Mocninné řady I.

Určete poloměr konvergence mocninné řady a konvergenci v krajních bodech konvergenčního intervalu.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$,
2. $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}(x+1)^n$,
3. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n$,
4. $\sum_{n=1}^{\infty} \alpha^{n^2} x^n, \quad 0 < \alpha < 1$,
5. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$,
6. $\sum_{n=1}^{\infty} \frac{n!}{a^{n^2}}x^n, \quad (a > 1)$,
7. $\sum_{n=1}^{\infty} \left(\frac{a^n}{n}+\frac{b^n}{n^2}\right) x^n, \quad a > 0, b > 0$,
8. $\sum_{n=1}^{\infty} \frac{x^n}{a^n+b^n}, \quad a > 0, b > 0$,
9. $\sum_{n=1}^{\infty} \frac{3^{-\sqrt{n}}x^n}{\sqrt{n^2+1}}$,
10. $\sum_{n=1}^{\infty} \left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right) x^n$,
11. $\sum_{n=1}^{\infty} \frac{[3+(-1)^n]^n}{n}x^n$.

Řešení:

1.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)^p}}{\frac{x^n}{n^p}}\right| = \lim_{n\to\infty} |x|\frac{n^p}{(n+1)^p} = |x| < 1 \Rightarrow R = 1, \ x \in (-1,1).$$

$$x = -1 : \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}, \quad \text{řada konverguje pro } p > 0 \text{ dle Leibnizova kritéria.}$$

$$x = 1 : \sum_{n=1}^{\infty} \frac{1}{n^p}, \ \text{tedy řada konverguje pro } p > 1.$$

$p \leq 0 : x \in (-1,1), \quad 0 < p \leq 1 : x \in [-1,1), \quad p > 1 : x \in [-1,1].$

2.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{n+1}+(-2)^{n+1}}{n+1}(x+1)^{n+1}}{\frac{3^n+(-2)^n}{n}(x+1)^n}\right| = \lim_{n\to\infty}\frac{(3^{n+1}+(-2)^{n+1})n}{(3^n+(-2)^n)(n+1)}|x+1|$$

$$= \lim_{n\to\infty}\frac{3^n}{3^n}\cdot\frac{3-2\left(\frac{-2}{3}\right)^n}{1+\left(\frac{-2}{3}\right)^n}\cdot\frac{n}{n+1}|x+1| = 3|x+1| < 1 \Rightarrow R = \frac{1}{3},\ x\in(-\frac{4}{3},-\frac{2}{3}).$$

$$x = -\frac{4}{3} : \sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}\left(\frac{-1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \Rightarrow \text{řada konverguje},$$

jelikož řada $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ konverguje dle Leibnizova kritéria a $\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \leq \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n < \infty.$

$$x = \frac{-2}{3} : \sum_{n=1}^{\infty}\frac{3^n+(-2)^n}{n}\left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n} + \sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n} \Rightarrow \text{řada diverguje},$$

jelikož řada $\sum_{n=1}^{\infty}\frac{1}{n}$diverguje a $\sum_{n=1}^{\infty}\frac{\left(\frac{2}{3}\right)^n}{n}$ konverguje.

$\Rightarrow$ Mocninná řada konverguje pro $x\in[-\frac{4}{3},-\frac{2}{3})$.

3.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2n+2)!}x^{n+1}}{\frac{(n!)^2}{(2n)!}x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}x\right| = \frac{|x|}{4} < 1 \Rightarrow R = 4,\ x\in(-4,4).$$

Jelikož $a_n = \frac{(n!)^2}{(2n)!}(\pm 4)^n \geq \pm\frac{(n!)^2}{(2^n n!)^2}(4)^n = 1$ nesplňuje nutnou podmínku konvergence ani pro $x=4$ ani pro $x=-4$, tak mocninná řada konverguje pro $x\in(-4,4)$.

4.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha^{(n+1)^2}x^{n+1}}{\alpha^{n^2}x^n}\right| = \lim_{n\to\infty}\alpha^{2n+1}|x| = 0 < 1 \Rightarrow R = \infty,\ x\in\mathbb{R}.$$

5. I.

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}|x|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n|x| = |x|e < 1 \Rightarrow R = \frac{1}{e},\ x\in(-\frac{1}{e},\frac{1}{e}).$$

II.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n+1}\right)^{(n+1)^2} x^{n+1}}{\left(1+\frac{1}{n}\right)^{n^2} x^n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{n+2}{n+1}\right)^{n^2+2n+1} x}{\left(\frac{n+1}{n}\right)^{n^2}}\right|$$

$$= |x| \lim_{n\to\infty}\left(\frac{n(n+2)}{(n+1)^2}\right)^{n^2}\cdot\left(\frac{n+2}{n+1}\right)^{2n+1}$$

$$= |x| \lim_{n\to\infty}\left(1-\frac{1}{n^2+2n+1}\right)^{n^2}\cdot\left(1+\frac{1}{n+1}\right)^{2n+1} = |x|e < 1 \Rightarrow R = \frac{1}{e}, x \in (-\frac{1}{e}, \frac{1}{e}),$$

$$x = \frac{1}{e} : \sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n},$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n = \lim_{n\to\infty} e^{n\ln\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)}$$

$$= \lim_{n\to\infty} e^{\frac{n\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}} = e^{\star} = e^{\frac{1}{2}} \neq 0 \ \Rightarrow \text{řada diverguje, totéž platí pro } x = -\tfrac{1}{e}.$$

$$\star = \lim_{n\to\infty}\frac{n\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)+\frac{n}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n+1}}{-\frac{1}{n^2}}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{-\frac{1}{n(1+n)}+\frac{1}{(n+1)^2}}{\frac{2}{n^3}} = \lim_{n\to\infty}\frac{(n-(n+1))n^3}{2n(n+1)^2} = \frac{1}{2}$$

$\Rightarrow$ mocninná řada konverguje pro $x \in (-\frac{1}{e}, \frac{1}{e})$.

6.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{a^{(n+1)^2}}x^{n+1}}{\frac{n!}{a^{n^2}}x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{a^{2n+1}}x\right| = 0, \Rightarrow R = \infty,\ x \in \mathbb{R}.$$

7. Označme $c = \max\{a,b\}$, pak

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{a^{n+1}}{n+1}+\frac{b^{n+1}}{(n+1)^2}\right)x^{n+1}}{\left(\frac{a^n}{n}+\frac{b^n}{n^2}\right)x^n}\right| = \left|\frac{\frac{a}{n+1}\left(\frac{a}{c}\right)^n+\frac{b}{(n+1)^2}\left(\frac{b}{c}\right)^n}{\frac{1}{n}\left(\frac{a}{c}\right)^n+\frac{1}{n^2}\left(\frac{b}{c}\right)^n}x\right|$$

$$= c|x| < 1 \Rightarrow \ x \in (-\frac{1}{c},\frac{1}{c}) = (-\frac{1}{\max\{a,b\}},\frac{1}{\max\{a,b\}}),$$

$$x = -\frac{1}{c} = \begin{cases}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{\left(\frac{b}{a}\right)^n(-1)^n}{n^2}, & a\ge b\\ \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{a}{b}\right)^n}{n}+\frac{(-1)^n}{n^2}, & a<b\end{cases} \ \Rightarrow \ \text{konverguje dle Leibnizova kritéria.}$$

$$x = \frac{1}{c} = \begin{cases}\sum_{n=1}^{\infty}\frac{1}{n}+\frac{\left(\frac{b}{a}\right)^n}{n^2}, & a\ge b \Rightarrow \text{divergence}\\ \sum_{n=1}^{\infty}\frac{\left(\frac{a}{b}\right)^n}{n}+\frac{1}{n^2}, & a<b \Rightarrow \text{konvergence}\end{cases}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x\in[-\frac{1}{a},\frac{1}{a}), \quad a\ge b \text{ a } x\in[-\frac{1}{b},\frac{1}{b}], \quad b>a.$$

8. Označme $c = \max\{a,b\}$, pak

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{a^{n+1}+b^{n+1}}}{\frac{x^n}{a^n+b^n}}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{a}{c}\right)^n+\left(\frac{b}{c}\right)^n}{a\left(\frac{a}{c}\right)^n+b\left(\frac{b}{c}\right)^n}x\right|$$

$$= \frac{1}{c}|x| < 1 \Rightarrow \ x\in(-c,c) = (-\max\{a,b\},\max\{a,b\}),$$

$$x = \pm c: \lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{c^n}{a^n+b^n} = 1 \ne 0 \Rightarrow \ \text{řada nekonverguje}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x\in(-\max\{a,b\},\max\{a,b\}).$$

9.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{-\sqrt{n+1}}}{\sqrt{(n+1)^2+1}}x^{n+1}}{\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}x^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{\frac{n^2+1}{n^2+2n+2}}}{3^{\sqrt{n+1}-\sqrt{n}}}x\right| = |x| < 1 \Rightarrow R = 1, x\in(-1,1),$$

$$x = \pm 1: \sum_{n=1}^{\infty}\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}(\pm1)^n \ \text{konverguje}, \ \text{jelikož} \ \left|\frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}\right| < \frac{1}{3^{\sqrt{n}}n} < \frac{1}{\sqrt{n}n} = \frac{1}{n^{\frac{3}{2}}}$$

$$\Rightarrow \text{mocninná řada konverguje pro } x\in[-1,1].$$

10.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n+1}\right)x^{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)x^n}\right| = \lim_{n\to\infty}\left(1+\frac{\frac{1}{n+1}}{\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)}\right)|x|$$

$$= |x| < 1 \Rightarrow R = 1, \ x\in(-1,1),$$

$$x = \pm1: \lim_{n\to\infty}a_n = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)(\pm1)^n \ne 0$$

$$\Rightarrow \text{mocninná řada konverguje pro } x\in(-1,1).$$

11.

$$\sum_{n=1}^{\infty} \frac{[3+(-1)^n]^n}{n} x^n = \sum_{n=1}^{\infty} \frac{[3+(-1)]^{2n-1}}{2n-1} x^{2n-1} + \sum_{n=1}^{\infty} \frac{[3+1]^{2n}}{2n} x^{2n} = \sum_{n=1}^{\infty} \frac{(2x)^{2n-1}}{2n-1} + \sum_{n=1}^{\infty} \frac{(4x)^{2n}}{2n} \Rightarrow$$

První řada konverguje pro $x \in (-\frac{1}{2}, \frac{1}{2})$, druhá pro $x \in (-\frac{1}{4}, \frac{1}{4})$,

tedy původní řada konverguje pro $x \in (-\frac{1}{4}, \frac{1}{4})$,

$$x = \pm\frac{1}{4} : \sum_{n=1}^{\infty} \frac{[3+(-1)^n]^n (\pm 1)^n}{4^n n} = \pm \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n}.$$

Jelikož první řada konverguje a druhá diverguje, tak původní řada diverguje

$\Rightarrow$ mocninná řada konverguje pro $x \in (-\frac{1}{4}, \frac{1}{4})$.

Mocninné řady II.

Sečtěte následující řady.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$,
2. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$,
3. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$,
4. $\sum_{n=1}^{\infty} nx^n$,
5. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$,
6. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$,
7. $1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{n}(2i)} x^n$, zderivujte a přenásobte výrazem $(1-x)$,
8. $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 x^n$,
9. $\sum_{n=1}^{\infty} n(n+1)x^n$,
10. $\sum_{n=2}^{\infty} \frac{n}{n-1} x^n$.

Řešení:

1.

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$$

$$S'(x) = 1 + x + x^2 + ... = \frac{1}{1-x}, \quad |x| < 1,$$

$$S(x) = \int \frac{1}{1-x} dx = -\ln(1-x) + C,$$

$$S(0) = \sum_{n=1}^{\infty} \frac{0^n}{n} = 0 = \ln(1) + C \Rightarrow C = 0 \Rightarrow S(x) = -\ln(1-x).$$

2.

$$S(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = x + \frac{x^3}{3} + \frac{x^5}{5} + ...$$
$$S'(x) = 1 + x^2 + x^4 + ... = \frac{1}{1-x^2}, \quad |x| < 1,$$
$$S(x) = \int \frac{1}{1-x^2} dx = \frac{1}{2} \int \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx = \frac{1}{2}(\ln(1+x) - \ln(1-x)) + C = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) + C,$$
$$S(0) = \sum_{n=1}^{\infty} \frac{0^{2n-1}}{2n-1} = 0 = \frac{1}{2} \ln(1) + C \Rightarrow C = 0 \Rightarrow S(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

3.

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$
$$S'(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$
$$S''(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... = S(x),$$
$$S''(x) - S(x) = 0$$
$$\lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1 \Rightarrow S(x) = C_1 e^x + C_2 e^{-x}, \Rightarrow S'(x) = C_1 e^x - C_2 e^{-x},$$
$$S(0) = 1 = C_1 + C_2, \ S'(0) = 0 = C_1 - C_2 \Rightarrow C_1 = C_2 = \frac{1}{2}, \Rightarrow$$
$$S(x) = \frac{1}{2}(e^x + e^{-x}).$$

4.

$$S(x) = \sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + ...$$
$$\frac{S(x)}{x} = 1 + 2x + 3x^2 + 4x^3 + ...$$
$$\int \frac{S(x)}{x} dx = x + x^2 + x^3 + ... = \frac{x}{1-x}$$
$$\frac{S(x)}{x} = \left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$
$$S(x) = \frac{x}{(1-x)^2}.$$

5.

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-...$$

$$S'(x)=1-x^2+x^4-...=\frac{1}{1+x^x}$$

$S(x)=$ arctg x.

6.

$$S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\frac{x}{2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+...$$

$$xS(x)=\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{3\cdot 4}+...$$

$$(xS(x))'=\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+...$$

$$(xS(x))''=1+x+x^2+x^3+...=\frac{1}{1-x}$$

$$(xS(x))'=\int\frac{1}{1-x}dx=-\ln(1-x)+C$$

$$\frac{0}{1}+\frac{0^2}{2}+\frac{0^3}{3}+...=0=\ln(1-0)+C\Rightarrow C=0$$

$$xS(x)=\int-\ln(1-x)dx\overset{p.p.}{=}-x\ln(1-x)-\int\frac{x}{1-x}dx$$

$$=-x\ln(1-x)+x+\ln(1-x)+C$$

$$\frac{0^2}{2}+\frac{0^3}{2\cdot 3}+\frac{0^4}{3\cdot 4}+...=0=-0\ln(1-0)+0+\ln(1-0)+C\Rightarrow C=0$$

$$S(x)=\frac{(1-x)\ln(1-x)}{x}+1.$$

7.

$$S(x) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(2i-1)}{\prod_{i=1}^{n}(2i)} x^n = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + ...$$

$$(S(x))' = \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}2x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}3x^2 + ...$$

$$(S(x))'(1-x) = \frac{1}{2}(1-x) + \frac{1\cdot 3}{2\cdot 4}2x(1-x) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}3x^2(1-x) + ...$$

$$(S(x))'(1-x) = \frac{1}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\cdot 2 - \frac{1}{2}\right)x + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 3 - \frac{1\cdot 3}{2\cdot 4}\cdot 2\right)x^2 ...$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{3}{4}\cdot 2 - 1\right)x + \frac{1\cdot 3}{2\cdot 4}\left(\frac{5}{6}\cdot 3 - 2\right)x^2 ...$$

$$\frac{1\cdot 3\cdot ...\cdot(2n-3)}{2\cdot 4\cdot ...\cdot(2n-2)}\left(\frac{2n-1}{2n}\cdot n - (n-1)\right) = \frac{1}{2}\cdot\frac{1\cdot 3\cdot ...\cdot(2n-3)}{2\cdot 4\cdot ...\cdot(2n-2)}$$

$$(S(x))'(1-x) = \frac{1}{2}\left(1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + ...\right) = \frac{1}{2}S(x)$$

$$\frac{(S(x))'}{S(x)} = \frac{1}{2(1-x)}$$

$$\ln(S(x)) = -\frac{1}{2}\ln(1-x) + C$$

$$S(x) = \frac{C}{\sqrt{1-x}}$$

$$S(0) = 1 + \frac{1}{2}0 + \frac{1\cdot 3}{2\cdot 4}0^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}0^3 + ... = 1 = \frac{C}{\sqrt{1-0}} \Rightarrow C = 1$$

$$S(x) = \frac{1}{\sqrt{1-x}}.$$

8.

$$S(x) = \sum_{n=1}^{\infty}(-1)^{n+1}n^2x^n = x - 4x^2 + 9x^3 - 16x^4 + ...$$

$$\frac{S(x)}{x} = 1 - 4x + 9x^2 - 16x^3 + ...$$

$$\int \frac{S(x)}{x}dx = x - 2x^2 + 3x^3 - 4x^4 + ...$$

$$\frac{\int \frac{S(x)}{x}dx}{x} = 1 - 2x + 3x^2 - 4x^3 + ...$$

$$\int \frac{\int \frac{S(x)}{x}dx}{x}dx = x - x^2 + x^3 - x^4 + ... = \frac{x}{1+x}$$

$$\frac{\int \frac{S(x)}{x}dx}{x} = \frac{1}{(1+x)^2}$$

$$\int \frac{S(x)}{x}dx = \frac{x}{(1+x)^2}$$

$$\frac{S(x)}{x} = \frac{(1+x)^2 - 2(1+x)x}{(1+x)^4} = \frac{1-x}{(1+x)^3}$$

$$S(x) = \frac{x - x^2}{(1+x)^3}.$$

9.

$$S(x) = \sum_{n=1}^{\infty} n(n+1)x^n = 2x + 2 \cdot 3x^2 + 3 \cdot 4x^3 + ..$$

$$\frac{S(x)}{x} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + ...$$

$$\int \frac{S(x)}{x}dx = 2x + 3x^2 + 4x^3 + ...$$

$$\int\int \frac{S(x)}{x}dxdx = x^2 + x^3 + x^4 + ... = \frac{x^2}{1-x}$$

$$\int \frac{S(x)}{x}dx = \left(\frac{x^2}{1-x}\right)' = \frac{2x(1-x) + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$

$$\frac{S(x)}{x} = \left(\frac{2x - x^2}{(1-x)^2}\right)' = \frac{2(1-x)^3 + 2(1-x)(2x - x^2)}{(1-x)^4} = \frac{2(1 - 2x + x^2) + 4x - 2x^2}{(1-x)^3}$$

$$S(x) = \frac{2x}{(1-x)^3}.$$

10.

$$S(x) = \sum_{n=2}^{\infty} \frac{n}{n-1} x^n = 2x^2 + \frac{3}{2}x^3 + \frac{4}{3}x^4 + ...$$

$$\frac{S(x)}{x} = 2x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + ...$$

$$\int \frac{S(x)}{x} dx = x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + ...$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...$$

$$\left(\frac{\int \frac{S(x)}{x} dx}{x} \right)' = 1 + x + x^2 + ... = \frac{1}{1-x}$$

$$\frac{\int \frac{S(x)}{x} dx}{x} = \int \frac{1}{1-x} dx = -\ln(1-x)$$

$$\int \frac{S(x)}{x} dx = -x \ln(1-x)$$

$$\frac{S(x)}{x} = -\ln(1-x) + \frac{x}{1-x}$$

$$S(x) = -x \ln(1-x) + \frac{x^2}{1-x}.$$