Embedding algebras into modules

David Stanovský

Charles University in Prague
Czech Republic

stanovsk@karlin.mff.cuni.cz
http://www.karlin.mff.cuni.cz/~stanovsk

Summer School on Algebra and Ordered Sets
Tále 2007

What is it all about?

	a	b	c	d	e	f	g
a	a	e	b	f	c	g	d
b	c	g	d	a	e	b	f
c	e	b	f	c	g	d	a
d	g	d	a	e	b	f	c
e	b	f	c	g	d	a	e
f	d	a	e	b	f	c	g
g	f	c	g	d	a	e	b

… a mess

What is it all about?

	a	b	c	d	e	f	g
a	a	e	b	f	c	g	d
b	c	g	d	a	e	b	f
c	e	b	f	c	g	d	a
d	g	d	a	e	b	f	c
e	b	f	c	g	d	a	e
f	d	a	e	b	f	c	g
g	f	c	g	d	a	e	b

... a mess

This is just $\underbrace{(\mathbb{Z}_7, *) \text{ with } x * y = 2x + 4y}_{\textit{reduct of an abelian group (linear representation)}}$.

Subreducts of modules

- $\mathbf{A}$ is called a *reduct* of $\mathbf{B}$, if all operations of $\mathbf{A}$ are terms of $\mathbf{B}$.
- $\mathbf{A}$ is called a *subreduct* of $\mathbf{B}$, if it is a subalgebra of a reduct of $\mathbf{B}$.

Subreducts of modules

- **A** is called a *reduct* of **B**, if all operations of **A** are terms of **B**.
- **A** is called a *subreduct* of **B**, if it is a subalgebra of a reduct of **B**.

Let $\mathbf{B} = (B, +, -, 0, \alpha\cdot : \alpha \in R)$ be an **R**-module:

Terms of **B** = expressions

$$t(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n \in R$.

Have we seen this talk a year ago?!

Have we seen this talk a year ago?!

NO!!! It was about *semimodules*.

Theorem (J. Ježek)

Every algebra is a subreduct of a semimodule.

Theorem (M. Stronkowski / DS)

An idempotent algebra is a subreduct of a semimodule over a commutative semiring $\Leftrightarrow$ *it is a Szendrei mode.*

Theorem (M. Stronkowski)

An entropic algebra satisfying Szendrei identities with onto operations is a subreduct of a semimodule over a commutative semiring.

Quasi-linear vs. quasi-affine

Linear representation = module term

$$f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

Affine representation = module polynomial

$$f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + \mathbf{c}$$

Quasi-linear algebras = subreducts of a modules
Quasi-affine algebras = polynomial subreducts of a modules

Quasi-linear = quasi-affine!

Theorem (Stronkowski, DS)

Quasi-affine algebras are quasi-linear.

Proof:

1. use Ježek's embedding of **A** into a semimodule **M**
2. take the smallest congruence α of **M** such that the factor is $+$-cancellative (thus $\mathbf{M}/\alpha$ is a subreduct of a module)
3. derive quasiidentities describing that $\alpha \cap A^2$ is trivial
4. check that quasi-affine algebras satisfy them

Which algebras are quasi-affine?

- They are *abelian*
    - diagonal is a congruence block on $\mathbf{A} \times \mathbf{A}$
    - for every term t and $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A$

$$t(\bar{a}, \bar{c}) = t(\bar{a}, \bar{d}) \quad \Rightarrow \quad t(\bar{b}, \bar{c}) = t(\bar{b}, \bar{d})$$

Which algebras are quasi-affine?

- They are *abelian*
  - diagonal is a congruence block on $\mathbf{A} \times \mathbf{A}$
  - for every term t and $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A$

$$t(\bar{a}, \bar{c}) = t(\bar{a}, \bar{d}) \quad \Rightarrow \quad t(\bar{b}, \bar{c}) = t(\bar{b}, \bar{d})$$

- (Quackenbush) Not all *abelian* algebras are *quasi-affine*: an infinite scheme of quasiidentities

Which algebras are quasi-affine?

- They are *abelian*
  - diagonal is a congruence block on $\mathbf{A} \times \mathbf{A}$
  - for every term t and $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in A$

$$t(\bar{a}, \bar{c}) = t(\bar{a}, \bar{d}) \quad \Rightarrow \quad t(\bar{b}, \bar{c}) = t(\bar{b}, \bar{d})$$

- (Quackenbush) Not all *abelian* algebras are *quasi-affine*: an infinite scheme of quasiidentities

- *Abelian* algebras are *quasi-affine*, if
  - (Kearnes, Szendrei) non-trivial idempotent Mal'tsev condition
    - (Herrman) congruence modular $\Rightarrow$ *affine*
    - (Kearnes, Szendrei) non-trivial congruence lattice identity $\Rightarrow$ *affine*
  - (Kearnes) idempotent simple
  - (TCT) finite simple

Subreducts of modules over commutative rings

Problem

Determine subreducts of modules over commutative rings.

Entropy (= all operations commute each other)

$$f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn})) =$$
$$g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn}))$$

Szendrei identities (= replace just one pair)

$$f(f(x_{11},\dots,x_{1n}),\dots,f(x_{n1},\dots,x_{nn})) =$$
$$f(f(x_{\pi(11)},\dots,x_{\pi(1n)}),\dots,f(x_{\pi(n1)},\dots,x_{\pi(nn)}))$$

where $\pi : ij \leftrightarrow ji$ for a single fixed ij.

Abelian entropic algebras satisfy Szendrei identities

$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) =$$
$$f(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3))$$

$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) =$$
$$f(f(x_1, y_1, x_3), f(x_2, y_2, y_3), f(z_1, z_2, z_3))$$

Abelian entropic algebras satisfy Szendrei identities

$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) =$$
$$f(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3))$$

$\Downarrow$ substitute $x_1 = x_3 = y_2 = \cdots = z_3 = x$

$$f(f(x, x_2, x), f(y_1, x, x), f(x, x, x)) =$$
$$f(f(x, y_1, x), f(x_2, x, x), f(x, x, x))$$

$\Downarrow$ abelianess

$$f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)) =$$
$$f(f(x_1, y_1, x_3), f(x_2, y_2, y_3), f(z_1, z_2, z_3))$$

Subreducts of modules over commutative rings

Reminder: they are *abelian* and *entropic*

Embeddable:

- (Stronkowski) cancellative entropic
- (easy) quasi-affine idempotent entropic

Subreducts of modules over commutative rings

Reminder: they are *abelian* and *entropic*

Embeddable:

- (Stronkowski) cancellative entropic
- (easy) quasi-affine idempotent entropic

Future plan:

for entropic:

abelian $\overset{???}{\Rightarrow}$ quasi-affine $\overset{???}{\Rightarrow}$ embeddable

for idempotent entropic:

abelian $\overset{???}{\Rightarrow}$ quasi-affine $\Rightarrow$ embeddable

References

New results:

D. Stanovský, M. Stronkowski, ***Subreducts of modules***, in progress.

Subreducts of modules:

R. Quackenbush, ***Quasi-affine algebras***, Algebra Universalis 20 (1985), 318–327.

Á. Szendrei, ***Modules in general algebra***, in: Contributions to General Algebra 10 (Proc. Klagenfurt Conf., 1997), Johannes Heyn, Klagenfurt, 1998; pp. 41–53.

All about modes:

A. Romanowska, JDH Smith, ***Modes.*** World Scientific, 2002.

A. Romanowska, ***Semi-affine modes and modals***, Sci. Math. Japon. 61 (2005), 159–194.

Á. Szendrei, ***Identities satisfied by convex linear forms***, Alg. Universalis 12 (1981), 103–122.

Embeddings into semimodules:

J. Ježek, ***Terms and semiterms***, Comment. Math. Univ. Carolinae 20 (1979), 447–460.

D. Stanovský, ***Idempotent subreducts of semimodules over commutative semirings***, submitted.

M. Stronkowski, ***Embeddings of entropic algebras***, PhD Thesis, Warsaw U. of Tech., 2006.