Commutator Theory for Loops

David Stanovský

Charles University, Prague, Czech Republic
stanovsk@karlin.mff.cuni.cz

joint work with Petr Vojtěchovský, University of Denver

July 2013

"Universal Algebra has had a disastrous impact on Loop Theory"

an unnamed loop theory guru

Feit-Thompson theorem

Theorem (Feit-Thompson, 1962)

Groups of odd order are solvable.

Can be extended?

- To which *class of algebras* ? (containing groups)
- What is *odd order* ?
- What is *solvable* ?

Feit-Thompson theorem

Theorem (Feit-Thompson, 1962)

Groups of odd order are solvable.

Can be extended?

- To which *class of algebras* ? (containing groups)
- What is *odd order* ?
- What is *solvable* ?

Theorem (Glauberman 1964/68)

Moufang loops of odd order are solvable.

Moufang loop = replace associativity by $x(z(yz)) = ((xz)y)z$

solvable = there are $N_i \trianglelefteq Q$ such that $1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and N_{i+1}/N_i are abelian groups.

Solvability, nilpotence – after R. H. Bruck (1950's)

Q is *solvable* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and N_{i+1}/N_i are abelian groups

Q is *nilpotent* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and $N_{i+1}/N_i \leq Z(Q/N_i)$

$Z(Q) = \{a \in Q : ax = xa, a(xy) = (ax)y, (xa)y = x(ay) \ \forall\ x, y \in Q\}$

Solvability, nilpotence – after R. H. Bruck (1950's)

Q is *solvable* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and N_{i+1}/N_i are abelian groups

Q is *nilpotent* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and $N_{i+1}/N_i \leq Z(Q/N_i)$

$Z(Q) = \{a \in Q : ax = xa, a(xy) = (ax)y, (xa)y = x(ay) \ \forall\ x, y \in Q\}$

Alternatively, if we had a commutator, define

$$Q^{(0)} = Q_{(0)} = Q, \qquad Q_{(i+1)} = [Q_{(i)}, Q_{(i)}], \qquad Q^{(i+1)} = [Q^{(i)}, Q]$$

- Q *solvable* iff $Q_{(n)} = 1$ for some n
- Q *nilpotent* iff $Q^{(n)} = 1$ for some n

We can set:

- $[N, N]$ is the smallest M such that N/M is an abelian group
- $[N, Q]$ is the smallest M such that $N/M \leq Z(Q/M)$
- $[N_1, N_2]$ is ???

Solvability, nilpotence – in universal algebra

Q is *solvable* if there are $\alpha_i \in \mathrm{Con}(Q)$ such that
$0_Q = \alpha_0 \le \alpha_1 \le ... \le \alpha_k = 1_Q$ and α_{i+1}/α_i is an abelian congr. in Q/α_i

Q is *nilpotent* if there are $\alpha_i \in \mathrm{Con}(Q)$ such that
$0_Q = \alpha_0 \le \alpha_1 \le ... \le \alpha_k = 1_Q$ and $\alpha_{i+1}/\alpha_i \le \zeta(Q/\alpha_i)$

$\zeta(Q) =$ ..... *abelian* means .....

Alternatively, since we have a commutator, define

$$\alpha^{(0)} = \alpha_{(0)} = 1_Q, \qquad \alpha_{(i+1)} = [\alpha_{(i)}, \alpha_{(i)}], \qquad \alpha^{(i+1)} = [\alpha^{(i)}, 1_Q]$$

- Q *solvable* iff $\alpha_{(n)} = 1$ for some n
- Q *nilpotent* iff $\alpha^{(n)} = 1$ for some n

Solvability, nilpotence – in universal algebra

Q is *solvable* if there are $\alpha_i \in \mathrm{Con}(Q)$ such that
$0_Q = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_Q$ and α_{i+1}/α_i is an abelian congr. in Q/α_i

Q is *nilpotent* if there are $\alpha_i \in \mathrm{Con}(Q)$ such that
$0_Q = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_Q$ and $\alpha_{i+1}/\alpha_i \leq \zeta(Q/\alpha_i)$

$\zeta(Q) =$ ..... *abelian* means .....

Alternatively, since we have a commutator, define

$$\alpha^{(0)} = \alpha_{(0)} = 1_Q, \qquad \alpha_{(i+1)} = [\alpha_{(i)}, \alpha_{(i)}], \qquad \alpha^{(i+1)} = [\alpha^{(i)}, 1_Q]$$

- Q *solvable* iff $\alpha_{(n)} = 1$ for some n
- Q *nilpotent* iff $\alpha^{(n)} = 1$ for some n

$[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$, hence

- $[\alpha, \alpha]$ is the smallest δ such that $C(\alpha, \alpha; \delta)$, or α/δ is ab. cg. in Q/δ
- $[\alpha, 1_Q]$ is the smallest δ such that $C(\alpha, 1_Q; \delta)$, or $\alpha/\delta \leq \zeta(Q/\delta)$

Abelianess, center, commutator

Smith-Gumm / Freese-McKenzie commutator theory (1970's-80's):

Centralizing relation for $\alpha, \beta, \delta \in \mathrm{Con}(A)$:

$C(\alpha, \beta; \delta)$ iff for every term t and every $x \,\alpha\, y$, $u_i \,\beta\, v_i$

$$t(x, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(x, v_1, \dots, v_n) \;\Rightarrow\; t(y, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(y, v_1, \dots, v_n)$$

We say

- A is *abelian* if $C(1_A, 1_A; 0_A)$

Abelianess, center, commutator

Smith-Gumm / Freese-McKenzie commutator theory (1970's-80's):

Centralizing relation for $\alpha, \beta, \delta \in \mathrm{Con}(A)$:

$C(\alpha, \beta; \delta)$ iff for every term t and every $x \,\alpha\, y$, $u_i \,\beta\, v_i$

$$t(x, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(x, v_1, \dots, v_n) \;\Rightarrow\; t(y, u_1, \dots, u_n) \stackrel{\delta}{\equiv} t(y, v_1, \dots, v_n)$$

We say

- A is *abelian* if $C(1_A, 1_A; 0_A)$
- α is *abelian* in A if $C(\alpha, \alpha; 0_A)$
- the *center* of A is the largest ζ such that $C(\zeta, 1_A; 0_A)$

Commutator $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$
(well behaved in congruence modular varieties, e.g., groups, loops, rings)

- α is *abelian* in A iff $[\alpha, \alpha] = 0_A$
- the *center* of A is the largest ζ such that $[\zeta, 1_A] = 0_A$

Translating to loops I

Good news

1. *A loop is abelian if and only if it is an abelian group.*
2. *The congruence center corresponds to the Bruck's center.*

Hence, nilpotent loops are really (centrally) nilpotent loops!

Translating to loops II

Bad news

Abelian congruences $\neq$ normal subloops that are abelian groups

N is an abelian group iff $[1_N, 1_N]_N = 0_N$, i.e., $[N, N]_N = 0$

N is *abelian in Q* iff $[\nu, \nu]_Q = 0_Q$, i.e., $[N, N]_Q = 0$

abelian $\neq$ abelian in Q !!!

Example: $Q = \mathbb{Z}_4 \times \mathbb{Z}_2$, redefine $(a, 1) + (b, 1) = (a * b, 0)$

$*$	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

- $N = \mathbb{Z}_4 \times \{0\} \trianglelefteq Q$
- $[N, N]_N = 0$, hence N is an abelian group
- $[N, N]_Q = N$, hence N is not abelian in Q

Translating to loops III

$\mathrm{TotMlt}(Q) = \langle L_a, R_a, M_a : a \in Q \rangle$
$\mathrm{TotInn}(Q) = \mathrm{TotMlt}(Q)_1$

Main Theorem

$\mathcal{V}$ a variety of loops, Φ a set of words that generates TotInn's in $\mathcal{V}$, then

$$[A, B] = Ng(\ \varphi_{u_1,\dots,u_n}(a)\ /\ \varphi_{v_1,\dots,v_n}(a) :\ \varphi \in \Phi,\ a \in A,\ u_i/v_i \in B\ )$$

for every $Q \in \mathcal{V}$ and $A, B \trianglelefteq Q$.

Examples:

- in loops, $\Phi = \{L_{a,b}, R_{a,b}, M_{a,b}, T_a, U_a\}$
- in groups, $\Phi = \{T_a\}$

Consequences: Two notions of solvability

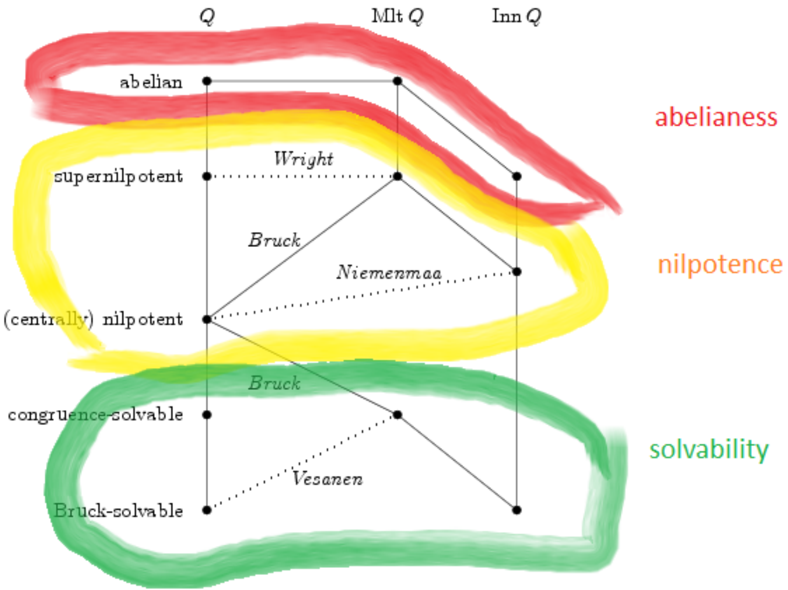
Q is *Bruck-solvable* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and N_{i+1}/N_i are abelian groups
i.e. $[N_{i+1}, N_{i+1}]_{N_{i+1}} \leq N_i$

Q is *congruence-solvable* if there are $N_i \trianglelefteq Q$ such that
$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$ and N_{i+1}/N_i are abelian in Q/N_i
i.e. $[N_{i+1}, N_{i+1}]_Q \leq N_i$

The loop from the last but one slide is

- Bruck-solvable
- NOT congruence-solvable

Solvability and nilpotence



Feit-Thompson revisited

Theorem (Glauberman 1964/68)

Moufang loops of odd order are Bruck-solvable.

Problem

Are Moufang loops of odd order congruence-solvable?

For Moufang loops,

- we know that abelian $\neq$ abelian in Q (in a 16-element loop)
- is it so that Bruck-solvable iff congruence-solvable?