Leftrightarrow$	Q. axioms
commutative ring	?	(?)	?
commutative, idempotent	(Q. modes)	$\Leftrightarrow$	(Q. modes)

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Problem

Is every abelian mode quasi-affine (i.e. Quackenbush axioms)?

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