$*$-linear theories of groupoids

Petar Đapić, Jaroslav Ježek, Petar Marković,
Ralph McKenzie and David Stanovský

University of Novi Sad
Serbia and Montenegro

Charles University in Prague
Czech Republic

Vanderbilt University, Nashville
U.S.A.

Motivation

Definition

A term t is called *linear*, if every variable in t occurs (at most) once.

Observation

In vector spaces, in the language $(+, -, 0, (\alpha\cdot))$, *every term is equivalent to a linear term* $\sum_i \alpha_i x_i$. *The term is unique up to commutativity and associativity and adding zeros.*

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Are there equational theories in the language of groupoids $(\cdot)$ such that every term is equivalent to a *unique* linear term?

2-linear theories of groupoids

We know equational theories of groupoids such that every *binary term* is equivalent to a unique linear term. In other words such that the free 2-generated groupoid consists of terms x, y, xy, yx.

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	a	b	c
a	a	c	c
b	c	b	c
c	a	b	c

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- Jerzy Dudek (1998) characterized all varieties of groupoids where the free 2-generated groupoid has at most 4 elements.

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$*$-linear $\Longrightarrow$ $var(t^*) \subseteq var(t)$.

Proof. If $x \in \mathrm{var}(t^*) \setminus \mathrm{var}(t)$, then a non-linear identity

$$t^*(x, z_1, \ldots, z_n) = t(z_1, \ldots, z_n, \ldots) = t^*(y, z_1, \ldots, z_n)$$

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Particularly, *$*$-linear $\Longrightarrow$ idempotent.*

Main theorem

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There are precisely six $*$-linear theories of groupoids.

Proof. Investigate possible free groupoids.

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There are precisely six $*$-linear theories of groupoids.

Proof. Investigate possible free groupoids.

1. Find all possible 2-generated and 3-generated free groupoids by filling the multiplication table.
2. Prove that most of them cannot be extended to more-generated free groupoids.
3. $*$-linear theories are generated by their 4-generated free groupoid.
4. Each of the remaining possible 3-generated groupoids is free for at most one $*$-linear theory.
5. Construct a $*$-linear theory for each of them.

A couple of technical definitions

- *n-linear theory* = every term in at most n variables is equivalent to a unique linear term
- *sharply n-linear theory* = n-linear and based by its at most n-variable equations

Clearly, sharply n-linear theory is determined by its n-generated free groupoid.

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- *n-linear theory* = every term in at most n variables is equivalent to a unique linear term
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Clearly, sharply n-linear theory is determined by its n-generated free groupoid.

- *extension of a groupoid* $\mathbf{G}$ = an equational theory which has $\mathbf{G}$ as a free groupoid

2-generated groupoids

Lemma (J. Dudek)

There are precisely twelve sharply 2-linear equational theories.

Their 2-generated free groupoids are the following groupoids and their duals.

$\mathbf{G}_0$	x	y	xy	yx
x	x	xy	yx	y
y	yx	y	x	xy
xy	y	yx	xy	x
yx	xy	x	y	yx

$\mathbf{G}_1$	x	y	xy	yx
x	x	xy	xy	x
y	yx	y	y	yx
xy	x	xy	xy	x
yx	yx	y	y	yx

$\mathbf{G}_2$	x	y	xy	yx
x	x	xy	xy	yx
y	yx	y	xy	yx
xy	x	y	xy	x
yx	x	y	y	yx

$\mathbf{G}_3$	x	y	xy	yx
x	x	xy	xy	yx
y	yx	y	xy	yx
xy	x	y	xy	yx
yx	x	y	xy	yx

$\mathbf{G}_4$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	x	xy	x
yx	y	yx	y	yx

$\mathbf{G}_5$	x	y	xy	yx
x	x	xy	x	xy
y	yx	y	yx	y
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

$\mathbf{G}_6$	x	y	xy	yx
x	x	xy	xy	xy
y	yx	y	yx	yx
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

3-generated groupoids

Lemma

- $\mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4$ *have no 3-linear extension.*
- $\mathbf{G}_5$ *has nine sharply 3-linear extensions.*
- $\mathbf{G}_6$ *has seven sharply 3-linear extensions.*

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- $\mathbf{G}_6$ *has seven sharply 3-linear extensions.*

Proof for $\mathbf{G}_1$. Let E extend $\mathbf{G}_1$. Since all identities in x, y have the same first and last variables on both sides, so does any identity in E. Hence $x \cdot yz = (xz \cdot z)(y(z \cdot xz)) = xz$, which is a linear identity, a contradiction.

$\mathbf{G}_1$	x	y	xy	yx
x	x	xy	xy	x
y	yx	y	y	yx
xy	x	xy	xy	x
yx	yx	y	y	yx

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Proof. Try all 21 possibilities for the normal form of the term $(x \cdot yz)(wz)$ and obtain by substitution either a contradiction with the table of $\mathbf{G}_5$, or a linear identity in E.

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Lemma

- $\mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4$ *have no 3-linear extension.*
- $\mathbf{G}_5$ *has nine sharply 3-linear extensions.*
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Lemma

- $\mathbf{G}_5$ *has no 4-linear extension.*

The only remaining possible 2-generated free groupoid is

$\mathbf{G}_6$	x	y	xy	yx
x	x	xy	xy	xy
y	yx	y	yx	yx
xy	xy	xy	xy	xy
yx	yx	yx	yx	yx

$\mathbf{Q}_1$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
a	a	d	e	d	e	p	d	e	q	p	p	q	q	p	q	p	q	p	p	q	q
b	g	b	f	g	r	f	g	s	f	r	r	r	s	s	s	r	r	r	s	s	s
c	h	i	c	t	h	i	u	h	i	t	u	t	t	u	u	t	t	u	u	t	u
d	d	d	j	d	j	j	d	j	j	j	j	j	j	j	j	j	j	j	j	j	j
e	e	l	e	l	e	l	l	e	l	l	l	l	l	l	l	l	l	l	l	l	l
f	n	f	f	n	n	f	n	n	f	n	n	n	n	n	n	n	n	n	n	n	n
g	g	g	k	g	k	k	g	k	k	k	k	k	k	k	k	k	k	k	k	k	k
h	h	m	h	m	h	m	m	h	m	m	m	m	m	m	m	m	m	m	m	m	m
i	o	i	i	o	o	i	o	o	i	o	o	o	o	o	o	o	o	o	o	o	o
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r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s
t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t
u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u

$\mathbf{Q}_2$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
a	a	d	e	d	e	p	d	e	q	j	p	l	q	p	q	p	q	p	p	q	q
b	g	b	f	g	r	f	g	s	f	r	k	r	s	n	s	r	r	r	s	s	s
c	h	i	c	t	h	i	u	h	i	t	u	t	m	u	o	t	t	u	u	t	u
d	d	d	j	d	j	j	d	j	j	j	j	j	j	j	j	j	j	j	j	j	j
e	e	l	e	l	e	l	l	e	l	l	l	l	l	l	l	l	l	l	l	l	l
f	n	f	f	n	n	f	n	n	f	n	n	n	n	n	n	n	n	n	n	n	n
g	g	g	k	g	k	k	g	k	k	k	k	k	k	k	k	k	k	k	k	k	k
h	h	m	h	m	h	m	m	h	m	m	m	m	m	m	m	m	m	m	m	m	m
i	o	i	i	o	o	i	o	o	i	o	o	o	o	o	o	o	o	o	o	o	o
j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j
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u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u

$\mathbf{Q}_3$	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
a	a	d	e	d	e	p	d	e	q	j	j	l	l	p	q	p	q	p	p	q	q
b	g	b	f	g	r	f	g	s	f	k	k	r	s	n	n	r	r	r	s	s	s
c	h	i	c	t	h	i	u	h	i	t	u	m	m	o	o	t	t	u	u	t	u
d	d	d	j	d	j	j	d	j	j	j	j	j	j	j	j	j	j	j	j	j	j
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c	h	i	c	t	h	i	u	h	i	t	u	t	m	u	o	t	m	u	o	t	u
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b	g	b	f	g	r	f	g	s	f	r	k	r	s	n	s	r	r	r	s	s	s
c	h	i	c	t	h	i	u	h	i	t	u	t	m	u	o	t	t	u	u	t	u
d	d	d	j	d	j	p	d	j	j	j	p	j	j	p	j	p	j	p	p	j	j
e	e	l	e	l	e	l	l	e	q	l	l	l	q	l	q	l	q	l	l	q	q
f	n	f	f	n	n	f	n	s	f	n	n	n	s	n	s	n	n	n	s	s	s
g	g	g	k	g	r	k	g	k	k	r	k	r	k	k	k	r	r	r	k	k	k
h	h	m	h	t	h	m	m	h	m	t	m	t	m	m	m	t	t	m	m	t	m
i	o	i	i	o	o	i	u	o	i	o	u	o	o	u	o	o	o	u	u	o	u
j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j
k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k
l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l
m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m
n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n
o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p	p
q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q	q
r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r	r
s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s
t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t
u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u	u

Q_7	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
a	a	d	e	d	e	p	d	e	q	j	p	l	q	p	q	p	q	p	p	q	q
b	g	b	f	g	r	f	g	s	f	r	k	r	s	n	s	r	r	r	s	s	s
c	h	i	c	t	h	i	u	h	i	t	u	t	m	u	o	t	t	u	u	t	u
d	d	d	j	d	j	p	d	j	j	j	p	j	j	p	j	p	j	p	p	j	j
e	e	l	e	l	e	l	l	e	q	l	l	l	q	l	q	l	q	l	l	q	q
f	n	f	f	n	n	f	n	s	f	n	n	n	s	n	s	n	n	n	s	s	s
g	g	g	k	g	r	k	g	k	k	r	k	r	k	k	k	r	r	r	k	k	k
h	h	m	h	t	h	m	m	h	m	t	m	t	m	m	m	t	t	m	m	t	m
i	o	i	i	o	o	i	u	o	i	o	u	o	o	u	o	o	o	u	u	o	u
j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j	j
k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k	k
l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l	l
m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m	m
n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n	n
o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
p	p	p	j	p	j	p	p	j	j	j	p	j	j	p	j	p	j	p	p	j	j
q	q	l	q	l	q	l	l	q	q	l	l	l	q	l	q	l	q	l	l	q	q
r	r	r	k	r	r	k	r	k	k	r	k	r	k	k	k	r	r	r	k	k	k
s	n	s	s	n	n	s	n	s	s	n	n	n	s	n	s	n	n	n	s	s	s
t	t	m	t	t	t	m	m	t	m	t	m	t	m	m	m	t	t	m	m	t	m
u	o	u	u	o	o	u	u	o	u	o	u	o	o	u	o	o	o	u	u	o	u

4-generated groupoids

Lemma

1. *They are regular and so is any 4-linear extension.*
2. *They are left or right non-permutational and so is any 4-linear extension.*
3. *Only* $\mathbf{Q}_1$, $\mathbf{Q}_2$ *and* $\mathbf{Q}_4$ *can have a 4-linear extension.*

Regular identity has the same variables on both sides.
Left non-permutational identity has the same order of the first ocurences of variables on both sides.

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UP TO THIS POINT, EVERYTHING COULD HAVE BEEN COMPUTED BY A COMPUTER.

Key steps

Lemma

Every $*$-linear equational theory is generated by its 4-generated free groupoid.

Proof. Take the shortest equation that holds in $\mathbf{F}(4)$ and not in E and derive a shorter one.

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Each of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_4$ has at most one 4-linear extension.

Proof for $\mathbf{Q}_2, \mathbf{Q}_4$. Let E_1, E_2 be two different equational theories extending $\mathbf{Q}_i$. There is a term t in 4 variables such that $t^* = \ell_1$ in E_1 and $t^* = \ell_2$ in E_2. Since E_1, E_2 have the same 3-variable equations, $\ell_1 = \ell_2$ in $\mathbf{Q}_i$ in any 3-variable substitution. Try all possibilities for ℓ_1, ℓ_2, find a contradiction in each case.

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So it only remains to construct the extensions.

$*$-linear theory extending $\mathbf{Q}_1$

- Normal form: *"delete all but the first occurence of every variable"*
- Example:

t	t^*
$x(yx \cdot z)$	$x(yz)$
$x(xy \cdot z)$	$x(yz)$

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- Base: $xx = x, \quad x(yx) = xy, \quad x(xy \cdot z) = x(yz)$
- Generator:

	a	b	c	d	e
a	a	b	d	d	a
b	b	b	c	c	b
c	c	c	c	c	c
d	d	d	d	d	d
e	a	b	c	d	e

$*$-linear theory extending $\mathbf{Q}_2$

- Normal form: *see blackboard*
- Example:

t	t^*
$x(yx \cdot z)$	$x(yz)$
$x(xy \cdot z)$	$(xy)z$

$*$-linear theory extending $\mathbf{Q}_2$

- Normal form: *see blackboard*
- Example:

t	t^*
$x(yx \cdot z)$	$x(yz)$
$x(xy \cdot z)$	$(xy)z$

- Base: $xx = x$, $x(yx) = xy$, $x(yx \cdot z) = x(yz)$, $(xy)(yz \cdot u) = (xy \cdot z)u$
- Generator:

	a	b	c	d	e
a	a	d	c	d	e
b	b	b	e	b	e
c	c	c	c	c	c
d	d	d	c	d	c
e	e	e	e	e	e

$*$-linear theory extending $\mathbf{Q}_4$

- Normal form: *see blackboard*
- Example:

t	t^*
$x(yx \cdot z)$	$(xy)z$
$x(xy \cdot z)$	$x(yz)$

$*$-linear theory extending $\mathbf{Q}_4$

- Normal form: *see blackboard*
- Example:

t	t^*
$x(yx \cdot z)$	$(xy)z$
$x(xy \cdot z)$	$x(yz)$

- Base: **inherently non-finitely based!**
- Generator:

	a	b	c	d
a	a	c	c	d
b	c	b	c	d
c	c	c	c	c
d	d	d	d	d

$*$-linear theories of semigroups

The intersection of the equational theory of semigroups and any of the three $*$-linear theories of groupoids yields the following equational theory of semigroups:

- Normal form: *"delete all but the first occurence of every variable"*
- Base: $xx = x, \quad x(yz) = (xy)z, \quad xyx = xy$
- Generator:

	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

It is easy to prove from the above results that this theory and its dual are the only equational theories of semigroups such that *every word is equivalent to a unique linear word*.

Thank you for your attention.