A description of Hilbert spaces as a dagger category

Thomas Vetterlein

Institute for Mathematical Methods in Medicine und Data-Based Modelling
Johannes Kepler University (Linz, Austria)

Joint work with Jan Paseka

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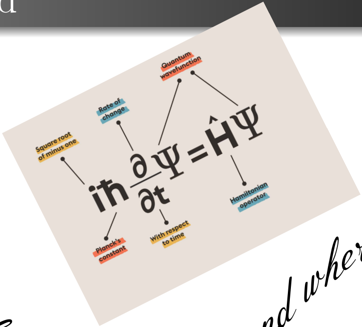
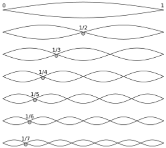
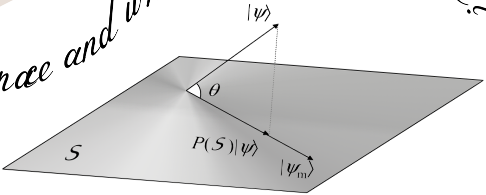
Background



What is the Hilbert space and where does it come from?





Find a simple framework leading in a natural manner to the most basic model of quantum physics.

Towards a simplified description of Hilbert spaces

We may describe complex Hilbert spaces as ...

- certain **algebras**,
- certain **relational structures**, or
- certain **categories**.

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C. Heunen, A. Kornell, Axioms for the category of Hilbert spaces, *Proc. Natl. Acad. Sci. USA*, 2022.

S. Lack, S. Tobin, A characterisation for the category of Hilbert spaces, *Appl. Categor. Struct.*, 2025.

Constructions for Hilbert spaces: composition

Orthogonal sum

For complex Hilbert spaces H_1 and H_2, let $H_1 \oplus H_2$ be the direct sum of the linear spaces and

$$(u_1 + u_2 \,|\, v_1 + v_2) \;=\; (u_1 \,|\, v_1) + (u_2 \,|\, v_2)\,, \qquad u_1, v_1 \in H_1, \;\; u_2, v_2 \in H_2.$$

Then $H_1 \oplus H_2$ is likewise a Hilbert space.

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More generally, we may form the direct limit of an isometric direct system of complex Hilbert spaces.

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Orthomodularity

Let S be a closed subspace of a complex Hilbert space. Then H is the orthogonal sum of S and $S^{\perp}$, i.e.

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The “atomic” Hilbert space

Let I be the 1-dimensional complex Hilbert space.

(i) Every non-zero isometry to I is unitary.

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The "atomic" Hilbert space

Let I be the 1-dimensional complex Hilbert space.

(i) Every non-zero isometry to I is unitary.

(ii) For every non-zero Hilbert space H, there is a non-zero linear map $I \to H$.

(iii) For every non-zero linear map $\varphi\colon I \to H$, there is a linear automorphism $h\colon I \to I$ such that $\varphi \circ h$ is an isometry.

Square roots of Hilbert space automorphisms

Strict square roots

Let φ be a unitary map on a complex Hilbert space H.
Then there is a unitary map ψ on H such that

- $\psi^2 = \varphi$,
- a closed subspace H is reducing for φ iff it is reducing for ψ.

Dagger categories

Definition (Abramsky, Coecke, Selinger, ...)

A dagger on a category $\mathcal{C}$ assigns to each morphism $f\colon X \to Y$ a morphism $f^*\colon Y \to X$ such that

- $(g \circ f)^* = f^* \circ g^*$ for any $f\colon X \to Y$ and $g\colon Y \to Z$,
- ${\mathrm{id}_A}^* = \mathrm{id}_A$ for any object A,
- $f^{**} = f$ for any $f\colon X \to Y$.

A category equipped with a dagger is called a dagger category.

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Let $\mathcal{H}il_{\mathbb{C}}$ consist of

- the complex Hilbert spaces;
- the bounded linear maps between them;
- the adjoint as the dagger.

Then $\mathcal{H}il_{\mathbb{C}}$ is a dagger category.

Some sorts of morphisms

Definition

In a dagger category, a morphism $f\colon A \to B$ is

- a dagger monomorphism if $f^* \circ f = \mathrm{id}_A$,
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Lemma

In $\mathcal{Hil}_{\mathbb{C}}$,

- the monomorphisms are the linear injections;
- the dagger monomorphisms are the isometries;
- the isomorphisms are the linear isomorphisms;
- the dagger isomorphism are the unitary maps.

Zero object

Definition

A zero object in a dagger category $\mathcal{C}$ is an object 0 such that there is a unique morphism $0_{0,A} \colon 0 \to A$ for each object A.

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Lemma

In $\mathcal{Hil}_{\mathbb{C}}$, the zero space is the zero object.

Biproducts

Let $\mathcal{C}$ be a dagger category with zero object.

Definition

By a dagger biproduct of $A, B \in \mathcal{C}$, we mean a coproduct

$$A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$$

such that ι_A, ι_B are dagger monomorphisms and ${\iota_B}^* \circ \iota_A = 0_{A,B}$.

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(i) For spaces $H_1, H_2 \in \mathcal{H}il_{\mathbb{C}}$, the orthogonal sum $H_1 \oplus H_2$, together with the inclusion maps, is a biproduct.

(ii) For any orthoclosed subspace S of $H \in \mathcal{H}il_{\mathbb{C}}$, H is the biproduct of S and $S^{\perp}$ via the inclusion maps.

Simple objects

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Lemma

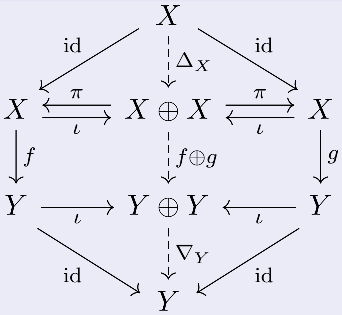
In $\mathcal{H}il_{\mathbb{C}}$, a space H is atomic if and only if H is 1-dimensional.

Theorem (cf. Herrlich, Strecker)

Consider a category with biproducts.

For $f, g\colon X \to Y$, let

$$f + g \;=\; \nabla_Y \circ (f \oplus g) \circ \Delta_X .$$



From a category to a linear structure

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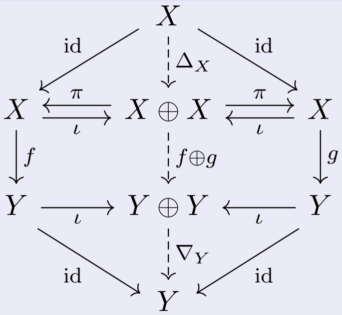

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- $(\mathcal{C}(X,Y); +, 0_{X,Y})$ is a commutative monoid.

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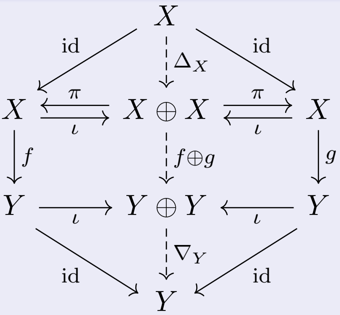

Then:

- $(\mathcal{C}(X,Y); +, 0_{X,Y})$ is a commutative monoid.
- Let $I \in \mathcal{C}$ be such that any non-zero endomorphism of I is an isomorphism. Then $(\mathcal{C}(I,I); +, 0_{I,I}, \circ, \mathrm{id}_I)$ is a division semiring.

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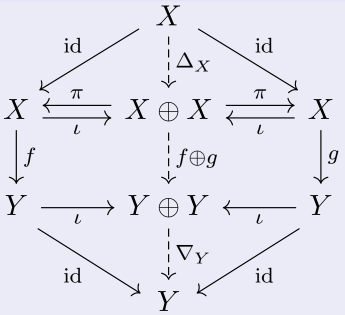

Then:

- $(\mathcal{C}(X, Y); +, 0_{X,Y})$ is a commutative monoid.
- Let $I \in \mathcal{C}$ be such that any non-zero endomorphism of I is an isomorphism. Then $(\mathcal{C}(I, I); +, 0_{I,I}, \circ, \mathrm{id}_I)$ is a division semiring.
- If, in addition, there are non-zero $f, g \in \mathcal{C}(I, I)$ such that $f + g = 0$, then $(\mathcal{C}(I, I); +, 0_{I,I}, \circ, \mathrm{id}_I)$ is a division ring.

From a dagger category to Hermitian spaces

Theorem (Paseka, V.)

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(H2) For every dagger monomorphism $f\colon A \to X$, there is a further dagger monomorphism $g\colon B \to X$ such that $A \xrightarrow{f} X \xleftarrow{g} B$ is a dagger biproduct.

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(H3) $\mathcal{C}$ contains an atomic object. In addition, we require:

- (a) For any atomic object I and non-zero object A in C, there is a non-zero morphism $u\colon I \to A$.
- (b) For any non-zero morphism $u\colon I \to A$ from an atomic object I to an object $A \in \mathcal{C}$, there is an automorphism h of I such that $u \circ h$ is a dagger monomorphism.

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Then ...

From a dagger category to Hermitian spaces

$F = (\mathcal{C}(I, I); +, 0_{I,I}, \circ^{\mathrm{op}}, \mathrm{id}_I)$ is a division ring.

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For $A \in \mathcal{C}$, put $\mathcal{V}(A) = \mathcal{C}(I, A)$. Then $\mathcal{V}(A)$, equipped with

- the addition $+$,
- the constant $0 = 0_{I,A}$,
- and the scalar multiplication given by

$$\alpha\, u \;=\; u \circ \alpha, \quad \alpha \in F, \;\; u \in \mathcal{V}(A),$$

is a linear space over F.

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$$(u \,|\, v) \;=\; v^* \circ u, \quad u, v \in \mathcal{V}(A),$$

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is a linear space over F. Moreover, the product

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makes $\mathcal{V}(A)$ into a Hermitian space.

Finally, for a morphism $f\colon A \to B$ in $\mathcal{C}$,

$$\mathcal{V}(f)\colon \mathcal{V}(A) \to \mathcal{V}(B),\; u \mapsto f \circ u$$

is an adjointable linear map. In fact, $\mathcal{V}$ is a dagger functor from $\mathcal{C}$ to the dagger category of Hermitian spaces over F.

Definition

In a dagger category, a morphism $p\colon A \to A$ is a projection if $p^* = p$ and $p^2 = p$.

From a dagger category to Hilbert spaces

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Let $f, g\colon A \to A$ be dagger automorphims. We say that g is a strict square root of f if:

- $g^2 = f$,
- a projection $p\colon A \to A$ commutes with f iff p commutes with g.

Theorem (PASEKA, V.)

Let $\mathcal{C}$ be a dagger category with a zero object such that conditions (H1)–(H3) hold and:

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Consequently, $\mathcal{C}$ is dagger equivalent to $\mathcal{H}il_{\mathbb{C}}$.

Conclusion

Five axioms characterise the dagger category of complex Hilbert spaces and bounded linear maps:

(H1) $\mathcal{C}$ has dagger biproducts.

(H2) For every dagger monomorphism $f\colon A \to X$, there is a further dagger monomorphism $g\colon B \to X$ such that $A \xrightarrow{f} X \xleftarrow{g} B$ is a dagger biproduct.

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