Uniform Generation of Clonoids

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Joint work with Peter Mayr

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Clonoids

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For algebras $\mathbb{A}$ and $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a **clonoid** from $\mathbb{A}$ to $\mathbb{B}$ if

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Example: $\mathbb{A} = (\mathbb{Z}_3; +, -, 0)$, $\mathbb{B} = (\{0,1\}; \wedge, \vee)$, C clonoid from $\mathbb{A}$ to $\mathbb{B}$.
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For $R \subseteq A^n, S \subseteq B^n$, let

$$\mathrm{Pol}(R, S) = \bigcup_{k \in \mathbb{N}} \{f \colon A^k \to B \mid f(R, \dots, R) \subseteq S\}$$

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Theorem (Couceiro, Foldes 2009)

Let $\mathbb{A}$ and $\mathbb{B}$ be algebras with $|A|$ finite. Let $C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}$. The following are equivalent.

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Since abelian groups are Mal'cev algebras, we obtain:

Upper Bound

For $\mathbb{A}$ and $\mathbb{B}$ finite abelian groups, the number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is finite or countably infinite.

Obtaining the upper bound

Theorem (Kreinecker 2019)

For prime p, the number of clonoids from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ is countably infinite.

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- Complete description of lattice of clonoids from $\mathbb{Z}_p$ to $\mathbb{Z}_p$.
- $p = 2$: Just 2 non-finitely generated clonoids.
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$$\begin{aligned} f(x_1, x_2) =& f(0,0) \\ & + 2^{-1}[f(x_1, 0) + f(x_1 + x_2, 0) - f(0,0) - f(x_2, 0) \\ & + f(0, x_2) + f(0, x_1 + x_2) - f(0,0) - f(0, x_1) \\ & + f(x_1, x_1) + f(x_2, x_2) - f(0,0) - f(x_1 + x_2, x_1 + x_2)]. \end{aligned}$$

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Notice that this formula holds independently of the choice of f.

Uniform generation

For $\mathbb{A}$ an R-module and $\mathbb{B}$ an S-module,
$f : A^k \to B$ is generated by n-ary minors if

$$\exists s \colon \{r \in R^{k\times k} \ : \ \operatorname{rank}(r) \leq n\} \to S \text{ such that}$$

$$f(x) = \sum_{r \in R^{k\times k}, \operatorname{rank}(r) \leq n} s(r) f(rx).$$

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For $U \subseteq B^{A^k}$ we say U is *uniformly generated* by n-ary minors if

$$\exists s\colon \{r \in R^{k\times k} \ : \ \mathrm{rank}(r) \leq n\} \to S \text{ such that for all } f \in U,$$

$$f(x) = \sum_{r\in R^{k\times k}, \mathrm{rank}(r)\leq n} s(r) f(rx).$$

That is, the choice of coefficients is independent of the choice of $f \in U$.

Theorem (Fioravanti 2020)

Let $\mathbb{A} = (\mathbb{Z}_p, +)$ and let $\mathbb{B}$ be a finite abelian group of coprime order.

- For $k \in \mathbb{N}$, $\{f : A^k \to B\}$ is uniformly generated by unary functions.
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For $f : A^2 \to B$ with $f(0,0) = 0$,

$$J(f)(x,y) := p^{-1} \sum_{a \in \mathbb{Z}_p} f(x + ay, 0) - f(ay, 0) = \begin{cases} f(x,y) & \text{if } y = 0, \\ 0 & \text{else.} \end{cases}$$

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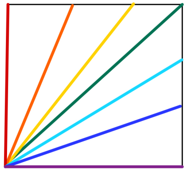

For $v \in A^2$ unary minors of f generate

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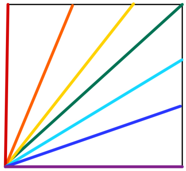

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Then $f = \sum f_v$ for v's generating distint lines.

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Theorem (Mayr, W. 2024)

Let $\mathbb{A} = (\mathbb{Z}_m, +)$ and let n be the maximal power of a prime dividing m. Let $\mathbb{B}$ be a finite abelian group with order coprime to m.

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Theorem (Mayr, W. 2024)

Let $\mathbb{A} = (\mathbb{Z}_{p^n}, +)$ for a prime p and $n \geq 1$. Let $\mathbb{B}$ coprime abelian group.

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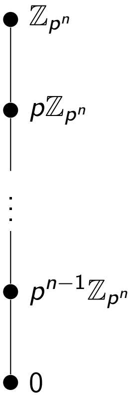

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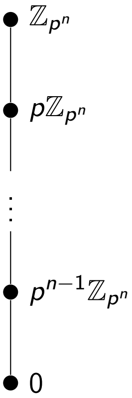

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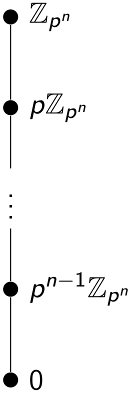

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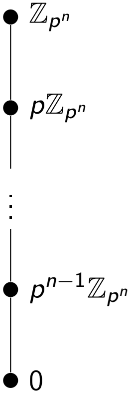

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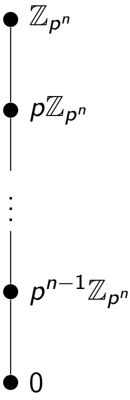

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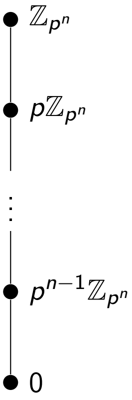

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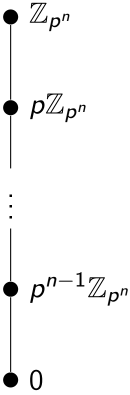

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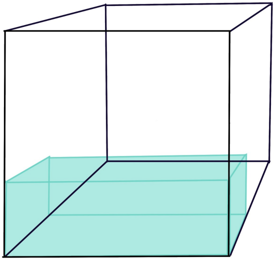

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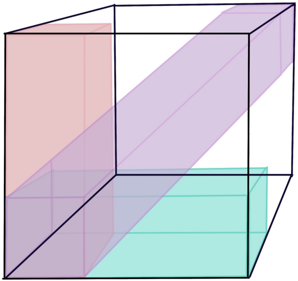

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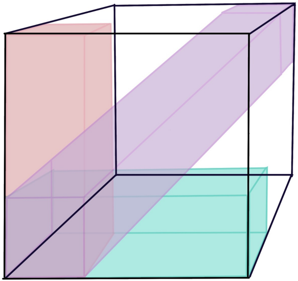

- For $N \neq M \in V, N \cap M \subseteq pA^k$, hence $f = \sum_N f_N$. ■

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Conjecture for Modules

For $\mathbb{A}$ and $\mathbb{B}$ finite modules, the number of clonoids from $\mathbb{A}$ to $\mathbb{B}$ is finite if and only if $\gcd(|A|, |B|) = 1$.

Thanks!