Contents

0. Introduction

1. Preface

Given a group G, the derived operation

$$x \circ y = xy^{-1}x$$

satisfies

$$\begin{array}{ll} (\textit{left symmetry}) & x \circ (x \circ y) = y, \\ (\textit{left distributivity}) & x \circ (y \circ z) = (x \circ y) \circ (x \circ z), \\ (\textit{idempotency}) & x \circ x = x. \end{array}$$

In other words, $G(\circ)$ forms a *left symmetric left distributive idempotent groupoid* (shortly an *LSLDI-groupoid*). The notion of (self)distributive systems was studied for a long time. Also the left distributivity, together with the left symmetry, appeared firstly already in 40's (of 20th century). However, it seemed formerly that non-idempotent selfdistributive structures did not play any significant rôle (whether symmetric or not). This was certainly true for the two-sided case (i.e. both left and, dually, right distributive), but in last years, one-sided distributive non-idempotent systems arised in various situations. On the other hand, almost nothing is known about non-idempotent left symmetric left distributive groupoids. The main purpose of our thesis is to contribute to the investigations of such structures.

The groupoid $G(\circ)$ mentioned above is called the *core* of the group G. The equations were not choosen accidently. We prove that the equational theory of cores is exactly the equational theory of LSLDI-groupoids. Although it may not seem so, cores are really very important examples of LSLDI-groupoids, as we can see bellow.

However, the core is not the only natural LSLDI-operation defined on a group. Given a group G and its involutory automorphism f, we put

$$x \diamond_f y = xf(x^{-1}y),$$

and if $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$ (denoting $\mathbb{Z}(G)$ the center of G), we put also

$$x \star_f y = xf(yx^{-1}).$$

It is easy to check that $G(\diamond_f)$ and $G(\star_f)$ are LSLDI-groupoids too. The first part of chapter II discusses the characterization of the varieties generated by these groupoids. We show that the variety generated by all $G(\diamond_f)$ coincides with the variety of left symmetric left distributive idempotent groupoids (all the same as $\circ$), and that, a bit surprizingly, this is not true for the variety generated by all $G(\star_f)$; the additional property on the automorphism, necessary for the left symmetry, seems to be too strong. E.g., observe that there is no operation $\star_f$ on a free group.

In the second part of this chapter we investigate some interesting examples of non-idempotent left symmetric left distributive systems. We generalize the presented operations to get rid of idempotency and we also find some new ones. We prove that, concerning the equational theory, the non-idempotent generalizations of the original operations behave in a similar way.

Chapter III is devoted to investigation of the structure of subdirectly irreducible non-idempotent LSLD-groupoids. We start with some basic properties, we characterize such groupoids containing left absorbing elements (in other words, we describe all one element extensions), we introduce groupoids of involutions as a special kind of the cores of symmetric groups, and finally, we obtain a description of all subdirectly irreducible non-idempotent LSLD-groupoids. After all, we apply the developed theory to characterize such groupoids with few non-idempotent elements. We significantly extend the results presented in the paper [Ke_2].

Our notation is rather standard and mostly follows [KN]. No preliminary knowledge of the topic is assumed. For various problems concerning left distributive groupoids see, above all, the books [De] and [KN].

2. Historical notes

The first explicit allusion to selfdistributivity is, perhaps, in the work of C. S. Peirce [Pe] from 1880. He discusses various forms of a distributive law and he pleads also that selfdistributivity, *"which has hitherto escaped notice, is not without interest."* Another early work, which mentions an example of a non-associative distributive structure, is [Sc] of E. Schröder (1887). The first known article fully pursued to selfdistributive structures is [BC] of C. Burstin and W. Mayer from 1929. They investigated two-sided distributive idempotent quasigroups. The first articles dealing with non-idempotent distributive groupoids were [Ru] of J. Ruedin (1966) in the two-sided case and [Ke_1] of T. Kepka (1981) in the one-sided case. A comprehensive survey for study of two-sided self-distributive systems is [JKN]. The newest and probably the most detailed book on distributive structures is [De] of P. Dehornoy. It contains also a plenty of references, as well as [KN].

The present work considers left symmetric one-sided distributive groupoids. They were introduced as *kei* in the paper [Ta] by M. Takasaki in 1943. During the time, they were referred also as symmetric sets [No_1], symmetric groupoids [Pi_1], [Pi_2] or involutory quandles [Jo_2]. In geometry, the notion of symmetric space was used in [Lo_1], [Lo_2]. On the other hand, concerning the non-idempotent LSLD-groupoids, probably the first and the only one paper about them is [Ke_2] of T. Kepka (1994).

Let us also note that the core of a group firstly appears in [Br] (more generally as the core of a Moufang loop). In russian it is called *сердцевина*; we refer e.g. to [Be] for results of the russian school concerning (mostly) left distributive quasigroups. We would like to advertise also the czech translation *duše* due to J. Nešetřil. The operations $\diamond_f$, $\star_f$ and some of their non-idempotent generalizations can be found in [Ke_2] and [KN] and they also should not be without interest.

3. Related problems

The aim of this section is to present a more or less complete brief overview of the known results concerning LSLDI-groupoids including also few interesting examples and applications. The known results are described in a plenty of short notes and they are quite fragmentary. Only few papers are really comprehensive and in such situation we recommend to read the original text (above all, [Pi_1], [Pi_2], [Jo_1]).

We often use contemporary terminology instead of the original one. For precise definitions see chapter I.

Foundation. As we have already said, the first investigations of LSLDI-groupoids are due to Mituhisa Takasaki in [Ta]. He presents no deep results, rather definitions and examples. E.g. he mentions the groupoid of all integers (or of all points in the plane) with an operation $a * b$ equal to the point symmetric to b through a. In fact, that are the cores of the groups $\mathbb{Z}(+)$ and $\mathbb{R}^2(+)$ and Takasaki already points that every abelian group forms an LSLDI-groupoid with the operation $a * b = 2a - b$. He studies also the equation $xy \cdot uv = xu \cdot yv$, now usually called *mediality*, and two-sided distributive structures.

Geometrical motivation. At the end of 60's, Ottmar Loos showed in [Lo_1] that the well-known notion of a *symmetric space* is equivalent to the structure consisting of a differentiable manifold equipped with a differentiable LSLDI-operation such that each point is an isolated fix point of its left translation. These investigations were continued in the book [Lo_2] and later in 70's by M. Kikkawa in [Ki_1], [Ki_2] and [Ki_3].

Nobuo Nobusawa: first steps. Involved by O. Loos, N. Nobusawa began to study LSLDI-groupoids from a purely algebraical point of view. In his first paper [No_1] he defined a *symmetric set* to be a mapping L from a finite set A to the symmetric group over A, sending an element a onto an involution L_a such that $L_a(a) = a$ and $L_{L_a(b)} = L_a L_b L_a$ for all $a, b \in A$. Clearly, putting $a * b = L_a(b)$, we obtain A as an LSLDI-groupoid, where L_a is the left translation of a. For a symmetric set A Nobusawa defined the *group of displacements* to be the subgroup of the symmetric group over A generated by all $L_a L_b$, $a, b \in A$. This group plays key rôle in further research of the structure of LSLDI-groupoids. As examples of symmetric sets he presented cores of groups. In the paper he investigated cycles (an analogy of powers in groups), *abelian* symmetric sets (such with abelian group of displacements), *homogeneous* symmetric sets (nowadays LSLDI-quasigroups), he described all LSLDI-groupoids with at most 5 elements and he also showed an example of a 27-element LSLDI-quasigroup.

Next article [KNN] was devoted to homogenity, i.e. to finite LSLDI-quasigroups. The main theorem of this paper says that given a finite group G generated by a G-invariant set A of its involutions, then A with the core operation is a quasigroup, if and only if its group of displacements is of even order. In particular, the assumption is satisfied, if A is the core of a group. The article contains many further results, let us state, above all, the analogy of the Lagrange theorem: if B is a subgroupoid of a finite LSLDI-quasigroup A, then $|A|$ is divisible by $|B|$.

Simple LSLDI-groupoids. Afterwards, N. Nobusawa turned to study simple symmetric sets. In the short note [No_2] he proves that the group of displacements of a simple LSLDI-groupoid A (not necessarily finite) is a minimal normal subgroup of the symmetric group over A. Hence, it is either simple or a direct product of two simple groups, which are conjugated. Moreover, if the second condition is true, then the group of displacements has precisely $|A|^2$ elements (and A is primitive, as shown in [Na]). Afterwards, the article continues by applying the result on the structure of symmetric groups.

Nobusawa's second paper concerning simplicity ([IN] with Y. Ikeda) discussed subgroupoids of the core of $SL_n(F)$, F a field. E.g. they stated that the group of displacements of the core of $SL_n(F)$ is isomorphic to $SL_n(F)/\{\pm 1\}$ for $n \geq 3$ or for $n = 2$ and $F \neq \mathbb{Z}_3$. The same holds for $PSL_n(F)$ too. Further, the core of $SL_n(\mathbb{Z}_q)$, q being a power of a prime p, is left ideal-free, iff $p \neq 2$ or n is odd; otherwise it consists of precisely two disjoint left ideals. Many examples are shown.

The topic was continued by few remarks of H. Nagao in [Na]. He discussed so called *transitive* (nowadays left ideal-free) and *primitive* (left ideal-free and there is no subset $I \subset A$ such that $AI \cap I = \emptyset$) symmetric sets. Every simple LSLDI-groupoid turns to be transitive and every primitive LSLDI-groupoid is simple. Further, given a group G generated by a set A of involutions in G such that A is a conjugation class in G and the subgroup generated by AA is a minimal normal subgroup of G, then A with the core operation is a simple LSLDI-groupoid.

The characterization of simple LSLDI-groupoids was solved by D. Joyce in [Jo_2]. It appeared as a particular case of characterization of simple quandles. He found that an LSLDI-groupoid is simple, if and only if it is either the core of a simple group, or it is a conjugation class of involutions in a simple non-abelian group (with the core operation).

Last article on simple LSLDI-groupoids is [No_5] of N. Nobusawa, again. He applies the developed theory to find a new proof (via symmetric sets) of the well-known theorem on simplicity of the orthogonal group.

A geometrical example. The paper [No_3] is devoted to the following LSLDI-groupoid. Let V be a vector space over a field F, $\mathrm{char}(F) \neq 2$, and let g be a non-degenerate symmetric bilinear form on V. Let $A = \{a \in V : g(a, a) = 1\}$. For $a \in A$ we put L_a to be the reflection in the hyperplane orthogonal to a. Now, let B be the set of all 1-dimensional subspaces of V generated by an element of A. We equip the set B with an operation $\langle a \rangle * \langle b \rangle = L_a(\langle b \rangle)$. Then $B(*)$ is an LSLDI-groupoid and the article presents some facts about its properties. E.g. it is left ideal-free, if there are a, b linearly independent such that $g(a, a) = g(b, b) \neq 0$. It is primitive, if $\dim V > 4$ or if $\dim V = 4$ and $F \neq \mathbb{Z}_3$.

Nobuo Nobusawa: last steps. After all, N. Nobusawa generalized LSLDI-groupoids by weakening the left symmetry. He defined the notion of a *pseudo-symmetric set*, in contemporary words left distributive idempotent (LDI) left quasigroup. In [No_4] he makes first observations and he deals, again, with simple LDI left quasigroups. Let A be one. If G is the subgroup of the symmetric group over A generated by all left translations in A, then G' (the commutator subgroup) is a minimal normal subgroup of G (cf. with simplicity of LSLDI-groupoids in [Na]). Moreover, the converse is true, providing that A is left ideal-free and p_A is trivial (for the definition see chapter I). There are also many examples (namely, special cases of group conjugation) and applications to simplicity of groups (via simplicity of their conjugation LDI left quasigroup).

Further article [No_6] contains some structure theory of pseudo-symmetric sets. Again, the group of displacements plays key rôle. There is shown a close connection between nilpotency and solvability of an LDI left quasigroup and its group of displacements. The paper [No_7] deals with LDI left quasigroups which are not a subgroupoid of a conjugation LDI left quasigroup. Finally, [No_8] presents an analogy of the group-theoretical Jordan-Hölder theorem for LDI left quasigroups.

Afterwards, N. Nobusawa abandoned the field of symmetric sets and turned to the the ring theory.

Structure theory — other authors. A big amount of structure results was introduced by R. S. Pierce in [Pi_1] and [Pi_2]. Both broad articles are full of interesting statements and thus we refer a kind reader mainly to the original text. The fundamental idea is a correspondence between LSLDI-groupoids and groups generated by involutions. The fundamental notion here is the subgroupoid $I(G) = \{x \in G : x^2 = 1\}$ of the core of a group G. Among others, Pierce investigates properties of $I(G)$ (categorial connections), he describes free LSLDI-groupoids (see also chapter II), he defines a graph $(A, \{(a,b) : ab \neq b\})$ of an LSLDI-groupoid A (it is useful mainly when dealing with *balanced* LSLDI-groupoids (satisfying $xy = y$ iff $yx = x$); note that cores are balanced), he presents results concerning a decomposition to disjoint left ideals, he investigates *cycles* (homomorphisms from $\mathbb{Z}(\circ)$), etc.

Finally, let us mention also the work [Do] of S. Doro using LSLDI-groupoids when investigating Moufang loops, and also the paper [Um] of N. Umaya. No younger paper fully devoted to the structure of LSLDI-groupoids is known to us.

Quandles. We would like to introduce the concept of a *quandle* found by David Joyce in [Jo_1]. It is an algebra with two binary operations $\triangleright, \triangleleft$ satisfying the equations

$$x \triangleright x = x, \quad (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z), \quad (x \triangleright y) \triangleleft y = (x \triangleleft y) \triangleright y.$$

Since quandles are quite deep-rooted, we conserve the right distributivity instead of the left one. As a fundamental example serves the group conjugation. Given a group G, we define the quandle $\mathrm{Conj}(G)$ by

$$x \triangleright y = y^{-1}xy, \quad x \triangleleft y = yxy^{-1} \quad \text{for all } x, y \in G.$$

In [Jo_1] D. Joyce proves that the equational theory of the group conjugation is precisely that of quandles (it means that the variety generated by all $\mathrm{Conj}(G)$, G a group, coincides with the variety of quandles). The free quandle over X appears to be the subquandle of $\mathrm{Conj}(G)$ generated by X, G being the free group over X.

Clearly, given a quandle Q, both $Q(\triangleright)$ and $Q(\triangleleft)$ are right distributive idempotent (RDI) groupoids. It is quite interesting that the question about equational theory of the group conjugation has an opposite solution in the case of one operation only. A. Drápal, T. Kepka and M. Musílek proved in [DKM] that the group conjugation (i.e. the variety generated by all $\mathrm{Conj}(G)(\triangleright)$, G a group) creates a proper subvariety of the variety of RDI-groupoids. It coincides with the variety generated by cancellative RDI-groupoids, but nothing more is known. Even this is a big open problem, if the equational theory is finitely based.

A symmetric structure on quandles we obtain by taking the equation $(x \triangleright y) \triangleright y = x$; such quandles are called *involutory*. It is easy to check that in every involutory quandle holds $x \triangleright y = x \triangleleft y$. Hence, the variety of involutory quandles is equivalent to the variety of LSLDI-groupoids (forget one operation and take the opposite groupoid). The rôle of involutory quandles in the quandle theory is not of the biggest importance, but it is not neglectible.

Structure of quandles. The comprehensive paper [Jo_1] starts with some structure theory of quandles and then continues by classifying knots via the (involutory) knot quandle. We briefly survey topics of the first part (for details see the original text).

As mentioned above, the variety generated by the group conjugation coincides with the variety of quandles. Also *abelian* quandles (satisfying mediality) are studied. The free abelian quandle over X appears to be the subquandle of $\mathrm{Conj}(G)$ generated by X, G

being the group presented by $\langle X;\ ab^{-1}c = cb^{-1}a,\ a,b,c \in X\rangle$. The free $(n+1)$-generated abelian involutory quandle is a subquandle of the core of $\mathbb{Z}^n(+)$ generated by $(0,\dots,0)$, $(1,0,\dots,0),\dots,(0,\dots,0,1)$. The 2-generated free involutory quandle is isomorphic to the core of $\mathbb{Z}(+)$. Further, action of a group is added to a quandle; this structure is call *augmented* quandle. This notion is widely used in the second part. Quandles are representable by conjugation classes and by coset classes. Involutory quandles are representable by sets covered by well-behaving families of mappings from the core of $\mathbb{Z}(+)$ (so called *involutory quandles with geodesics*).

In the paper [Jo$_2$] we can find charaterization of simple quandles and, in particular, of simple involutory quandles, i.e. of LSLDI-groupoids (cf. the paragraph devoted to simplicity).

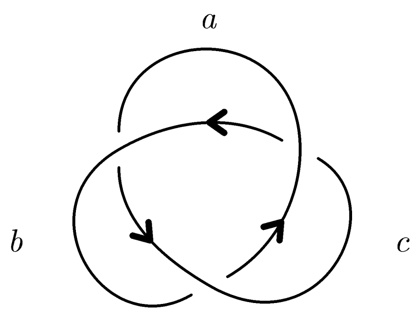
The knot quandle. This paragraph gathers from the article [Jo$_1$]. A *knot* is a subspace of a (3-dimensional) sphere S^3 homeomorphic to a circle. Two knots K_1 and K_2 are *equivalent*, if there is an (auto)homeomorphism of the sphere such that its restriction to K_1 is a homeomorphism of K_1 and K_2. One of the most important questions in the knot theory is to classify knots, in particular, to find invariants with respect to the equivalence of knots.

Usually only *tame* knots are studied (tame means equivalent to a close polygonal curve). For them, the classical result is the fundamental group of a knot K, defined as the fundamental group of the topological space $S^3 \setminus K$. It is a classifying invariant, i.e. two knots are equivalent, iff their fundamental groups are isomorphic.

D. Joyce found another such classifying invariant — the knot quandle. It is defined in a similar way like the fundamental group. Since the definition is useless for a reader not skilled in algebraical topology, we present another description, which is equivalent to the original one.

Given a *regular projection* of a tame knot, i.e. a mapping to a (2-dimensional) plane such that there are only finitely many crossings and none of them is a three-fold crossing. This projection divides the knot to arcs $a_1,\dots,a_n$ (by an *arc* we mean a segment from one underpass through some overpasses to the next underpass). Denote A the set of arcs. Now, we choose an orientation of a knot and we define the *knot quandle* to be a quandle generated by A and presented by equations (1) $a \triangleleft b = c$ for every underpass, where "a comming under b gives c" and where b is going over a from the right to the left (looking in the direction of a given by the orientation of the knot), and by equations (2) $a \triangleright b = c$ in the same situation, but b going over a from the left to the right. The *involutory knot quandle* is defined in the same way; but, since $\triangleright$ and $\triangleleft$ coincide, the orientation is useless.

(i)



trefoil

The trefoil quandle has a presentation

$$\langle a,b,c;\ a \triangleright b = c,\ b \triangleright c = a,\ c \triangleright a = b\rangle .$$

Hence, one can easily compute that the involutory trefoil quandle is the core of $\mathbb{Z}_3$.

(ii)

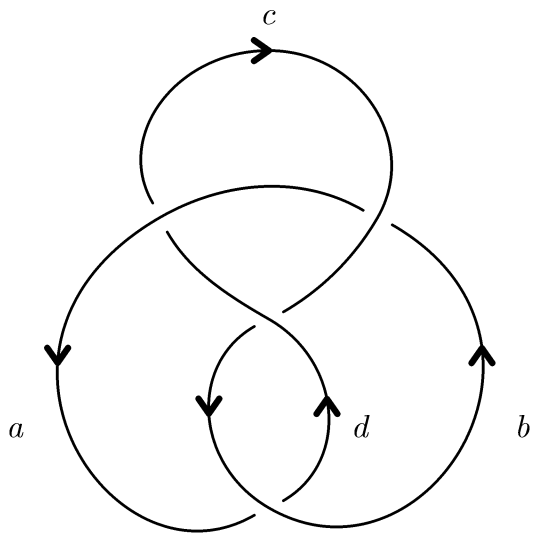


figure-eight

The the figure-eight quandle has a presentation

$$\langle a, b, c, d;\ a \rhd b = d,\ b \lhd c = a,\ c \rhd d = b,\ d \lhd a = c \rangle\,.$$

Hence, one can easily compute that the involutory figure-eight quandle is the core of $\mathbb{Z}_5$ (it is generated by arcs, not the set of arcs!).

It appears that two knots are equivalent, iff their knot quandles are isomorphic. Thus, the knot quandle remains invariant with respect to a regular projection. Unfortunately, for involutory knot quandles only the right-left implication is true; but the counterexample to the converse is quite complicated. The main advantage of involutory knot quandles is their easy computability.

Proof of the statements are quite deep and significantly use known results on fundamental groups.

Further knot applications. Let us mention two supplementary references concerning knots. The paper [AF] of H. Azcan and R. Fenn deals with involutory quandles with geodesics (see [Jo_1], chapter 11) and their application in the knot theory, in particular, in spherical representation of a knot. The paper [Ka] of L. H. Kauffman is devoted to foundations of the virtual knot theory. Among others, he defines the (involutory) virtual knot quandle (in a similar way like D. Joyce in [Jo_1] for standard knots).

The non-idempotent case. As has been already announced, perhaps the only one paper concerning non-idempotent left symmetric left distributive groupoids is [Ke_2] of Tomáš Kepka. In his short note he states some fundamental structural properties. Varieties of LSLD-groupoids and subdirectly irreducible such groupoids are investigated. See also the book [KN] for wider offer of related problems.

I. Basic facts

The purpose of this section is to give a basic insight into the structure of LSLD-groupoids. We present here facts used allover the work. Some of them can be found in [Ke_2].

By a *groupoid* we mean a non-empty set equipped with a binary operation, usually denoted as multiplication. A *left symmetric left distributive* groupoid (shortly an *LSLD-groupoid*) is a groupoid satisfying the equations

(*left symmetry*) $$x \cdot xy = y$$

(*left distributivity*) $$x \cdot yz = xy \cdot xz$$

If the groupoid satisfies also

(*idempotency*) $$xx = x$$

it is called *idempotent* (shortly an *LSLDI-groupoid*). Given a groupoid G and $x \in G$, we denote L_x a transformation of G defined by $L_x(y) = xy$ for all $y \in G$; it is called the *left translation* of x in G. By an *involution* we mean a permutation f such that $f = f^{-1}$.

I.1.1 Lemma.

(1) *A groupoid G is left symmetric, iff every its left translation is an involution.*
(2) *A groupoid G is left distributive, iff every its left translation is an endomorphism of G.*
(3) *A groupoid G is LSLD, iff every its left translation is an involutory automorphism of G.*
(4) *A left symmetric groupoid G is left distributive, iff $L_{xy} = L_x L_y L_x$ for all $x, y \in G$.*

Proof.

(1) The left symmetry says that $L_x L_x(y) = y$ for all $x, y \in G$, i.e. $L_x L_x = id_G$ for all $x \in G$.
(2) The left distributivity says that $L_x(yz) = L_x(y) L_x(z)$ for all $x, y, z \in G$.
(3) An obvious consequence of (1) and (2).
(4) The left distributivity says that $L_x L_y(z) = L_{xy} L_x(z)$ for all $x, y, z \in G$, i.e. $L_x L_y = L_{xy} L_x$ for all $x, y \in G$. Thus, $L_{xy} = L_x L_y L_x^{-1} = L_x L_y L_x$, since $L_x^{-1} = L_x$ by (1). □

I.1.2 Example. The following are the only (up to an isomorphism) two-element LSLD-groupoids.

S	a	b
a	a	b
b	a	b

T	0	$\tilde{0}$
0	$\tilde{0}$	0
$\tilde{0}$	$\tilde{0}$	0

Since left translations are, by I.1.1 (1), involutions, they might be either both identities, like in **S**, or both transpositions, like in **T**, or one is the identity (e.g. L_a) and the other one (i.e. L_b) is the transposition. It can be checked easily that **S** and **T** are LSLD-groupoids, but, in the third case, we have $b = b \cdot (b \cdot b) = (b \cdot b) \cdot (b \cdot b) = a \cdot a = a$ and hence the distributive law cannot be satisfied.

In all tables of the present work, we usually express idempotent elements by latin alphabet and the non-idempotent ones by natural numbers. Also, we denote xx by $\widetilde{x}$. It has a good reason, as we will see later.

A set $S \subseteq G$ is called *definable*, if there exists a formula φ (in the language consisting of one binary operational symbol) with one free variable such that $S = \{x \in G : \varphi(x)\}$. A set $I \subseteq G$ is said to be a *left ideal* in G, if $GI \subseteq I$. If $\{a\}$ is a left ideal in G, then a is called *left absorbing*.

I.1.3 Lemma. *Let t be a (binary) term with $n+1$ variables. Then every left distributive groupoid satisfies the equation $t(yx_1, \dots, yx_n) = yt(x_1, \dots, x_n)$.*

Proof. By induction on the length of t. If t is a single variable, the statement is trivial. Otherwise $t = t_1t_2$ and $t_i(yx_1, \dots, yx_n) = yt_i(x_1, \dots, x_n)$ holds, $i = 1, 2$. Then

$$t(yx_1, \dots, yx_n) = t_1(yx_1, \dots, yx_n)t_2(yx_1, \dots, yx_n) =$$
$$yt_1(x_1, \dots, x_n) \cdot yt_2(x_1, \dots, x_n) = y \cdot t_1(x_1, \dots, x_n)t_2(x_1, \dots, x_n) = yt(x_1, \dots, x_n).$$

□

I.1.4 Lemma on definable sets. *Every definable subset of an LSLD-groupoid is either empty, or a left ideal.*

Proof. Let G be an LSLD-groupoid and $S = \{x \in G : \varphi(x)\}$ for a formula $\varphi(x)$. Given arbitrary $x \in S$, $y \in G$, we need to prove that $yx \in S$, i.e. that $\varphi(yx)$ holds.

Let φ be of the form $(Q_1u_1)(Q_2u_2)\dots(Q_nu_n)\psi(u, u_1, \dots, u_n)$, where $Q_1, \dots, Q_n$ are quantifiers and ψ is a formula with $n+1$ free variables without quantifiers. Since left translations are permutations, it is enough to prove the following statement: for all $x, x_1, \dots, x_n, y \in G$

(†) if $\psi(x, x_1, \dots, x_n)$ holds, then $\psi(yx, yx_1, \dots, yx_n)$ holds too.

Let ψ be an atomic formula of the form $(t = s)$, t, s being terms, and $\psi(x, x_1, \dots, x_n)$ is true. Then, using I.1.3, $t(yx, yx_1, \dots, yx_n) = yt(x, x_1, \dots, x_n) = ys(x, x_1, \dots, x_n) = s(yx, yx_1, \dots, yx_n)$ and thus (†) holds.

Let ψ be of the form ψ_1 & ψ_2 and for ψ_1, ψ_2 (†) already holds. Then for ψ obviously (†) holds too.

Let ψ be of the form $\neg\psi_1$ and for ψ_1 (†) already holds. Let $\psi(x, x_1, \dots, x_n)$ be true and $\psi(yx, yx_1, \dots, yx_n)$ not. Then $\psi_1(yx, yx_1, \dots, yx_n)$ is true and thus $\psi_1(y \cdot yx, y \cdot yx_1, \dots, y \cdot yx_n) = \psi_1(x, x_1, \dots, x_n)$ holds (induction assumption). Hence, a contradiction with $\psi = \neg\psi_1$. □

I.1.5 Remark. The statement remains valid also in a bit more general form. Every subset of an LSLD-groupoid definable in the language $\{\cdot, \in J_1, \dots, \in J_n\}$, $J_1, \dots, J_n$ being left ideals and $\in J_1, \dots, \in J_n$ the corresponding unary relational symbols, is either empty or a left ideal too. The given proof may be easily extended into the general one, but we don't do it because of technical difficulty and useless of the result in further text.

I.1.6 Definition. Let G be an LSLD-groupoid. We define

(1) sets $I_G = \{x \in G : xx = x\}$ and $K_G = \{x \in G : xx \neq x\} = G \setminus I_G$;
(2) equivalences $p_G = \{(x,y) \in G \times G : L_x = L_y\}$ and $q_G = \{(a,b) \in I_G \times I_G : L_a|_{K_G} = L_b|_{K_G}\} \cup id_{K_G}$;
(3) a relation $ip_G = \{(x,y) \in G \times G : x = y \text{ or } xx = y\}$;
(4) a mapping $o_G : G \to G$, $o_G(x) = xx$ for all $x \in G$.

I.1.7 Lemma. *Let G be an LSLD-groupoid. Then*

(1) *I_G and K_G are either empty or left ideals;*
(2) *p_G and q_G are congruences on G and $ip_G \subseteq p_G$;*
(3) *ip_G is a congruence on G, G/ip_G is idempotent and ip_G is the smallest congruence such that the corresponding factor is idempotent; moreover, if A is a non-trivial block of ip_G, then $A \simeq \mathbf{T}$;*
(4) *o_G is an involutory automorphism of G.*

Proof.

(1) Both sets are definable.
(2) Let $x, y \in G$ be such that $L_x = L_y$. Clearly, $L_{xz} = L_{yz}$ for all $z \in G$, since $xz = yz$. Further, $zx \cdot u = zx \cdot (z \cdot zu) = z(x \cdot zu) = z(xz \cdot xu) = z(yz \cdot yu) = \cdots = zy \cdot u$ for all $z, u \in G$. Hence $L_{zx} = L_{zy}$ for all $z \in G$ too and p_G is a congruence.
Let $a, b \in I_G$ be such that $L_a|_{K_G} = L_b|_{K_G}$. Then $L_{az}|_{K_G} = L_{bz}|_{K_G}$ for all $z \in G$, since for every $k \in K_G$ we have $az \cdot k = a(z \cdot ak) = a(z \cdot bk) = b(z \cdot bk) = bz \cdot k$. Also $L_{za}|_{K_G} = L_{zb}|_{K_G}$ for all $z \in G$, since for every $k \in K_G$ we have $a \cdot zk = b \cdot zk$, and thus $za \cdot k = z(a \cdot zk) = z(b \cdot zk) = zb \cdot k$. Consequently, q_G is a congruence. The rest follows from $xz = x(x \cdot xz) = xx \cdot (x \cdot xz) = xx \cdot z$ for all $z \in G$.
(3) Let $(x, xx) \in ip_G$. Then $xx \cdot xx = x \cdot xx = x$, hence the relation ip_G is symmetric. Transitivity is trivial, so it is an equivalence. Further, for every $z \in G$ holds $xz = xx \cdot z$ due to (2), and $(zx, z \cdot xx) \in ip_G$ according to $z \cdot xx = zx \cdot zx$. Thus ip_G is a congruence. Clearly, G/ip_G is idempotent and ip_G is the smallest such congruence. Since $xx \cdot xx = x$, a non-trivial block of ip_G consists of two elements and so, by I.1.2, it is isomorphic to $\mathbf{T}$.
(4) o_G is a homomorphism, since $o_G(xy) = xy \cdot xy = x \cdot yy = xx \cdot yy = o_G(x)o_G(y)$ for all $x, y \in G$. It is an involution according to (3). □

I.1.8 Corollary. $\mathbf{T}$ *is the only (up to an isomorphism) simple non-idempotent LSLD-groupoid (i.e. possessing no non-trivial congruence).*

I.1.9 Example. The following are all (up to an isomorphism) three-element LSLDI-groupoids. The left translation of an element x might be either the identity, or the transposition of the two other elements. It is easy to check, which combinations give a left ditributive groupoid. Note that $\mathbf{S}_1$ is a right zero semigroup and $\mathbf{S}_3$ is a commutative distributive quasigroup. It is also the smallest Steiner triple system and it appears as an example of a non-associative distributive structure already in [Sc].

$\mathbf{S}_1$	a	b	c
a	a	b	c
a	a	b	c
a	a	b	c

$\mathbf{S}_2$	a	b	c
a	a	c	b
a	a	b	c
a	a	b	c

$\mathbf{S}_3$	a	b	c
a	a	c	b
a	c	b	a
a	b	a	c

I.1.10 Example. The following are all (up to an isomorphism) three-element non-idempotent LSLD-groupoids. The left translations of non-idempotent elements are determined by I.1.7 (3) and the left translation of the idempotent one can be either the identity, or the transposition. It is easy to check that both groupoids are left distributive.

$\mathbf{T}_1$	e	0	$\tilde{0}$
e	e	0	$\tilde{0}$
$0, \tilde{0}$	e	$\tilde{0}$	0

$\mathbf{T}_2$	e	0	$\tilde{0}$
e	e	$\tilde{0}$	0
$0, \tilde{0}$	e	$\tilde{0}$	0

The left translations of x and xx are always written into one row, because they are, by I.1.7 (2), equal.

I.1.11 Lemma. *Let G be an LSLD-groupoid. Then $\mathbf{T}$ is a homomorphic image of G, if and only if G is isomorphic to $\mathbf{T} \times (G/ip_G)$.*

Proof. Choose $g : G \to \mathbf{T}$ and put $h : G \to G/ip_G$ the natural projection. Then $f : G \to \mathbf{T}\times(G/ip_G)$, defined by $f(x) = (g(x), h(x))$ for every $x \in G$, is a homomorphism. Since $\mathbf{T}$ is idempotent-free, so is G. Thus f is onto, because h is so, and $f(x) = (0, h(x))$ iff $f(xx) = (1, h(x))$. Further, if $g(x) = g(y)$, then $(x, y) \notin ip_G$, since $\mathbf{T}$ is idempotent-free, and therefore $h(x) \neq h(y)$. Hence f is injective and thus an isomorphism. The converse is clear. □

II. Left symmetric left distributive operations on a group

1. The idempotent case

On a given group G we define the following three operations. For all $x, y \in G$, we put

(a) $x \circ y = xy^{-1}x$;
(b) $x \diamond_f y = xf(x^{-1}y)$ for f an involutory automorphism of G;
(c) $x \star_f y = xf(yx^{-1})$ for f an involutory automorphism of G satisfying $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$, where $\mathbb{Z}(G)$ denotes the center of G.

II.1.1 Lemma. *The operations $\circ$, $\diamond_f$ and $\star_f$ are left symmetric, left distributive and idempotent.*

Proof.

(a) $x \circ (x \circ y) = xx^{-1}yx^{-1}x = y$.
$(x \circ y) \circ (x \circ z) = xy^{-1}xx^{-1}zx^{-1}xy^{-1}x = xy^{-1}zy^{-1}x = x \circ (y \circ z)$.
$x \circ x = xx^{-1}x = x$.

(b) $x \diamond_f (x \diamond_f y) = xf(x^{-1}xf(x^{-1}y)) = xx^{-1}ff(y) = y$.
$(x \diamond_f y) \diamond_f (x \diamond_f z) = xf(x^{-1}y)f((xf(x^{-1}y))^{-1}xf(x^{-1}z)) =$
$= xf(x^{-1}y)y^{-1}xf(x^{-1}x)x^{-1}z = xf(x^{-1}yf(y^{-1}z)) = x \diamond_f (y \diamond_f z)$.
$x \diamond_f x = xf(x^{-1}x) = xf(1) = x$.

(c) $x \star_f (x \star_f y) = xf(xf(yx^{-1})x^{-1}) = xf(x)yx^{-1}f(x^{-1}) = yx^{-1}xf(x)f(x^{-1}) = y$, since $xf(x) \in \mathbb{Z}(G)$.
$(x \star_f y) \star_f (x \star_f z) = xf(yx^{-1})f(xf(zx^{-1})(xf(yx^{-1}))^{-1}) =$
$= xf(y)f(x^{-1})f(x)zx^{-1}xy^{-1}f(x^{-1}) = xf(yf(zy^{-1})x^{-1}) = x \star_f (y \star_f z)$.
$x \star_f x = xf(xx^{-1}) = xf(1) = x$. □

We consider the following four varieties:

(a) the variety $\mathcal{A}$ generated by all $G(\circ)$, G a group;
(b) the variety $\mathcal{B}$ generated by all $G(\diamond_f)$ such that G is a group and f its involutory automorphism;
(c) the variety $\mathcal{C}$ generated by all $G(\star_f)$ such that G is a group and f its involutory automorphism satisfying $xf(x) \in \mathbb{Z}(G)$ for every $x \in G$;
(d) the variety $\mathcal{V}$ of LSLDI-groupoids.

Firstly, we are going to construct two infinitely generated groupoids $A \in \mathcal{A}$ and $B \in \mathcal{B}$ being free in $\mathcal{V}$ and thus to show that the varieties $\mathcal{A}, \mathcal{B}$ and $\mathcal{V}$ coincide.

Let $\mathbf{FG}_X$ be the free group generated by a set X and let $\mathbf{F}_X$ be the free LSLDI-groupoid generated by X. For the rest of the chapter, we let Z to be a fixed countable infinite set. Put $\widetilde{Z} = \{\widetilde{x} : x \in Z\}$, a disjoint copy of Z, and $Y = Z \cup \widetilde{Z}$.

(a) Let A be the subgroupoid of $\mathbf{FG}_Z(\circ)$ generated by the set Z.
(b) Let B be the subgroupoid of $\mathbf{FG}_Y(\diamond_f)$ generated by the set Z, where $f : \mathbf{FG}_Y \to \mathbf{FG}_Y$ is the automorphism determined by $f(x) = \widetilde{x}$, $f(\widetilde{x}) = x$, $x \in Z$.

II.1.2 Lemma. *Let X be a non-empty set. Each element of the free LSLDI-groupoid $\mathbf{F}_X$ can be written in the form*

$$(\dagger) \qquad x_1(x_2(\dots(x_{n-1}x_n))),$$

where $x_1, \dots, x_n \in X$ *and* $x_i \neq x_{i+1}$, $i = 1, \dots, n-1$.

Proof. Firstly, observe that (using left symmetry and left distributivity)

$$xy \cdot z = (xy)(x \cdot xz) = x(y(xz)). \tag{✠}$$

Now, we prove by induction that each term $t \in \mathbf{F}_X$ can be written in the form (†). The statement is trivial for $t \in X$. Given two terms $t = x_1(x_2(\dots(x_{n-1}x_n)))$, $s = y_1(y_2(\dots(y_{m-1}y_m)))$ in the form (†), by using n-times (✠) we get

$$\begin{gathered} ts = x_1(x_2(\dots(x_{n-1}x_n)))s = x_1([x_2(\dots(x_{n-1}x_n))](x_1s)) = \\ x_1(x_2([x_3(\dots(x_{n-1}x_n))](x_2(x_1s)))) = \\ \dots = x_1(x_2(\dots(x_{n-1}(x_n(x_{n-1}(\dots(x_2(x_1s)))))))). \end{gathered}$$

If $x_1 \neq y_1$, we have ts in the form (†). Otherwise, by left symmetry, we get rid of the equal elements and then we obtain ts in the form (†). □

II.1.3 Lemma. *All terms of the form* (†) *represent in the groupoids A and B pairwise different elements (identifying term variables $x_1, x_2, \dots$ with the (countable infinite) basis Z).*

Proof. (a) Given $x_1, \dots, x_n \in A$ such that $x_i \neq x_{i+1}$, $i = 1, \dots, n-1$, put $\varepsilon = (-1)^n$. Then

$$x_1 \circ (x_2 \circ (\dots(x_{n-1} \circ x_n))) = x_1 x_2^{-1} \dots x_{n-1}^{\varepsilon} x_n^{-\varepsilon} x_{n-1}^{\varepsilon} \dots x_2^{-1} x_1.$$

Clearly, this word is not reducible and thus all such words must be pairwise different.

(b) Given $x_1, \dots, x_n \in A$ such that $x_i \neq x_{i+1}$, $i = 1, \dots, n-1$, put $\varepsilon = 0$ for n odd and $\varepsilon = 1$ for n even. Then

$$x_1 \diamond_f (x_2 \diamond_f (\dots(x_{n-1} \diamond_f x_n))) = x_1 f(x_1^{-1}) f(x_2) x_2^{-1} x_3 f(x_3^{-1}) \dots f^{\varepsilon}(x_{n-1}^{-1}) f^{\varepsilon}(x_n).$$

Clearly, this word is not reducible and thus all such words must be pairwise different. □

II.1.4 Corollary. *Each term $t \in \mathbf{F}_X$, X a non-empty set, has the (unique) normal form* (†).

II.1.5 Corollary. *The groupoids A, B and $\mathbf{F}_Z$ are isomorphic and thus free in $\mathcal{V}$.*

Proof. The isomorphisms are

$$\left.\begin{array}{r} x_1 \circ (x_2 \circ (\dots(x_{n-1} \circ x_n))) \\ x_1 \diamond_f (x_2 \diamond_f (\dots(x_{n-1} \diamond_f x_n))) \end{array}\right\} \longleftrightarrow x_1(x_2(\dots(x_{n-1}x_n))).$$

□

Now, we focus attention on the variety $\mathcal{C}$. Given an arbitrary group G and its involutory automorphism f satisfying $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$, we claim that $G(\star_f)$ satisfies the equation

$$x \star_f (y \star_f (z \star_f (x \star_f (y \star_f (z \star_f (x \star_f (y \star_f (z \star_f t)))))))) = x \star_f (y \star_f (z \star_f t)) \tag{‡}$$

for all $x, y, z, t \in G$. This equation cannot hold in a free LSLDI-groupoid according to II.1.4, and hence $\mathcal{C} \neq \mathcal{V}$. The proof of (‡) will be given later in more general form.

II.1.6 Theorem.

(1) *The varieties* $\mathcal{A}, \mathcal{B}, \mathcal{V}$ *are identical.*
(2) *The variety* $\mathcal{C}$ *is a proper subvariety of* $\mathcal{V}$, *since* $\mathcal{C}$ *satisfies the additional equation* (‡).

Proof. See the previous text. □

II.1.7 Remark. Another proof that $\mathcal{A} = \mathcal{V}$ appears already in [Pi$_1$, 4.12]. Put G the countable infinite free power of $\mathbb{Z}_2$. The subgroupoid of $G(\circ)$ generated by all involutions in G (with exception of the neutral element) is isomorphic to the free LSLDI-groupoid $\mathbf{F}_Z$.

2. THE NON-IDEMPOTENT CASE

In the present section we find non-idempotent generalizations of the operations $\circ$, $\diamond_f$ and $\star_f$ and we also investigate two new LSLD-operations which idempotency deprive of interest. For a given group G and all $x, y \in G$, we put

(a) $x \circ_f y = xf(y)x$ for f an involutory antiautomorphism of G, i.e. an involution $f : G \to G$ such that $f(xy) = f(y)f(x)$, satisfying $f(x^2)x^2 = 1$ and $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$;
(b) $x \diamond_{f,a} y = xf(x^{-1}y)a$ for f an involutory automorphism of G and $a \in G$ satisfying $a^2 = 1$, $f(a) = a$;
(c) $x \star_{f,a} y = xf(y)af(x^{-1})$ for f an involutory automorphism of G and $a \in G$ satisfying $a^2 = 1$, $ax = xf(a)$ and $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$;
(d) $x \bullet_f y = xf(y)x$ for f an involutory automorphism of G satisfying $f(x^2)x^2 = 1$ and $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$;
(e) $x \odot_a y = xax^{-1}y$ for $a \in G$ such that $a^2 = 1$.

II.2.1 Remark. It is easy to check similarly like in II.1.1 that the introduced operations are left symmetric and left distributive. They are idempotent, if and only if

(a) $f(x) = x^{-1}$ for all $x \in G$; in such a case $G(\circ_f) = G(\circ)$.
(b) $a = 1$; in such a case $G(\diamond_{f,a}) = G(\diamond_f)$.
(c) $a = 1$; in such a case $G(\star_{f,a}) = G(\star_f)$.
(d) $f(x) = x^{-1}$ for all $x \in G$; in such a case the group G must be abelian.
(e) $a = 1$; in such a case $G(\odot_a)$ is a right zero semigroup (i.e. a groupoid satisfying the equation $xy = y$).

On the other hand, if the groupoids are not idempotent, then

(a) $x \in G(\circ_f)$ is idempotent iff $f(x) = x^{-1}$;
(b) $G(\diamond_{f,a})$ is idempotent-free;
(c) $G(\star_{f,a})$ is idempotent-free;
(d) $x \in G(\bullet_f)$ is idempotent iff $f(x) = x^{-1}$;
(e) $G(\odot_a)$ is idempotent-free.

Again, we consider $\mathcal{A}'$ ($\mathcal{B}'$, $\mathcal{C}'$, $\mathcal{D}'$, $\mathcal{E}'$, resp.) to be the varieties generated by all $G(\circ_f)$ ($G(\diamond_{f,a})$, $G(\star_{f,a})$, $G(\bullet_f)$, $G(\odot_a)$, resp.). Clearly, $\mathcal{A} \subset \mathcal{A}'$, $\mathcal{B} \subset \mathcal{B}'$, $\mathcal{C} \subset \mathcal{C}'$. We are going to show that $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{E}'$ coincide with the variety $\mathcal{W}$ of LSLD-groupoids, but that this is not true for the varieties $\mathcal{C}'$ and $\mathcal{D}'$.

Let $\mathbf{F}'_X$ be the free LSLD-groupoid generated by a set X. Note that $\mathbf{F}'_1 = \mathbf{T}$.

II.2.2 Theorem. *Let X be a non-empty set. Then $\mathbf{F}'_X$ is isomorphic to $\mathbf{T} \times \mathbf{F}_X$. Consequently, the variety $\mathcal{W}$ is generated by $\mathcal{V} \cup \{\mathbf{T}\}$.*

Proof. Since $\mathbf{T}$ is a homomorphic image of $\mathbf{F}'_X$, by I.1.11, $\mathbf{F}'_X \simeq \mathbf{T} \times H$ for an LSLDI-groupoid H. Because $\mathbf{F}_X$ is free for LSLDI-groupoids, it turns to be $H \simeq \mathbf{F}_X$. □

II.2.3 Corollary. *Each term $t \in \mathbf{F}'_X$, X a non-empty set, has the (unique) normal form*

$$x_1(x_2(\dots(x_{n-1}x_n))), \tag{†'}$$

where $x_1, \dots, x_n \in X$ and $x_i \neq x_{i+1}$, $i = 1, \dots, n-2$.

Proof. Applying II.1.2, each term can be written in the form (†'). By II.2.2 it is unique, because

$$\{x_1(x_2(\dots(x_{n-1}x_n))), x_1(x_2(\dots(x_{n-1}(x_nx_n))))\}$$

corresponds to

$$\mathbf{T} \times \{x_1(x_2(\dots(x_{n-1}x_n)))\}.$$

□

II.2.4 Corollary. *The varieties $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{W}$ coincide.*

Proof. With respect to II.2.2 and $\mathcal{A} = \mathcal{B} = \mathcal{V}$, it suffices to prove that $\mathbf{T} \in \mathcal{A}'$ and $\mathbf{T} \in \mathcal{B}'$. It is easy to check that, in the first case, $\mathbf{T}$ is isomorphic to the subgroupoid $\{1, 3\}$ of $\mathbb{Z}_4(\circ_{id})$ and, in the second case, $\mathbf{T}$ is isomorphic to $\mathbb{Z}_2(\diamond_{id,1})$. □

Now, we are going to construct an infinitely generated groupoid $E \in \mathcal{E}'$ being free in $\mathcal{W}$ and thus to show that the varieties $\mathcal{E}'$ and $\mathcal{W}$ coincide. Let E be the subgroupoid of $(\mathbb{Z}_2 * \mathbf{FG}_Z)(\odot_a)$ generated by the (countable infinite) set Z, where $*$ means the free product of two groups and a is the non-neutral element of $\mathbb{Z}_2$.

II.2.5 Lemma. *The groupoid E is isomorphic to $\mathbf{F}'_Z$ and thus free in $\mathcal{W}$.*

Proof. The isomorphism is

$$x_1(x_2(\dots(x_{n-1}x_n))) \quad \longleftrightarrow \quad x_1 \odot_a (x_2 \odot_a (\dots(x_{n-1} \odot_a x_n))),$$

since the word

$$x_1 \odot_a (x_2 \odot_a (\dots(x_{n-1} \odot_a x_n))) = x_1 a x_1^{-1} x_2 a x_2^{-1} \dots x_{n-1} a x_{n-1}^{-1} x_n$$

is reducible if and only if $x_{n-1} = x_n$, and in such a case it obviously cannot be obtained by any other term in the form (†'). □

Now, we focus attention on the variety $\mathcal{C}'$. Given an arbitrary group G, its involutory automorphism f and $a \in G$ satisfying $a^2 = 1$, $ax = xf(a)$ and $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$, we claim that $G(\star_{f,a})$ satisfies the equation

$$\begin{gathered} x \star_{f,a} (y \star_{f,a} (z \star_{f,a} (x \star_{f,a} (y \star_{f,a} (z \star_{f,a} (x \star_{f,a} (y \star_{f,a} (z \star_{f,a} t)))))))) = \\ x \star_{f,a} (y \star_{f,a} (z \star_{f,a} t)) \end{gathered} \tag{‡'}$$

for all $x, y, z, t \in G$. This equation cannot hold in a free LSLD-groupoid according to II.2.3, and hence $\mathcal{C}' \neq \mathcal{W}$. Note that (‡) we obtain when choosing $a = 1$.

Proof of ($\ddagger'$). Put $u = xf(y)z$ and $v = f(z^{-1})y^{-1}af(x^{-1})$. Since (1) $arf(a)s = rf(a)^2s = rf(a^2)s = rs$ for arbitrary $r, s \in G$ and (2) $uf(u) \in \mathbb{Z}(G)$, we have

$$x \star_{f,a} (y \star_{f,a} (z \star_{f,a} (x \star_{f,a} (y \star_{f,a} (z \star_{f,a} (x \star_{f,a} (y \star_{f,a} (z \star_{f,a} t)))))))) =$$
$$uf(u)uf(t)af(z^{-1})f(a)y^{-1}af(x^{-1})f(a)z^{-1}af(y^{-1})f(a)x^{-1}af(z^{-1})f(a)y^{-1}af(x^{-1}) \stackrel{(1)}{=}$$
$$uf(u)uf(t)f(z^{-1})y^{-1}f(x^{-1})z^{-1}f(y^{-1})x^{-1}f(z^{-1})y^{-1}af(x^{-1}) =$$
$$uf(u)uf(t)f(u^{-1})u^{-1}v \stackrel{(2)}{=} uf(t)uf(u)f(u)^{-1}u^{-1}v = uf(t)v =$$
$$xf(y)zf(t)f(z^{-1})y^{-1}af(x^{-1}) \stackrel{(1)}{=} xf(y)zf(t)af(z^{-1})f(a)y^{-1}af(x^{-1}) =$$
$$x \star_{f,a} (y \star_{f,a} (z \star_{f,a} t))$$

□

Finally, we investigate the variety $\mathcal{D}'$. We claim that for an arbitrary group G and its involutory automorphism f such that $f(x^2)x^2 = 1$ and $xf(x) \in \mathbb{Z}(G)$ for all $x \in G$, the groupoid $G(\bullet_f)$ satisfies the equation

($\sharp$) $x \bullet_f (y \bullet_f (z \bullet_f (x \bullet_f (y \bullet_f (z \bullet_f (s \bullet_f (z \bullet_f (y \bullet_f (x \bullet_f (z \bullet_f (y \bullet_f (x \bullet_f t)))))))))))) = s \bullet_f t$

for all $x, y, z, s, t \in G$. This equation cannot hold in a free LSLD-groupoid according to II.2.3, and hence $\mathcal{D}' \neq \mathcal{W}$.

Proof of ($\sharp$). Put $u = xf(y)z$ and $v = zf(y)x$. Since (1) $uf(u), vf(v) \in \mathbb{Z}(G)$ and (2) $u^2f(u^2) = f(v^2)v^2 = 1$, we have

$$x \bullet_f (y \bullet_f (z \bullet_f (x \bullet_f (y \bullet_f (z \bullet_f (s \bullet_f (z \bullet_f (y \bullet_f (x \bullet_f (z \bullet_f (y \bullet_f (x \bullet_f t)))))))))))) =$$
$$uf(u)sf(v)vf(t)uf(u)sf(v)v \stackrel{(1)}{=} sf(t)u^2f(u)^2sf(v)^2v^2 \stackrel{(2)}{=} sf(t)s = s \bullet_f t$$

□

II.2.6 Theorem.

(1) *The varieties $\mathcal{A}', \mathcal{B}', \mathcal{E}'$ and $\mathcal{W}$ are identical.*
(2) *The varieties $\mathcal{C}'$ and $\mathcal{D}'$ are proper subvarieties of $\mathcal{W}$, since $\mathcal{C}'$ satisfies the additional equation ($\ddagger'$) and $\mathcal{D}'$ satisfies the additional equation ($\sharp$).*

Proof. See the previous text. □

II.2.7 Remark. Let $\mathcal{I}$ be the variety of idempotent groupoids. Obviously, $\mathcal{W} \cap \mathcal{I} = \mathcal{V}$ and thus $\mathcal{A}' \cap \mathcal{I} = \mathcal{A}$ and $\mathcal{B}' \cap \mathcal{I} = \mathcal{B}$. The question is, to describe $\mathcal{C}' \cap \mathcal{I}$ and $\mathcal{D}' \cap \mathcal{I}$. Also the variety generated by cores of abelian groups might be of interest.

III. Subdirectly irreducible non-idempotent left symmetric left distributive groupoids

1. PRELIMINARIES

A non-trivial groupoid G possessing the smallest non-trivial congruence, i.e. a congruence $\mu_G \neq id_G$ such that $\mu_G \subseteq \nu$ for every non-trivial congruence ν on G, is called *subdirectly irreducible* and μ_G is then called the *monolith* of G. In this section we describe basic properties of subdirectly irreducible non-idempotent LSLD-groupoids.

III.1.1 Lemma. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid. Then*

(1) *every left ideal J in G satisfies either $J \supseteq K_G$ or $J \cap K_G = \emptyset$;*
(2) *ip_G is the monolith of G;*
(3) *$L_a|_{K_G} \neq L_b|_{K_G}$, whenever $a, b \in I_G$, $a \neq b$;*
(4) *any two automorphisms f, g of G satisfying $f|_{K_G} = g|_{K_G}$ are equal.*

Proof.

(1) Let $J \subset K_G$ be a left ideal. Then $L = K_G \setminus J$ is a left ideal too, because all left translations are K_G-invariant permutations by I.1.7 (1). Take the congruences $\rho_J = (ip_G \cap (J \times J)) \cup id_G$ and $\rho_L = (ip_G \cap (L \times L)) \cup id_G$. Since J, L contain at least two elements, both congruences are non-trivial and $\rho_J \cap \rho_L = id_G$. Hence, a contradiction with subdirect irreducibility of G.
(2) Let $\mu_G \subseteq ip_G$ be the monolith of G. Put $J = \{x \in K_G : (x, xx) \in \mu_G\}$. Then J is a left ideal, since μ_G is a congruence. Consequently, by (1), $J = K_G$ and $\mu_G = ip_G$.
(3) According to I.1.7 (2), $q_G = \{(a, b) \in I_G \times I_G : L_a|_{K_G} = L_b|_{K_G}\} \cup id_{K_G}$ is a congruence. It must be trivial, since $q_G \cap ip_G = id_G$. Thus $L_a|_{K_G} = L_b|_{K_G}$ iff $a = b$.
(4) We need to show that $f(a) = g(a)$ for every $a \in I_G$. Observe that $f|_{K_G} = g|_{K_G}$ iff $f^{-1}|_{K_G} = g^{-1}|_{K_G}$, because every automorphism h of G is K_G-invariant. Now, given $a \in I_G$ and $k \in K_G$, we have $f(a)k = f(a)ff^{-1}(k) = f(af^{-1}(k))$. Since $af^{-1}(k) = ag^{-1}(k) \in K_G$, there is $f(af^{-1}(k)) = g(ag^{-1}(k)) = g(a)k$. Thus $L_{f(a)}|_{K_G} = L_{g(a)}|_{K_G}$ and, by (3), $f(a) = g(a)$. □

III.1.2 Corollary. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid. Then K_G/ip_{K_G} is an LSLDI-groupoid and it contains no proper definable subset, as well as K_G does.*

Proof. Combine I.1.4 and III.1.1 (1). □

III.1.3 Corollary. *Let G be a subdirectly irreducible idempotent-free LSLD-groupoid. Then both G and G/ip_G are left ideal-free.*

Proof. See III.1.1 (1). □

III.1.4 Extension theorem. *Let G, H be non-idempotent LSLD-groupoids. If G is a subdirectly irreducible subgroupoid of H, all elements of $H \setminus G$ are idempotent and $p_H = p_G \cup id_H$, then H is subdirectly irreducible.*

Proof. Let ν be a congruence on H. If $(x, y) \in \nu$ for $x, y \in G$, then $ip_G \subseteq \nu$ by subdirect irreducibilty of G and thus $ip_H = ip_G \cup id_H \subseteq \nu$. If $(x, y) \in \nu$ for $x \in H \smallsetminus G$, $x \neq y \in H$, then $L_x \neq L_y$ and therefore, by III.1.1 (3), there is $k \in K_G = K_H$ such that $xk \neq yk$. Hence $(xk, yk) \in \nu$, $xk, yk \in G$, and the first case finishes the proof. □

III.1.5 Example. $\mathbf{S}$ and $\mathbf{T}$ are simple (and hence subdirectly irreducible) LSLD-groupoids, since they consist of two elements only.

III.1.6 Example. $\mathbf{S}_2$ and $\mathbf{S}_3$ are subdirectly irreducible LSLDI-groupoids, but $\mathbf{S}_1$ is not. Moreover, $\mathbf{S}_3$ is simple.

III.1.7 Example. Both $\mathbf{T}_1$ and $\mathbf{T}_2$ are subdirectly irreducible non-idempotent LSLD-groupoids.

III.1.8 Example. Let $G(+) = \mathbb{Z}_{2^n}(+)$, $n = 1, 2, \dots, \infty$. Let $a \in G$ be such that $a \neq 0 = 2a$. Clearly, $f(x) = -x$ is an involutory automorphism of G satisfying $f(a) = a$ and thus $G(\diamond_{f,a}) = G(\star_{f,a})$ is an idempotent-free LSLD-groupoid (see II.2.1). It is easy to check that for $n = 1, \infty$ they are subdirectly irreducible.

2. GROUPOIDS WITH LEFT ABSORBING ELEMENTS

We are going to describe all subdirectly irreducible non-idempotent LSLD-groupoids containing a left absorbing element. We will see that such a groupoid is subdirectly irreducible, iff the subgroupoid obtained by kicking off the absorbing element is so.

III.2.1 Lemma. *Let G be an LSLD-groupoid, $f \in \{id_G, o_G\}$ and $A_f = \{a \in I_G : L_a = f\}$. Then A_f is a left ideal. In particular, if G is subdirectly irreducible non-idempotent, then $|A_f| \leq 1$ and $a \in A_f$ is left absorbing.*

Proof. A_f is definable by the formula

$$(xx = x) \ \& \ (\forall y)(xy = f(y))$$

(where $f(y)$ means either y, or yy) and thus A_f is a left ideal. The rest is a consequence of III.1.1 (3). □

III.2.2 Definition. Let G be an LSLD-groupoid and f its involutory automorphism. $G[f]$ will designate the groupoid on the set $G \cup \{f\}$ such that G is a subgroupoid of $G[f]$ and $xf = ff = f$ and $fx = f(x)$ for all $x \in G$.

$G[f]$	f	$\dots$	x	y	$\dots$
f	f		$f(x)$	$f(y)$	
$\dots$		$\dots$			$\dots$
x	f		xx	xy	
y	f		yx	yy	
$\dots$		$\dots$			$\dots$

III.2.3 Lemma. *Let G be an LSLD-groupoid and f its involutory automorphism. Then $G[f]$ is an LSLD-groupoid, iff $L_x = L_{f(x)}$ for all $x \in G$.*

Proof. Obviously, $G[f]$ is left symmetric and idempotent. Thus, according to I.1.1 (4), it is left distributive, iff for all $x, y \in G$ (i) $L_f = L_y L_f L_y$, (ii) $L_{f(y)} = L_f L_y L_f$ and (iii)

$L_{xy} = L_x L_y L_x$. Since G is an LSLD-groupoid, (iii) is true. (ii) clearly holds for any involutory automorphism f. Finally, (i) is satisfied iff $f(x) = yf(yx)$ for all $x, y \in G$ and the observation

$$f(x) = y \cdot f(yx) \;\Leftrightarrow\; x = f(y) \cdot yx \;\Leftrightarrow\; f(y)x = yx \;\Leftrightarrow\; L_{f(y)}(x) = L_y(x)$$

completes the proof. □

III.2.4 Lemma. *Let G be a subdirectly irreducible LSLD-groupoid, $f \in \{id_G, o_G\}$ and $L_a \neq f$ for all $a \in I_G$. Then $G[f]$ is a subdirectly irreducible LSLD-groupoid.*

Proof. By I.1.7 and III.2.3, $G[f]$ is an LSLD-groupoid. Let ν be a congruence on $G[f]$. If $(a, f) \in \nu$ for $a \in I_G$, there is $x \in G$ such that $L_a(x) \neq f(x)$ and thus $(L_a(x), f(x)) = (ax, fx) \in \nu \cap (G \times G)$; hence $\mu_G \subseteq \nu$. If $(k, f) \in \nu$ for $k \in K_G$, then $(kk, kf) = (kk, f) \in \nu$ and thus $(k, kk) \in \nu$; hence $\mu_G \subseteq \nu$. Consequently, $G[f]$ is subdirectly irreducible and $\mu_{G[f]} = \mu_G \cup \{(f, f)\}$. □

III.2.5 Lemma. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid containing a left absorbing element a. Then $G \simeq H[f]$, where $H = G \setminus \{a\}$ and $f = L_a|_H$. Moreover, H is subdirectly irreducible and $f \in \{id_H, o_H\}$.*

Proof. Put $\nu = \{(x, f(x)) : x \in H\} \cup id_G$. Then ν is an equivalence, since f is an involution. If $(x, f(x)) \in \nu$, then for every $y \in H$

(i) $(yx, yf(x)) \in \nu$, because $f(yx) = f(y)f(x) = yf(x)$ by III.2.3;
(ii) $(ax, af(x)) \in \nu$, because $ax = f(x)$ and $af(x) = f(f(x))$;
(iii) $(xy, f(x)y) \in \nu$, because $xy = f(x)y$ by III.2.3;
(iv) $(xa, f(x)a) \in \nu$, because $xa = a = f(x)a$.

Hence ν is a congruence and so $\nu = id_G$ or $ip_G \subseteq \nu$. In the first case $f = id_H$. In the second case $f(x) = xx$ for all $x \in K_G$ and the uniqueness of an automorphism defined on K_G (III.1.1 (4)) gives $f = o_H$.

Now, assume that ρ is a non-trivial congruence on H. Put $\sigma = \rho \cup \{(a, a)\}$. For every $(x, y) \in \rho$ we have $(xa, ya) = (a, a) \in \sigma$ and $(ax, ay) = (f(x), f(y)) \in \rho \subseteq \sigma$, because $f \in \{id_H, o_H\}$. Hence, σ is a non-trivial congruence on G, $ip_G \subseteq \sigma$ and thus $ip_H \subseteq \rho$. Consequently, H is subdirectly irreducible. □

III.2.6 Lemma. *Let G, H be subdirectly irreducible non-idempotent LSLD-groupoids containing no left absorbing element, $f \in \{id_G, o_G\}$. Then*

(1) *$G \simeq H$ iff $G[f] \simeq H[f]$.*
(2) *$G[o_G][id_{G[o_G]}] \simeq G[id_G][o_{G[id_G]}]$;*

Proof.

(1) An isomorphism $\varphi : G[f] \to H[f]$ must send the left absorbing element of $G[f]$ onto a left absorbing element of $H[f]$. Since no such is in H, f is mapped onto f and $\varphi|_G$ is an isomorphism of G and H. The converse is clear.
(2) Clear. □

III.2.7 Adjoining theorem. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid.*

(1) *If G contains no left absorbing element, then*

$$G,\; G[id_G],\; G[o_G] \text{ and } G[id_G][o_{G[id_G]}]$$

are pairwise non-isomorphic subdirectly irreducible LSLD-groupoids.

(2) *If G contains a left absorbing element, then there exists the unique (up to an isomorphism) LSLD-groupoid H with no left absorbing element such that G is isomorphic to one of the groupoids*

$$H[id_H],\ H[o_H] \text{ and } H[id_H][o_{H[id_H]}].$$

Moreover, H is subdirectly irreducible.

Proof.

(1) According to III.2.1, $L_a \notin \{id_G, o_G\}$ for every $a \in I_G$. Hence, by III.2.4, G, $G[id_G]$, $G[o_G]$ and $G[id_G][o_{G[id_G]}]$ are subdirectly irreducible LSLD-groupoids. Any isomorphism between two of them must send a left absorbing element onto a left absorbing element with the same left translation. Consequently, they are pairwise non-isomorphic.

(2) Suppose that G contains a set A of left absorbing elements, $|A| \geq 3$. Since the left translation of a left absorbing element is, by III.2.5, either id_G or o_G, there exist $a, b \in A$, $a \neq b$, such that $L_a = L_b$. This contradicts III.1.1 (3). Now, apply III.2.5 and III.2.6. □

III.2.8 Corollary. *Let G be a subdirectly irreducible LSLD-groupoid with $|I_G| = 1$. Then $G \simeq H[f]$ for a subdirectly irreducible idempotent-free LSLD-groupoid H and $f \in \{id_H, o_H\}$.*

III.2.9 Example. Note that $\mathbf{T}_1 \simeq \mathbf{T}[id_{\mathbf{T}}]$ and $\mathbf{T}_2 \simeq \mathbf{T}[o_{\mathbf{T}}]$ are the only three-element subdirectly irreducible non-idempotent LSLD-groupoids. On the other hand, $\mathbf{S}_3$ is a three-element subdirectly irreducible LSLDI-groupoid containing no left absorbing element.

III.2.10 Example. $\mathbf{T}[id_{\mathbf{T}}][o_{\mathbf{T}_1}]$ is the unique four-element subdirectly irreducible non-idempotent LSLD-groupoid.

Proof.

(i) Let $|I_G| = 2$. Since I_G is a two-element LSLDI-groupoid, $I_G \simeq \mathbf{S}$ (see I.1.2). Thus the left translation of an idempotent element is either id_G or o_G. Hence, by III.2.1 and III.2.7, $G \simeq \mathbf{T}[id_{\mathbf{T}}][o_{\mathbf{T}_1}]$.

(ii) Let $I_G = \emptyset$. Then $|G/ip_G| = 2$ and thus $G/ip_G \simeq \mathbf{S}$. This contradicts III.1.3, since $\mathbf{S}$ is not left ideal-free. □

3. Groupoids of involutions

The present section deals with groupoids of involutions — special subgroupoids of the core of a symmetric group. We prove a theorem on embedability of the idempotent part of a subdirectly irreducible LSLD-groupoid G into a groupoid of involutions and we show an upper bound on $|I_G|$.

III.3.1 Definition.

(i) Let ε be an equivalence on a non-empty set X. We designate $\mathrm{Inv}(X, \varepsilon)$ the subgroupoid of the core of the symmetric group over X consisting of all ε-invariant involutions.

(ii) An equivalence ε is called a *pairing* (a *semipairing*, resp.), if every block of ε consists of (at most, resp.) two elements. For a semipairing ε we denote $o_\varepsilon \in \mathrm{Inv}(X,\varepsilon)$ such involution that $(x, o_\varepsilon(x)) \in \varepsilon$ and $o_\varepsilon(x) = x$ iff $\{x\}$ is a block of ε.

(iii) For a semipairing ε on a non-empty set X we put $\mathrm{Inv}^*(X,\varepsilon) = \mathrm{Inv}(X,\varepsilon) \setminus \{id_X, o_\varepsilon\}$.

III.3.2 Lemma. *Let ε be an equivalence on a non-empty set X. Then* $\mathrm{Inv}(X,\varepsilon)$ *is an LSLDI-groupoid and id_X is a left neutral right absorbing element of* $\mathrm{Inv}(X,\varepsilon)$.

Proof. $\mathrm{Inv}(X,\varepsilon)$ is really a subgroupoid of the core of a symmetric group, since $f \circ g = fgf$ is obviously an ε-invariant involution on X, provided that f, g are so. Further, $id_X \circ f = f$ and $f \circ id_X = ff = id_X$ for every $f \in \mathrm{Inv}(X,\varepsilon)$. □

III.3.3 Lemma. *Let ε be a semipairing on a non-empty set X. Then o_ε is a left neutral right absorbing element of* $\mathrm{Inv}(X,\varepsilon)$.

Proof. Since every $f \in \mathrm{Inv}(X,\varepsilon)$ is ε-invariant, $o_\varepsilon \circ f = o_\varepsilon f o_\varepsilon = f$ and $f \circ o_\varepsilon = f o_\varepsilon f = o_\varepsilon$. □

III.3.4 Corollary. $\mathrm{Inv}^*(X,\varepsilon)$ *is a subgroupoid of* $\mathrm{Inv}(X,\varepsilon)$.

III.3.5 Theorem. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid with at least one idempotent element. Then the mapping*

$$\varphi : I_G \to \mathrm{Inv}(K_G, ip_{K_G}), \quad \varphi(a) = L_a|_{K_G} \;\; \textit{for every} \;\; a \in I_G$$

is an injective homomorphism.

Proof. The mapping φ is well-defined, because every left translation in G is ip_{K_G}-invariant involution. Further, $\varphi(ab) = L_{ab}|_{K_G} = (L_a L_b L_a)|_{K_G} = \varphi(a) \circ \varphi(b)$ for all $a, b \in I_G$, according to I.1.1 (4). Finally, injectivity follows from III.1.1 (3). □

For a cardinal number κ choose a pairing ε_κ on κ and put $\mathrm{Inv}(\kappa) = \mathrm{Inv}(\kappa, \varepsilon_\kappa)$. Denote $\lambda_\kappa = |\,\mathrm{Inv}(\kappa)|$.

III.3.6 Corollary. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid. Then I_G is either empty or isomorphic to a subgroupoid of* $\mathrm{Inv}(|K_G|)$. *Hence*

$$|G| \le \lambda_{|K_G|} + |K_G|.$$

Later we will show that the upper bound on $|I_G|$ is the best possible.

III.3.7 Theorem.

(1) $\lambda_2 = 2$, $\lambda_4 = 6$, $\lambda_{2n} = 2\lambda_{2n-2} + (2n-2)\lambda_{2n-4}$ *for every* $3 \le n < \omega$.

(2) $\lambda_\kappa = 2^\kappa$ *for every* $\kappa \ge \omega$.

Proof.

(1) Let $X_n = \{0, \dots, n-1, \widetilde{0}, \dots, \widetilde{n}-1\}$ and $\varepsilon_n = \bigcup_{k=0}^{n-1} \{k, \widetilde{k}\}^2$. We compute the number of ε_n-invariant involutions on X_n. Clearly

$$\mathrm{Inv}(X_1, \varepsilon_1) = \{id_{X_1}, o_{\varepsilon_1}\},$$
$$\mathrm{Inv}(X_2, \varepsilon_1) = \{id_{X_2}, o_{\varepsilon_2} = (0\ \widetilde{0})(1\ \widetilde{1}), (0\ \widetilde{0}), (1\ \widetilde{1}), (0\ 1)(\widetilde{0}\ \widetilde{1}), (0\ \widetilde{1})(\widetilde{0}\ 1)\}.$$

Now, suppose $n \ge 3$. For $f \in \mathrm{Inv}(X_n, \varepsilon_n)$ one of the following holds:

(i) $f(0) = 0$. In such a case $f(\widetilde{0}) = \widetilde{0}$ and the other values can be choosen arbitrary. Hence there are λ_{2n-2} possibilities.
(ii) $f(0) = \widetilde{0}$. In such a case $f(\widetilde{0}) = 0$ and the other values can be choosen arbitrary. Hence there are λ_{2n-2} possibilities.
(iii) $f(0) = x \notin \{0, \widetilde{0}\}$. In such a case $f(\widetilde{0}) = \widetilde{x}$, $f(x) = 0$, $f(\widetilde{x}) = \widetilde{0}$ (denoting $\widetilde{\widetilde{x}} = x$) and the other values can be choosen arbitrary. Hence there are λ_{2n-4} possibilities, while x can be choosen in $2n-2$ ways.
Consequently, $|\operatorname{Inv}(X_n, \varepsilon_n)| = \lambda_{2n} = 2\lambda_{2n-2} + (2n-2)\lambda_{2n-4}$.

(2) There are at most 2^κ involutions on κ, because there are at most $\kappa^\kappa = 2^\kappa$ permutations of κ. On the other hand, there are at least 2^κ ε_κ-invariant involutions: take all involutions of the form $(x_1\ y_1)(x_2\ y_2)\ldots$, while $\{x_1, y_1\}, \{x_2, y_2\}, \ldots$ being pairwise different blocks of ε_κ. Since ε_κ consists of κ two-element blocks, $\lambda_\kappa = 2^\kappa$. $\square$

4. GENERAL DESCRIPTION

The statements III.3.5 and III.3.6 say that a subdirectly irreducible non-idempotent LSLD-groupoid G has the following form ($f, g \in I \simeq I_G$, $x, y \in K_G$).

$$
\begin{array}{c|cccccccc}
G & \ldots & f & g & \ldots & x & y & \ldots \\
\hline
\ldots & \ldots & & & \ldots & & & \ldots \\
f & & f \circ f & f \circ g & & f(x) & f(y) & \\
g & & g \circ f & g \circ g & & g(x) & g(y) & \\
\ldots & \ldots & & & \ldots & & & \ldots \\
x & & ? & ? & & xx & xy & \\
y & & ? & ? & & yx & yy & \\
\ldots & \ldots & & & \ldots & & & \ldots
\end{array}
$$

In the present section we deliver a full description of G. In other words, the question-marks in the table are replaced with (uniquely determined) elements of I.

III.4.1 Definition. Let G be an LSLD-groupoid. We designate

$$\mathrm{IA}(G) = \operatorname{Inv}(G, ip_G) \cap \operatorname{Aut}(G)$$
$$\mathrm{IA}^*(G) = \operatorname{Inv}^*(G, ip_G) \cap \operatorname{Aut}(G)$$

the subgroupoids of $\operatorname{Inv}(G, ip_G)$ consisting of all involutory automorphisms of G (in the $*$-case, different from the obvious ones).

III.4.2 Theorem. *Let K be an idempotent-free LSLD-groupoid and I be a subgroupoid of $\mathrm{IA}(K)$. Put $G = I \cup K$. Then the following conditions are equivalent.*

(1) *There exists an operation $\boxdot$ on G such that $G(\boxdot)$ is an LSLD-groupoid, $K_G = K$, $I_G = I$ and $f \boxdot x = f(x)$ for all $f \in I$, $x \in K$.*
(2) *$L_x f L_x \in I$ for all $f \in I$, $x \in K$.*

Moreover, if the conditions are satisfied, the operation is unique and $x \boxdot f = L_x f L_x$ for all $f \in I$, $x \in K$.

Proof. Let $x \in K$. We need to extend the left translation L_x of x in K to a left translation L_x^G of x in G. According to the equation $x \boxdot (f \boxdot y) = (x \boxdot f) \boxdot (x \boxdot y)$, we require

$L_x f = L_x^G(f) L_x$ for every $f \in I$, and so we must put necessarily $x \boxdot f = L_x f L_x$ for every $f \in I$. This is possible, iff $L_x f L_x \in I$ for all $f \in I$, $x \in K$.

In this way defined groupoid G is obviously left symmetric. We must verify the left distributivity.

For $x, y \in K$, $f \in I$ we have $x \boxdot (y \boxdot f) = L_x L_y f L_y L_x = L_x L_y L_x L_x f L_x L_x L_y L_x = L_{xy} L_x f L_x L_{xy} = (x \boxdot y) \boxdot (x \boxdot f)$, further $x \boxdot (f \boxdot y) = L_x f(y) = L_x f L_x (xy) = (x \boxdot f) \boxdot (x \boxdot y)$, and finally $f \boxdot (x \boxdot y) = f(xy) = f(x) f(y) = (f \boxdot x) \boxdot (f \boxdot y)$.

For $x \in K$, $f, g \in I$ we have $x \boxdot (f \boxdot g) = L_x f g f L_x = L_x f L_x L_x g L_x L_x f L_x = (x \boxdot f) \boxdot (x \boxdot g)$, further $f \boxdot (x \boxdot g) = f L_x g L_x f = L_{f(x)} f g f L_{f(x)} = (f \boxdot x) \boxdot (f \boxdot g)$, since $(f L_x)(y) = f(xy) = f(x) f(y) = (L_{f(x)} f)(y)$ for every $y \in K$, and finally $f \boxdot (g \boxdot x) = f(g(x)) = (fgf)(f(x)) = (f \boxdot g) \boxdot (f \boxdot x)$. $\square$

III.4.3 Definition. The groupoid G from III.4.2 is denoted $I \sqcup K$. The groupoid $\mathrm{IA}(K) \sqcup K$ is called the *full extension* of K (clearly, it always satisfies III.4.2 (2)).

$I \sqcup K$	$\dots$	f	g	$\dots$	x	y	$\dots$
$\dots$	$\dots$			$\dots$			$\dots$
f		$f \circ f$	$f \circ g$		$f(x)$	$f(y)$	
g		$g \circ f$	$g \circ g$		$g(x)$	$g(y)$	
$\dots$	$\dots$			$\dots$			$\dots$
x		$L_x f L_x$	$L_x g L_x$		xx	xy	
y		$L_y f L_y$	$L_y g L_y$		yx	yy	
$\dots$	$\dots$			$\dots$			$\dots$

III.4.4 Example. Let ε be a pairing on a non-empty set K. We equip the set K with such operation that $L_x = o_\varepsilon$ for all $x \in K$. Easily, K is an LSLD-groupoid and $\mathrm{IA}(K) \simeq \mathrm{Inv}(|K|)$. The full extension $G = \mathrm{Inv}(|K|) \sqcup K$ is subdirectly irreducible and it is the largest subdirectly irreducible LSLD-groupoid with $|K|$ non-idempotent elements (see III.3.6).

Proof of subdirect irreducibility. Let ν be a congruence on G.

(i) If $(x, xx) \in \nu$ for some $x \in K$, then multiplying by $f = (x\ y)(xx\ yy)$ from the left for all $y \in K$ we get $(y, yy) = (f(x), f(xx)) \in \nu$. Hence $ip_G \subseteq \nu$.

(ii) If $(x, y) \in \nu$ for some $x, y \in K$, $y \neq x \neq yy$, then multiplying by $f = (y\ yy)$ from the left we get $(x, yy) = (f(x), f(y)) \in \nu$ and thus $(y, yy) \in \nu$. Then (i) makes the job.

(iii) If $(f, g) \in \nu$ for some $f, g \in I$, $f \neq g$, then there exist $x \in K$ such that $f(x) \neq g(x)$. Multiplying by x from the right we get $(f(x), g(x)) \in \nu$ and (i) or (ii) works.

(iv) If $(f, x) \in \nu$ for some $f \in I$, $x \in K$, then multiplying by x from the left we get $(f, xx) = (xf, xx) \in \nu$ and thus $(x, xx) \in \nu$. Then (i) solves the problem. $\square$

III.4.5 Remark. The full extension of an idempotent-free LSLD-groupoid K without a proper definable subset (cf. III.1.2) need not to be necessarily subdirectly irreducible (see the next section for an example). On the other hand, if $\mathrm{IA}(K)$ is rich enough, even somewhat bad-seeming K can form a subdirectly irreducible LSLD-groupoid (cf. III.4.4).

III.4.6 Main theorem. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid. Then there is a subgroupoid I of $\mathrm{IA}(K_G)$ such that G is isomorphic to $I \sqcup K_G$.*

Proof. By III.3.6, I_G is isomorphic (via φ) to a subgroupoid I of $\mathrm{Inv}(K_G, ip_{K_G})$. Then $I \subseteq \mathrm{IA}(K)$ (according to I.1.1). By III.3.5, $fx = (\varphi f)(x)$ for all $f \in I_G$, $x \in K_G$. Now, applying the uniqueness from III.4.2, we get $G \simeq I \sqcup K_G$. $\square$

III.4.7 Remark. When characterizing subdirectly irreducible non-idempotent LSLD-groupoids, according to the adjoining theorem III.2.7, it is enough to describe only such groupoids containing no left absorbing element. Therefore we will usually use $\mathrm{IA}^*(K_G)$ instead of $\mathrm{IA}(K_G)$.

III.4.8 Theorem. *Let G be a subdirectly irreducible non-idempotent LSLD-groupoid. Then every subgroupoid $H \supset G$ of the full extension $\mathrm{IA}(G) \sqcup G$ is subdirectly irreducible.*

Proof. Reformulated extension theorem III.1.4. □

5. Few non-idempotent elements

We are going to apply the developed theory to characterize subdirectly irreducible LSLD-groupoids with 2, 4 and 6 non-idempotents. Also existence of small subdirectly irreducible idempotent-free LSLD-groupoids is going to be discussed. For this purpose, we develop a general method, roughly described as follows.

Given k an even (see III.1.1 (3)) number of demanded non-idempotent elements. According to III.4.6, the searched groupoids are of the form $I \sqcup K$ such that K is an idempotent-free LSLD-groupoid containing no proper definable subset (III.1.2) and I is a subgroupoid of $\mathrm{IA}(K)$. In a view of III.4.7, we can suppose that I is a subgroupoid of $\mathrm{IA}^*(K)$.

(1) Describe all LSLDI-groupoids with $k/2$ elements containing no proper definable subset (III.1.2).
(2) Expand the found groupoids in all possible ways to k-element idempotent-free LSLD-groupoids containing no proper definable subset.
(3) Describe $\mathrm{Inv}^*(k)$ and its subgroupoids.
(4) Compute $\mathrm{IA}^*(K)$ for every found LSLD-groupoid K.
(5) Investigate all groupoids $I \sqcup K$, K being a found LSLD-groupoid and I being a subgroupoid of $\mathrm{IA}^*(K)$, whether they are well-defined (III.4.2) and subdirectly irreducible.

Note that, in a view of III.4.8, once found subdirectly irreducible $K \sqcup I$, then for all $J \supseteq I$ it is enough to check the property III.4.2 (2) only.

When characterizing subdirectly irreducible idempotent-free LSLD-groupoids, we can make few simplifications.

(1) Describe all left-ideal free LSLDI-groupoids with $k/2$ elements (III.1.3).
(2) Expand the found groupoids in all possible ways to k-element idempotent-free LSLD-groupoids containing no proper definable subset.
(3) For every found LSLD-groupoid check, whether it is subdirectly irreducible.

III.5.1 Two non-idempotents. According to III.3.7, a subdirectly irreducible LSLD-groupoid with two non-idempotent elements consists of at most four elements. We have already discussed in III.1.5, III.2.9 and III.2.10 that

$$\mathbf{T},\ \mathbf{T}[id_{\mathbf{T}}],\ \mathbf{T}[o_{\mathbf{T}}]\ \text{ and }\ \mathbf{T}[id_{\mathbf{T}}][o_{\mathbf{T}_1}]$$

are the only ones (up to an isomorphism).

III.5.2 Four non-idempotents. Let $ip = \bigcup_{k=0}^{1}\{k,\widetilde{k}\}^2$ be a pairing on the set $K = \{0,\widetilde{0},1,\widetilde{1}\}$.

(1) By I.1.2, $\mathbf{S}$ is the only one two-element LSLDI-groupoid.

(2) There are three possibilities to expand $\mathbf{S}$ to an idempotent-free LSLD-groupoid such that $ip_K = ip$.

K_1	0	$\widetilde{0}$	1	$\widetilde{1}$
$0,\widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$1,\widetilde{1}$	$\widetilde{0}$	0	$\widetilde{1}$	1

K_2	0	$\widetilde{0}$	1	$\widetilde{1}$
$0,\widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$1,\widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1

K_3	0	$\widetilde{0}$	1	$\widetilde{1}$
$0,\widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$
$1,\widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1

K_2 contains a proper subset $\{1,\widetilde{1}\}$ definable by the formula $(\exists y)(yx = x)$, and so it cannot form a subdirectly irreducible LSLD-groupoid. Neither do the other.

(3) According to III.3.7, $|\operatorname{Inv}^*(4)| = 4$. We designate $a = (0\ \widetilde{0})$, $b = (1\ \widetilde{1})$, $c = (0\ 1)(\widetilde{0}\ \widetilde{1})$, $d = (0\ \widetilde{1})(\widetilde{0}\ 1)$ the elements of $I = \operatorname{Inv}^*(K, ip)$.

$I(\circ)$	a	b	c	d
a	a	b	d	c
b	a	b	d	c
c	b	a	c	d
d	b	a	c	d

It is easy to see that I contains three subgroupoids

$$I_1 = \{a,b\},\ I_2 = \{c,d\},\ I_3 = \{a,b,c,d\}.$$

(4) Easily, $\mathrm{IA}^*(K_1) = \mathrm{IA}^*(K_3) = I$.

(5) Neither K_1, nor K_3, is subdirectly irreducible. Now, we check all $I_i \sqcup K_j$. Since both $I_1 \sqcup K_1$, $I_1 \sqcup K_3$ contain a proper left ideal $\{0,\widetilde{0}\}$, they cannot be subdirectly irreducible. The groupoid $I_2 \sqcup K_1$ possesses left absorbing elements c, d, because $L_x = o_{K_1}$ and thus $L_x c L_x = c$ for any $x \in K_1$. So $I_2 \sqcup K_1$ cannot be subdirectly irreducible because of III.2.7. It is easy to check that the remaining combinations are both well-defined and subdirectly irreducible.

Consequently, $I_2 \sqcup K_3$, $I_3 \sqcup K_1$ and $I_3 \sqcup K_3$ (and the groupoids obtained from them by the adjoining theorem III.2.7) are the only (up to an isomorphism) subdirectly irreducible LSLD-groupoids with four non-idempotent elements.

III.5.3 Six non-idempotents. Let $ip = \bigcup_{k=0}^{2}\{k,\widetilde{k}\}^2$ be a pairing on the set $K = \{0,\widetilde{0},1,\widetilde{1},2,\widetilde{2}\}$.

(1) By I.1.9, $\mathbf{S}_1$ and $\mathbf{S}_3$ are the only three-element LSLDI-groupoids containing no proper definable subset.

(2) There are the following possibilities to expand them to an idempotent-free LSLD-groupoid such that $ip_K = ip$. Clearly, none of them contains a proper definable subset.

K_1	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0,\widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2
$1,\widetilde{1}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2
$2,\widetilde{2}$	$\widetilde{0}$	0	$\widetilde{1}$	1	$\widetilde{2}$	2

K_2	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0,\widetilde{0}$	$\widetilde{0}$	0	$\widetilde{1}$	1	2	$\widetilde{2}$
$1,\widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	$\widetilde{2}$	2
$2,\widetilde{2}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{2}$	2

K_3	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0,\widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$	2	$\widetilde{2}$
$1,\widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	2	$\widetilde{2}$
$2,\widetilde{2}$	0	$\widetilde{0}$	1	$\widetilde{1}$	$\widetilde{2}$	2

K_4	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$
$0,\widetilde{0}$	$\widetilde{0}$	0	$\widetilde{2}$	2	$\widetilde{1}$	1
$1,\widetilde{1}$	$\widetilde{2}$	2	$\widetilde{1}$	1	$\widetilde{0}$	0
$2,\widetilde{2}$	$\widetilde{1}$	1	$\widetilde{0}$	0	$\widetilde{2}$	2

(3) According to III.3.7, $|\operatorname{Inv}^*(6)| = 18$. It is an easy (but a bit long) computation that $I = \operatorname{Inv}^*(K, ip)$ contains precisely the following subgroupoids:

$$
\begin{aligned}
I_1 &= \{(x\ \widetilde{x}) : x = 0,1,2\},\\
I_2 &= \{(x\ \widetilde{x})(y\ \widetilde{y}) : x,y = 0,1,2,\ x \neq y\},\\
I_{3,1} &= \{(x\ y)(\widetilde{x}\ \widetilde{y}) : x,y = 0,1,2,\ x \neq y\},\\
I_{3,2} &= \{(0\ \widetilde{1})(\widetilde{0}\ 1), (0\ \widetilde{2})(\widetilde{0}\ 2), (1\ 2)(\widetilde{1}\ \widetilde{2})\},\\
I_{3,3} &= \{(0\ \widetilde{1})(\widetilde{0}\ 1), (1\ \widetilde{2})(\widetilde{1}\ 2), (0\ 2)(\widetilde{0}\ \widetilde{2})\},\\
I_{3,4} &= \{(0\ \widetilde{2})(\widetilde{0}\ 2), (1\ \widetilde{2})(\widetilde{1}\ 2), (0\ 1)(\widetilde{0}\ \widetilde{1})\},\\
I_3 &= \{(x\ y)(\widetilde{x}\ \widetilde{y}), (x\ \widetilde{y})(\widetilde{x}\ y) : x,y = 0,1,2,\ x \neq y\} = I_{3,1} \cup I_{3,2} \cup I_{3,3} \cup I_{3,4},\\
I_{4,1} &= \{(x\ \widetilde{y})(\widetilde{x}\ y)(z\ \widetilde{z}) : \{x,y,z\} = \{0,1,2\}\},\\
I_{4,2} &= \{(0\ 1)(\widetilde{0}\ \widetilde{1})(2\ \widetilde{2}), (0\ 2)(\widetilde{0}\ \widetilde{2})(1\ \widetilde{1}), (1\ \widetilde{2})(\widetilde{1}\ 2)(0\ \widetilde{0})\},\\
I_{4,3} &= \{(0\ 1)(\widetilde{0}\ \widetilde{1})(2\ \widetilde{2}), (1\ 2)(\widetilde{1}\ \widetilde{2})(0\ \widetilde{0}), (0\ \widetilde{2})(\widetilde{0}\ 2)(1\ \widetilde{1})\},\\
I_{4,4} &= \{(0\ 2)(\widetilde{0}\ \widetilde{2})(1\ \widetilde{1}), (1\ 2)(\widetilde{1}\ \widetilde{2})(0\ \widetilde{0}), (0\ \widetilde{1})(\widetilde{0}\ 1)(2\ \widetilde{2})\},\\
I_4 &= \{(x\ \widetilde{y})(\widetilde{x}\ y)(z\ \widetilde{z}), (x\ y)(\widetilde{x}\ \widetilde{y})(z\ \widetilde{z}) : \{x,y,z\} = \{0,1,2\}\} = I_{4,1} \cup \dots \cup I_{4,4},\\
&I_{3,i} \cup I_{4,i}, \quad i = 1,2,3,4,\\
&\text{all unions of } I_1, I_2, I_3, I_4.
\end{aligned}
$$

Clearly, $|I_1| = |I_2| = |I_{3,i}| = |I_{4,i}| = 3$, $i = 1,\dots,4$ and $|I_3| = |I_4| = 6$.

(4) Easily,

$$
\begin{aligned}
\mathrm{IA}^*(K_1) &= \mathrm{IA}^*(K_3) = I,\\
\mathrm{IA}^*(K_2) &= I_1 \cup I_2,\\
\mathrm{IA}^*(K_4) &= I_{3,1} \cup I_{4,1}.
\end{aligned}
$$

(5) No K_j, $j = 1,\dots,4$, is subdirectly irreducible. The table on the next page shows, which combinations $I_i \sqcup K_j$ are well-defined ("wd") and subdirectly irreducible ("SI"). The computation is easy, but long, and therefore we leave it to reader's own.

Note that all obtained subdirectly irreducible groupoids are pairwise non-isomorphic. The following table shows the number of pairwise non-isomorphic subdirectly irreducible LSLD-groupoids with the given number of idempotent and non-idempotent elements (including ones with left absorbing elements).

$\lvert K_G\rvert, \lvert I_G\rvert$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	2	1																		
4	0	0	1	2	3	4	2														
6	0	0	0	0	0	0	4	8	4	8	16	8	6	12	6	4	8	4	2	4	2

$\sqcup$	K_1	K_2	K_3	K_4
I_1	wd	wd	wd	
I_2	wd	wd	wd	
$I_{3,1}$	wd			wd
$I_{3,2}, I_{3,3}, I_{3,4}$	wd			
I_3	SI		SI	
$I_{4,1}$	wd			wd
$I_{4,2}, I_{4,3}, I_{4,4}$	wd			
I_4	SI		SI	
$I_{3,1} \cup I_{4,1}$	wd			wd
$I_{3,i} \cup I_{4,i}$	wd			
$I_1 \cup I_2$	wd	wd	wd	
$I_i \cup I_j$	SI		SI	
$I_i \cup I_j \cup I_k$	SI		SI	
$\mathrm{Inv}^*(K, ip)$	SI		SI	

(i, j, k being such that the table has sense)

III.5.4 Eight non-idempotents. The same procedure leads to the description of all subdirectly irreducible LSLD-groupoids with eight non-idempotent elements. Absence of any new ideas makes useless to do it without a computer. Thus we present only the first step — characterization of four-element LSLDI-groupoids containing no proper definable subset.

(1) Let $S = \{0, 1, 2, 3\}$ be some. Then all left translations have the same number of fix points, otherwise it has proper definable subsets of those with the same number of fix points. Since $aa = a$ for all $a \in S$, left translations may have four or two fix points. The first case leads to the right zero semigroup S_1 (see the table bellow). So, without lose of generality, let $0, 1$ be the only fix points of L_0, i.e. $L_0 = (2\ 3)$. Further, $10 = 01 \cdot 00 = 0 \cdot 10$ (left distributivity) and hence 10 is a fix point of L_0. Therefore $10 = 0$ (since $11 = 1$) and so $L_1 = L_0$. Now, also $L_2 = L_3$, otherwise the set $\{0, 1\}$ is definable by the formula $(\exists y \neq x)(L_x = L_y)$. Consequently, since 2 is a fix point of L_2 and 3 is a fix point of L_3, the only possibility is $L_2 = L_3 = (0\ 1)$. It is easy to check that the obtained groupoid S_2 is suitable.

S_1	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

S_2	0	1	2	3
0	0	1	3	2
1	0	1	3	2
2	1	0	2	3
3	1	0	2	3

Consequently, there is no subdirectly irreducible idempotent-free LSLD-groupoid with eight elements, because neither S_1, nor S_2, is left ideal-free.

III.5.5 Ten non-idempotents. It is easy to see that if G is an LSLD-groupoid containing no proper definable subset, then all its left ideals have the same number of elements. Hence, a p-element, p a prime, such LSLDI-groupoid is either a right zero semigroup, or it is left ideal-free. Again, we are going to prove that there is no ten-element subdirectly irreducible idempotent-free LSLD-groupoid. We follow the general method described in the beginning of this section.

(1) Using similar arguments like in III.5.4 (but with much more cases), we can prove that the only (up to an isomorphism) five-element LSLDI-groupoids containing no proper definable subset are the following. S_4 is an LSLDI-quasigroup and thus it is left ideal-free.

S_3	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	0	1	2	3	4

S_4	0	1	2	3	4
0	0	2	1	4	3
1	3	1	4	0	2
2	4	3	2	1	0
3	2	4	0	3	1
4	1	0	3	2	4

(2) Assume that $K = \{0, \widetilde{0}, 1, \widetilde{1}, 2, \widetilde{2}, 3, \widetilde{3}, 4, \widetilde{4}\}$ is an idempotent-free LSLD-groupoid with $ip_K = \bigcup_{k=0}^{4} \{k, \widetilde{k}\}^2$. Since $K/ip_K \simeq S_4$, without lose of generality we put $01 = \widetilde{2},\ 03 = \widetilde{4},\ 12 = \widetilde{4},\ 10 = \widetilde{3}$. Then $\widetilde{1}\widetilde{0} = 3,\ \widetilde{1}\widetilde{2} = 4$ and thus $20 = \widetilde{4},\ 21 = \widetilde{3}$, since L_0 is an automorphism. Also $30 = \widetilde{2},\ 21 = \widetilde{4},\ 40 = \widetilde{1},\ 42 = \widetilde{3}$, since L_2 is an automorphism. Now, the groupoid K is uniquely determined.

(3) We see that $\nu = \{0, 1, 2, 3, 4\}^2 \cup \{\widetilde{0}, \widetilde{1}, \widetilde{2}, \widetilde{3}, \widetilde{4}\}^2$ is a congruence on K and that $\nu \cap ip_K = id_K$. Hence K cannot be subdirectly irreducible.

III.5.6 Twelve non-idempotents. The same can be done for twelve non-idempotent elements again. Amazingly, we obtain a subdirectly irreducible idempotent-free LSLD-groupoid $\mathbf{K}$.

$\mathbf{K}$	0	$\widetilde{0}$	1	$\widetilde{1}$	2	$\widetilde{2}$	3	$\widetilde{3}$	4	$\widetilde{4}$	5	$\widetilde{5}$
$0, \widetilde{0}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{4}$	4	$\widetilde{5}$	5	$\widetilde{2}$	2	$\widetilde{3}$	3
$1, \widetilde{1}$	0	$\widetilde{0}$	$\widetilde{1}$	1	$\widetilde{5}$	5	$\widetilde{4}$	4	$\widetilde{3}$	3	$\widetilde{2}$	2
$2, \widetilde{2}$	$\widetilde{4}$	4	$\widetilde{5}$	5	$\widetilde{2}$	2	3	$\widetilde{3}$	$\widetilde{0}$	0	$\widetilde{1}$	1
$3, \widetilde{3}$	5	$\widetilde{5}$	4	$\widetilde{4}$	2	$\widetilde{2}$	$\widetilde{3}$	3	1	$\widetilde{1}$	0	$\widetilde{0}$
$4, \widetilde{4}$	$\widetilde{2}$	2	3	$\widetilde{3}$	$\widetilde{0}$	0	1	$\widetilde{1}$	$\widetilde{4}$	4	5	$\widetilde{5}$
$5, \widetilde{5}$	3	$\widetilde{3}$	$\widetilde{2}$	2	$\widetilde{1}$	1	0	$\widetilde{0}$	4	$\widetilde{4}$	$\widetilde{5}$	5

Note that $\mathbf{S}_3 \simeq (\mathbf{K}/ip_{\mathbf{K}})/\varepsilon$, ε being the pairing $\{0,1\}^2 \cup \{2,3\}^2 \cup \{4,5\}^2$, is the three-element commutative LSLDI-quasigroup. Unfortunately, no general method for finding subdirectly irreducible idempotent-free LSLD-groupoid in a similar way has been found yet.

6. FURTHER PROBLEMS

We state some problems that remain open and might be of interest.

(1) Develop a structure theory of both left and right symmetric left distributive groupoids. (Clearly, they must be idempotent.) How does look their variety?
(2) Characterize subdirectly irreducible idempotent-free LSLD-groupoids. Maybe, (1) can be useful.

(3) Use computer search for more examples of small subdirectly irreducible LSLD-groupoids.
(4) Investigate the group derived operations introduced in the section II.2 to obtain more examples of subdirectly irreducibile LSLD-groupoids (cf. III.1.8).
(5) Characterize all expansions of one element (resp. of a subset) of an LSLD-groupoid preserving the left symmetry and the left distributivity.
(6) Find better characterization of subdirectly irreducible non-idempotent LSLD-groupoids than III.4.6. Maybe, (2), (3), (4) or (5) can be useful.
(7) Describe all subdirectly irreducible LSLDI-groupoids.

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