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ON EQUATIONAL THEORY OF GROUP CONJUGATION

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Abstract. Given a group, there is a natural operation of conjugation, defined by $x * y = xyx^{-1}$. We study the variety generated by all $G(*)$, G a group. In particular, we are concerned about the question, whether this variety has finitely based equational theory.

We study the equations satisfied by group conjugation. Strictly speaking, for a group G we put $x*y = xyx^{-1}$ for all $x, y \in G$ and we ask, which equations hold in all groupoids $G(*)$, G a group. Particularly, does there exist a finite base of equations they satisfy?

It is easy to see that every $G(*)$ is

(1) *idempotent* (I), i.e. it satisfies the equation $xx \approx x$,
(2) *left distributive* (LD), i.e. it satisfies the equation $x \cdot yz \approx xy \cdot xz$,
(3) a *left quasigroup*, i.e. for every $a \in G$ the left translation $L_a : G \to G$, $x \mapsto ax$, is bijective.

It turns out that LDI left quasigroups satisfy the same equations as all $G(*)$ do (it means, they generate the same variety). However, it wasn't clear, whether they satisfy some additional equations, not following from LDI. In 1999, D. Larue found in [6] an infinite independent set of such equations of the form

$$\begin{aligned}(xy \cdot y)(x \cdot z) &\approx (xy)(yx \cdot z)\\ (yxy \cdot xy \cdot y)(yx \cdot z) &\approx (yxy \cdot xy)(yyx \cdot z)\\ (yyxy \cdot yxy \cdot xy \cdot y)(yyx \cdot z) &\approx (yyxy \cdot yxy \cdot xy)(yyyx \cdot z)\\ &\quad \text{etc.}\end{aligned}$$

We use this result to get a conjecture which implies that the equations of conjugation are not finitely based. We also provide a couple of properties of the variety generated by all $G(*)$.

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Our terminology and notation is rather standard, consult e.g. [1]. Letters in terms are right associated, e.g. $xyz = x \cdot yz = x(yz)$. A groupoid is said to be left cancellative (left divisible, resp.), if for every $a \in G$ the left translation L_a is one-one (onto, resp.).

1. The variety generated by conjugation

Let $\mathbf{FG}(X)$ denote a free group over a set X and $\mathbf{F}(X)$ the subgroupoid of $\mathbf{FG}(X)(*)$ generated by the set X.

Proposition 1. (A. Drápal, T. Kepka, M. Musílek)

1. *$\mathbf{F}(X)$ is free over X in the variety generated by all $G(*)$, G a group.*
2. *If $|X| \leq \aleph_0$, then $\mathbf{F}(X)$ can be embedded into $\mathbf{F}(x, y)$.*
3. *The groupoid $\mathbf{F}(x, y)$ is right cancellative.*

Proof. See [3]. □

Lemma 2. (T. Kepka) *Every LDI left cancellative groupoid can be embedded into an LDI left quasigroup.*

Proof. See [4]. □

Lemma 3. (D. Larue) *Every equation satisfied by LDI left cancellative groupoids is satisfied by LDI left divisible groupoids.*

Proof. See [6]. □

It is easy to check that if A is an LDI left quasigroup with pairwise distinct left translations, then A can be embedded into $\mathrm{Sym}(A)(*)$ (where $\mathrm{Sym}(A)$ denotes the symmetric group over A) by mapping an element a onto its left translation L_a. However, the LDI left quasigroup

	0	1	2
0	0	1	2
1	0	1	2
2	1	0	2

cannot be embedded into any $G(*)$, since $x * y = y$ iff $y * x = x$ for all $x, y \in G$, while $0 \cdot 2 = 2$ and $2 \cdot 0 = 1$.

Lemma 4. *Every LDI left quasigroup $A(\cdot)$ is a homomorphic image of a subgroupoid of $\mathbf{FG}(A)(*)$.*

Proof. Put $B = \{wxw^{-1} : x \in A, w \in \mathbf{FG}(A)\}$. Clearly, $B(*)$ is a subgroupoid of $\mathbf{FG}(A)(*)$.

For every $a, b \in A$, let $a\backslash b$ denote the (unique) $c \in A$ such that $a \cdot c = b$.

Now, we define by induction a mapping $f : B \to A$. For $x \in A$ put $f(x) = x$. If $f(w)$ is already defined, put $f(xwx^{-1}) = x \cdot f(w)$ and $f(x^{-1}wx) = x\backslash f(w)$. Clearly, f is a well-defined surjective mapping.

To prove $f(r * s) = f(r) \cdot f(s)$ for all $r, s \in B$, we use induction on the length of r. For every $s \in B$ and $x \in A$ we have $f(x * s) = f(xsx^{-1}) = x \cdot f(s) = f(x) \cdot f(s)$ directly from the definition of f. Now, let $r = xwx^{-1}$ and $f(w * u) = f(w) \cdot f(u)$ for every $u \in B$. Then $f(r * s) = f(xwx^{-1}sxw^{-1}x^{-1}) = x \cdot (f(w) \cdot (x\backslash f(s))) = (x \cdot f(w)) \cdot (x \cdot (x\backslash f(s))) = (x \cdot f(w)) \cdot f(s) = f(xwx^{-1}) \cdot f(s) = f(r) \cdot f(s)$. Finally, let $r = x^{-1}wx$ and $f(w * u) = f(w) \cdot f(u)$ for every $u \in B$. Then $x \cdot f(r * s) = x \cdot f(x^{-1}wxsx^{-1}w^{-1}x) = x \cdot (x\backslash(f(w) \cdot (x \cdot f(s)))) = f(w) \cdot (x \cdot f(s)) = (x \cdot (x\backslash f(w))) \cdot (x \cdot f(s)) = x \cdot ((x\backslash f(w)) \cdot f(s)) = x \cdot (f(r) \cdot f(s))$, and we obtain the desired using left cancellativity of A. □

The following theorem is an easy consequence of the preceeding lemmas.

Theorem 5. *The following classes of groupoids generate the same variety:*

(1) *all $G(*)$, G a group;*
(2) *the subgroupoid of $\mathbf{FG}(x, y)(*)$ generated by x, y;*
(3) *LDI left quasigroups;*
(4) *LDI left cancellative groupoids;*
(5) *LDI left divisible groupoids;*
(6) *LDI (both left and right) cancellative groupoids.*

Let us denote this variety $\mathfrak{Conj}$.

Remark. It follows from Kepka's lemma that LDI left quasigroups generate the quasivariety of LDI left cancellative groupoids (given by LDI and the quasiequation $xy \approx xz \;\to\; y \approx z$). However, the quasivariety generated by LDI left divisible groupoids is strictly bigger, since there are LDI left divisibles which are not left cancellative. On the other hand, the quasivariety generated by all $G(*)$ is strictly smaller, since it satisfies the quasiequation $xy \approx y \;\to\; yx \approx x$.

We call a groupoid *medial*, if it satisfies the equation $xy \cdot uv \approx xu \cdot yv$. Indeed, medial idempotent (MI) groupoids are always left distributive.

Let R be a commutative ring with unit, M a module over R and $k \in R$. For all $x, y \in M$, put $x \circ_k y = (1 - k)x + ky$. It is easy to see that $M(\circ_k)$ is an MI groupoid. It is left cancellative (left divisible, resp.), iff the mapping $x \mapsto kx$ is injective (surjective, resp.) on M.

Theorem 6. *The variety of medial idempotent groupoids is generated by each of the following sets of groupoids:*

(1) *the set of all $\mathbb{Q}(\circ_k)$, $k \in \mathbb{Z}$ (where $\mathbb{Q}$ denotes the field of rational numbers);*
(2) *the set containing a single groupoid $R(\circ_k)$, where R is a field extension of the field $\mathbb{Q}$ and $k \in R$ is not algebraic over $\mathbb{Q}$.*

Proof. Denote $\mathfrak{M}$ the assumed generating set. Let an equation $t(x_1, \dots, x_n) \approx s(x_1, \dots, x_n)$ holds in $\mathfrak{M}$. We prove that $MI \vdash t \approx s$.

Let m denote the greater of the depths of t, s. Let t' (s', resp.) be a term such that the corresponding binary tree is complete of depth m and $I \vdash t \approx t'$ ($I \vdash s \approx s'$, resp.). E.g., we may choose $(xy \cdot (uv \cdot u))'$ to be $(xy \cdot xy)(uv \cdot uu)$ or $(xx \cdot yy)(uv \cdot uu)$. Clearly, $\mathfrak{M} \models t' \approx s'$ and it is enough to prove that $MI \vdash t' \approx s'$.

We say that a certain occurence of a variable in a term has property D_i, if we reach it in the corresponding complete binary tree by turning i-times right and $(m-i)$-times left. Let ξ_{ij} (ζ_{ij}, resp.) denote the number of variables x_j having the property D_i in the term t' (s', resp.). It follows that $t'(a_1, \dots, a_n) = \sum_{i=0}^m k^i(1-k)^{m-i}(\sum_{j=1}^n \xi_{ij} a_j)$ and $s'(a_1, \dots, a_n) = \sum_{i=0}^m k^i(1-k)^{m-i}(\sum_{j=1}^n \zeta_{ij} a_j)$ in $R(\circ_k)$ for every field R and $k, a_1, \dots, a_n \in R$

We show that $\xi_{ij} = \zeta_{ij}$. Put $\delta_{ij} = \xi_{ij} - \zeta_{ij}$. From $\mathfrak{M} \models t' \approx s'$ it follows that $\sum_{i=0}^m \sum_{j=1}^n k^i(1-k)^{m-i} a_j \ \delta_{ij} = 0$ for every assumed k and $a_1, \dots, a_n$. Fix $l \in \{1, \dots, n\}$ and put $a_l = 1$ and $a_j = 0$ for $j \neq l$. In the first case, choosing $k = 2, \dots, m+2$, we obtain $m+1$ linear equations for $\delta_{0l}, \dots, \delta_{ml}$. It is easy to see that the only solution in $\mathbb{Q}$ is $\delta_{0l} = \dots = \delta_{ml} = 0$. In the second case, since no polynomial can have k as its root, we get all δ_{il} equal 0 directly.

We have proven that t' and s' may differ only by permuting occurences of variables satisfying certain D_i. So it is enough to prove the following. If u, v are two terms such that their corresponding binary trees are complete of depth m and they difer just by transposing two occurences of the variables x, y satisfying D_i for some i, then $MI \vdash u \approx v$.

By induction on m. For $m \leq 2$ it is clear. If $m > 2$, write $u = u_1u_2 \cdot u_3u_4$ and $v = v_1v_2 \cdot v_3v_4$. Without loss of generality, suppose x occurs first.

(a) Both x, y occur in the left (right, resp.) subterm of u. Use induction assumption on the left (right, resp.) subterm.
(b) x occurs in u_1, y occurs in u_3. By mediality, $u \approx u_1u_3 \cdot u_2u_4$, by induction assumption on u_1u_3 we get $u \approx v_1v_3 \approx v_2v_4$ and again mediality yields $MI \vdash u \approx v$.
(c) x occurs in u_1, y occurs in u_4. Let w be the term obtained from u_1u_2 by transposition of the occurence of x and the rightmost variable with

D_i. Certainly, then this x is in the right subterm of w. Now, use induction assumption to get $MI \vdash u_1u_2 \approx w$, then proceed as in (b) on the term $w \cdot u_3u_4$ to get $w' \approx v_3v_4$ (transposing that x and y) and finally, use induction assumption on w' to get $MI \vdash w' \approx v_1v_2$.

(d) x occurs in u_2, y occurs in u_3. Proceed as in (c). We must take special care on the case when x is the rightmost variable in u_2 and y is the leftmost variable in u_3.

□

Corollary 7. *The variety of medial idempotent groupoids is a subvariety of the variety* $\mathfrak{Conj}$.

Proof. $R(\circ_k)$ is an LDI left quasigroup for every field R and $k \in R \setminus \{0\}$. □

It follows from the Proposition 1 that $\mathfrak{Conj}$ is a subvariety of the variety generated by LDI right cancellative groupoids. We show that the inclusion is proper (and thus answer a question of A. Drápal, T. Kepka, M. Musílek from [3]). The following groupoid is an LDI right quasigroup which does not satisfy the first Larue's equation from the list in the introduction: $((0 \cdot 1) \cdot 1)(0 \cdot 2) = 4 \neq 3 = (0 \cdot 1)((1 \cdot 0) \cdot 2)$.

	0	1	2	3	4	5	6	7	8
0	0	2	6	6	6	0	2	0	2
1	3	1	5	5	5	3	1	3	1
2	6	7	2	4	3	8	0	1	5
3	8	5	4	3	2	1	7	6	0
4	1	0	3	2	4	6	5	8	7
5	5	3	1	1	1	5	3	5	3
6	2	6	0	0	0	2	6	2	6
7	7	4	8	8	8	7	4	7	4
8	4	8	7	7	7	4	8	4	8

The groupoid was found by a computer search using an automated model builder SEM by J. Zhang and H. Zhang [7] and it is a smallest such example. However, the following problem remains unsolved.

Problem. (A. Drápal, T. Kepka, M. Musílek; D. Larue) Do LDI right cancellative groupoids generate the variety of LDI groupoids? In particular, are free LDI groupoids right cancellative? How about LDI right divisible groupoids?

2. The equational theory of conjugation

Let A be an LDI groupoid and $c \in A$ such that $cx \neq c$ for all $x \in A \smallsetminus \{c\}$. We denote $A^{[c]}$ the following groupoid: the universe is $A \cup \{c'\}$ for $c' \notin A$ and the operation of $A^{[c]}$ is an extension of the operation on A defined by $c'x = cx$ for all $x \in A$, $xc' = xc$ for all $x \in A \smallsetminus \{c\}$ and $c'c' = cc' = c'$. It is easy to check that $A^{[c]}$ is an LDI groupoid.

$A^{[c]}$	c'	c	x
c'	c'	c	cx
c	c'	c	cx
x	xc	xc	x

In further text, let R always be an extension of the field $\mathbb{Q}$ containing all square roots of positive integers. For $k, c \in R$, we denote $R_k^{[c]}$ the groupoid $R(\circ_k)^{[c]}$.

Theorem 8. (D. Larue) *There are (logically) independent equations ε_n, $n \in \mathbb{N}$, satisfied in $\mathfrak{Conj}$ but not in the variety of LDI groupoids. Moreover, there exist $k_n \in R$ such that for every $c \in R$ the groupoid $R_{k_n}^{[c]}$ satisfies all the equations ε_i, $i \in \mathbb{N} \smallsetminus \{n\}$, but not the equation ε_n.*

Proof. See [6]. □

Remark. The equations ε_n can be taken e.g. from the list in the introduction omitting every $(2\ell)^2$-th one, $\ell \in \mathbb{N}$. In such a case one can choose $k_n = 1 + \frac{1}{\sqrt{\ell}}$, whenever ε_n is ℓ-th in the list.

Conjecture 9. (D. Larue) *Let $c \in R$. Let an equation ε holds in $\mathfrak{Conj}$. Then $R_k^{[c]} \not\models \varepsilon$ for at most finitely many $k \in R$.*

The theorem and validity of the conjecture directly imply that $\mathfrak{Conj}$ is not finitely based — if there were a finite base $\mathfrak{B}$, then $R_k^{[c]} \models \mathfrak{B}$ for all but finitely many k, hence $R_k^{[c]} \models \varepsilon_n$ for all n and all but finitely many k, so we get a contradiction with the theorem.

Below, $x_1, x_2, \ldots$ always denote variables. By n-ary term t we mean a binary term in variables $x_1, \ldots, x_n$. Without loss of generality, suppose x_n is the rightmost variable of t. Denote t_1 the left subterm of t, t_2 the left subterm of the right subterm of t and so forth. Hence, t is a product $t_1(t_2(\ldots(t_m x_n)))$ for some m.

Let $\varphi : A^{[c]} \to A$ be the mapping defined by $\varphi(x) = x$ for all $x \in A$ and $\varphi(c') = c$. It is easy to check that φ is an onto homomorphism.

Lemma 10. *Let t, s be n-ary terms such that $A \models t \approx s$. Then $A^{[c]} \not\models t \approx s$, iff there are $a_1, \dots, a_n \in A^{[c]}$ such that one of the following conditions takes place:*

(1) $t(a_1, \dots, a_n) = c'$ *and* $s(a_1, \dots, a_n) = c$;
(2) $t(a_1, \dots, a_n) = c$ *and* $s(a_1, \dots, a_n) = c'$.

Proof. Take $a_1, \dots, a_n \in A^{[c]}$ such that $u = t(a_1, \dots, a_n) \neq s(a_1, \dots, a_n) = v$. By assumption, $\varphi(u) = \varphi(v)$, hence $\{u, v\} = \{c, c'\}$. □

First of all, observe that for $a, b \in A^{[c]}$ we have $ab = c'$, iff $b = c'$ and $a \in \{c, c'\}$. Thus, for n-ary terms $t = t_1(t_2(\dots(t_m x_n)))$ and $s = s_1(s_2(\dots(s_l x_n)))$, the condition (1) is equivalent to $a_n = c'$ and $t_i(a_1, \dots, a_n) \in \{c, c'\}$ for all i and $s_i(a_1, \dots, a_n) \notin \{c, c'\}$ for at least one i. Let $b_i = \varphi(a_i)$, $i = 1, \dots, n$. If (1) holds, then $b_n = c$ and $t_i(b_1, \dots, b_n) = c$ for all i and $s_i(a_1, \dots, a_n) \neq c$ for at least one i. On the other hand, given such $b_1, \dots, b_n \in A$, then $a_1 = b_1, \dots, a_{n-1} = b_{n-1}, a_n = c'$ satisfy the condition (1). Consequently, (1) is equivalent to existence of $b_1, \dots, b_{n-1} \in A$ such that

(i) $t_i(b_1, \dots, b_{n-1}, c) = c$ for all i;
(ii) $s_i(b_1, \dots, b_{n-1}, c) \neq c$ for at least one i.

Note that for every n-ary term r there are polynomials $\rho_1, \dots, \rho_n$ over the field R so that in $R(\circ_k)$

$$r(a_1, \dots, a_n) = \sum_{j=1}^{n} \rho_j(k) a_j.$$

Lemma 11. *Let $t = t_1(t_2(\dots(t_m x_n))), s = s_1(s_2(\dots(s_l x_n)))$ be n-ary terms such that $R(\circ_k) \models t \approx s$. Let $\tau_{i1}, \dots, \tau_{in}$ be the polynomials corresponding to the terms t_i and $\sigma_{i1}, \dots, \sigma_{in}$ be the polynomials corresponding to the terms s_i. Then the following conditions are equivalent:*

(1) *There are $a_1, \dots, a_n \in R_k^{[c]}$ so that $t(a_1, \dots, a_n) = c'$ and $s(a_1, \dots, a_n) = c$.*
(2) *There are $b_1, \dots, b_{n-1} \in R$ so that $t_i(b_1, \dots, b_{n-1}, c) = c$ for all i and $s_i(b_1, \dots, b_{n-1}, c) \neq c$ for at least one i.*
(3)

$$rank \begin{pmatrix} \tau_{11}(k) & \dots & \tau_{1,n-1}(k) \\ & \dots & \\ \tau_{m1}(k) & \dots & \tau_{m,n-1}(k) \end{pmatrix} < rank \begin{pmatrix} \tau_{11}(k) & \dots & \tau_{1,n-1}(k) \\ & \dots & \\ \tau_{m1}(k) & \dots & \tau_{m,n-1}(k) \\ \sigma_{11}(k) & \dots & \sigma_{1,n-1}(k) \\ & \dots & \\ \sigma_{l1}(k) & \dots & \sigma_{l,n-1}(k) \end{pmatrix}$$

Proof. The equivalence of (1) and (2) was proved above (in general). Now, (2) is equivalent to the fact that the system of m linear equations over the field R

$$\left(\begin{array}{ccc|c} \tau_{11}(k) & \dots & \tau_{1,n-1}(k) & (1-\tau_{1n}(k))c \\ & \dots & & \dots \\ \tau_{m1}(k) & \dots & \tau_{m,n-1}(k) & (1-\tau_{mn}(k))c \end{array}\right)$$

has a solution $b_1, \dots, b_{n-1} \in R$, but the system of l linear equation over the field R

$$\left(\begin{array}{ccc|c} \sigma_{11}(k) & \dots & \sigma_{1,n-1}(k) & (1-\sigma_{1n}(k))c \\ & \dots & & \dots \\ \sigma_{l1}(k) & \dots & \sigma_{l,n-1}(k) & (1-\sigma_{ln}(k))c \end{array}\right)$$

has not the solution $b_1, \dots, b_{n-1}$. This gives the equivalence of (2) and (3). □

Corollary 12. *Under the same assumptions, define matrices $T(k) = (\tau_{ij}(k))$, $S(k) = (\sigma_{ij}(k))$ and $U(k) = \binom{T(k)}{S(k)}$. Then the following conditions are equivalent:*

(1) $R_k^{[c]} \models t \approx s$;
(2) $rank(T(k)) = rank(S(k)) = rank(U(k))$.

So, we got an equivalent form of Larue's conjecture.

Conjecture 13. *Let an equation $t \approx s$ hold in $\mathfrak{Conj}$. Then $rank(T(k)) < rank(U(k))$ for at most finitely many $k \in R$.*

3. A conjecture

Conjecture 14. *Let $w \in \mathbf{F}(X)$. Then there are two words $u, v \in \mathbf{F}(X)$, each of them shorter than w, such that $w = u * v$.*

The conjecture implies two interesting facts. First, there is an algorithm to decide, whether a given word from $\mathbf{FG}(X)$ belongs to $\mathbf{F}(X)$. Next, there is a simple direct proof of the fact that every LDI left divisible groupoid (and hence also every LDI left quasigroup) is a homomorphic image of some $G(*)$.

4. On the role of idempotency

It is interesting that the equational theories of LD groupoids and LD left cancellative groupoids coincide (P. Dehornoy [2]), but the equational theory of LDI groupoids is strictly smaller than that of LDI left cancellative groupoids. On the other hand, the equational theory of LD left cancellative groupoids is strictly smaller then that of LD left divisibles, but the equational theories of LDI left cancellative and LDI left divisible groupoids coincide. (P. Dehornoy proved even more — free LD groupoids are left cancellative.)

Consider the first Larue's equation ε_1 $(xy \cdot y)(xz) \approx (xy)(yx \cdot z)$ (found independently in [6] and [3]).

Proposition 15. (1) *ε_1 is satisfied in every LD left divisible groupoid.*

(2) *ε_1 is satisfied in every LDI left cancellative groupoid.*

(3) *There is an LD left cancellative groupoid which does not satisfy ε_1.*

Proof. (1) Let A be an LD left divisible groupoid and $a, b, c \in A$. Take $d \in A$ such that $c = bd$. Then $(ab \cdot b)(ac) = (ab \cdot b)(ab \cdot ad) = (ab)(b \cdot ad) = (ab)(ba \cdot bd) = (ab)(ba \cdot c)$.

(2) $b((ab \cdot b)(ac)) = (b(ab \cdot b))(b \cdot ac) = ((b \cdot ab)(bb))(ba \cdot bc) = ((b \cdot ab)b)((ba \cdot b)(ba \cdot c)) = (b \cdot ab)(b \cdot (ba \cdot c)) = b((ab)(ba \cdot c))$ and use left cancellativity. Alternatively, verify the equation for all $G(*)$, G a group.

(3) Since the equational theories of LD groupoids and LD left cancellative groupoids coincide, it is enough to check that $R_k^{[c]} \not\models \varepsilon_1$ for every field R, $c \in R$ and $k \in \{\frac{1}{2}, 2\}$. However, it is not easy to find a particular example of an LD left cancellative groupoid which does not satisfy ε_1. Here it is: take a group G, an element $e \in G$ and an endomorphism f of G such that $ef^2(x) = f^2(x)e$ for all $x \in G$ and $ef(e)e = f(e)ef(e)$. Put $x \circ y = xf(y)ef(x^{-1})$ for all $x, y \in G$. Then $G(\circ)$ is an LD groupoid (see [5], IV.2.2). Moreover, $G(\circ)$ is left cancellative (left divisible, resp.) iff f is injective (surjective, resp.). Now, take for G the symmetric group on ω, $e = (0\ 1)$ and f the endomorphism given by the transformation $i \mapsto i + 1$ of ω. Then $G(\circ)$ is an LD left cancellative groupoid and choosing $a = (0\ 1\ 2)$, $b = (0\ 2\ 1)$ and $c = (0\ 1)$, one can verify $G(\circ) \not\models \varepsilon_1$. □

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