Role of weak congruences in theoretical and practical applications

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In this talk, we will mention two aspects of applying weak congruences.
The first is connected to the representation of various classes of groups and group-like algebras. Namely, we characterized several classes of groups by their weak congruence lattices, for example, abelian groups, Hamiltonian groups, nilpotent groups, solvable groups, etc.

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Another aspect is connected to applications with so-called Ω-algebras, which are ordinary algebras with generalized equality. Particular Ω-algebras like Ω -vector spaces have a big role in the approximate solving of systems of relational equations.
Both aspects will be briefly presented with some examples.

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Clearly, every congruence on a subalgebra of $\mathcal{A}$ is a weak congruence on $\mathcal{A}$, and vice versa, every nonempty weak congruence θ on $\mathcal{A}$ is a congruence on a subalgebra $\mathcal{B}_\theta$ of $\mathcal{A}$, where $B_\theta := \{x \in A \mid x\,\theta\,x\}$.

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Vojvodić, Šešelja: On the lattice of weak congruence relations. Algebra Universalis 25 (1988), 121-130.

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Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.

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$\mathcal{A}$ is said to have the **congruence intersection property** (**CIP**) if for any $\rho \in$Con $\mathcal{B}$, $\theta \in$Con $\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in$Sub $\mathcal{A}$,

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Hence, $\mathcal{A}$ *has the CIP if and only if* Δ *is a distributive element of the lattice* $Cw\mathcal{A}$*, if and only if* $n_\Delta : \rho \mapsto \rho \vee \Delta$ *is a homomorphism from* $\mathrm{Con_w}(\mathcal{A})$ *onto* $\uparrow\Delta$.

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Theorem

The following are equivalent for an algebra $\mathcal{A}$: $\mathcal{A}$ has the CEP if and only if in $\mathrm{Con_w}(\mathcal{A})$, for $\rho, \theta \in \mathrm{Con}\,\mathcal{B}$, $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$,
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An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

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An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

Theorem

If an algebra $\mathcal{A}$ has the CIP and the CEP, then any lattice identity holds on $\mathrm{Con_w}(\mathcal{A})$ if and only if it holds on $\mathrm{Sub}\,\mathcal{A}$ and on $\mathrm{Con}\,\mathcal{A}$.

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Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\operatorname{Con} \mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\operatorname{Con}_w(\mathcal{A}) \cong \operatorname{Con} \mathcal{B} \cong L$.

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The above construction by which the diagonal relation of the algebra corresponds to the bottom of the lattice is called the **trivial representation**.

Weak congruence lattice representation problem 1

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

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A representation by which the diagonal relation corresponds to an element different from the bottom of the lattice is said to be **non-trivial**.

Weak congruence lattice representation problem 2

Let L be an algebraic lattice. Is there a non-trivial representation of this lattice by a weak congruence lattice?

Δ-suitable elements.

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Some necessary and some sufficient conditions for a lattice and an element to be representable by a weak congruence lattice of an algebra.

Weak congruences on groups

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For every group $\mathcal{G}$ there is a 1-1 correspondence between weak congruences and ordered pairs (H, K) of subgroups of $\mathcal{G}$, such that $K \triangleleft H$.

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Analogously, $\mathcal{G}$ satisfies the $*$CIP if and only if

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A finite group $\mathcal{G}$ is a Dedekind group if and only if it satisfies the CIP.

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Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

The following statements on a group G are equivalent:

(1) *G is a Dedekind group.*

(2) $\mathrm{Con_w}(G)$ *is modular.*

(3) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.

(4) *G has the* CIP *and the* CEP.

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G. Czédli, B. Šešelja, A. Tepavčević, Semidistributive elements in lattices; application to groups and rings, Algebra Univers. 58 (2008) 349-355.

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Corollary

A group G is cyclic if and only if the subgroup lattice is distributive and satisfies the maximal condition (ascending chain condition).

Corollary

A group is locally cyclic if and only if its weak congruence lattice is distributive.

Theorem

A group G is abelian if and only if $\mathrm{Con_w}(G)$ is an A-lattice.

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A group G is solvable if and only if the lattice $\mathrm{Con_w}(G)$ has a subnormal series of intervals consisting of A-lattices.

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A group G is nilpotent if and only if the lattice $\mathrm{Con_w}(G)$ has a finite series of intervals

$$[\{e\}^2, H_1^2], [H_1^2, H_2^2], \dots, [H_i^2, H_{i+1}^2], \dots, [H_k^2, G^2], \qquad (1)$$

so that for every $i \in \{0, 1, \dots, k\}$ the following holds:

(a) $\Delta_{H_i} \blacktriangleleft \Delta$;

(b) *in the sublattice $[H_i^2, G^2]$ as a lattice with normal elements determined by $H_i^2 \vee \Delta$, for every $\delta \in C([H_i^2, H_i^2 \vee \Delta])$, the interval $[H_i^2, \overline{H_i^2 \vee \Delta_{H_{i+1}} \vee \delta}]$ is an A-lattice determined by $H_i^2 \vee \Delta_{H_{i+1}} \vee \delta$.*

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Ω-sets and structures

In 1977., M.P. Fourman and D.S. Scott introduced models for intuitionistic predicate logic. These were Ω-*sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

A complete Heyting algebra Ω is fixed.
Recall that a complete Heyting algebra is a complete lattice $(\Omega, \leqslant)$ such that for $x, y \in \Omega$, the set $\{z \in \Omega \mid z \wedge x \leqslant y\}$ has a largest element.
An **Ω-set** is a pair (A, E), where A is a nonempty set and E is an Ω-valued equality, i.e., a function $E : A^2 \to \Omega$ fulfilling:

$E(a,b) = E(b,a)$ (symmetry) and
$E(a,b) \wedge E(b,c) \leqslant E(a,c)$ (transitivity).

Intuitively,
$E(a,b)$ – the "truth value" of the formula $a = b$ and
$E(a,a)$ – the truth value of the formula $a \in A$.

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If (A, E) is an Ω-set, then the mapping $\mu : A \to \Omega$, such that
$\mu(x) := E(x, x), \quad$ for every $x \in A$
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The function E in an Ω-set is **separated** if

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Observe that *separated symmetric and transitive relation on a set A is the equality relation on a subset of A.*

Ω-algebras

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If $\mu : X \to \Omega$ is an Ω-valued set on X then for $p \in \Omega$, the set

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Proposition

The collection $\{\mu_p \mid p \in \Omega\}$ of all cuts of the function $\mu : X \to \Omega$ is a closure system on X.

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R is **transitive** if

$$R(x, y) \geqslant R(x, z) \wedge R(z, y) \text{ for all } x, y, z \in A.$$

Let $\mu : A \rightarrow \Omega$ and $R : A^2 \rightarrow \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

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Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

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A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

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A lattice-valued equivalence R on A is a **lattice-valued equality**, if it satisfies the **separation property**.

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For every n-ary operation $f \in F$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n));$$

and $R(c, c) = 1$.

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If (A, E) is an Ω-set and $p \in \Omega$, then the cut μ_p is a subset of A, and the cut E_p is an equivalence relation on μ_p.
In addition, the collection of all cuts $\{E_p \mid p \in \Omega\}$ of E is a closure system, a subposet of the lattice of all weak equivalences on A.

Ω-algebra; cuts of subalgebras and congruences

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Proposition

Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold for every $p \in \Omega$:
(i) The cut μ_p of μ is a subalgebra of $\mathcal{A}$, and
(ii) The cut E_p of E is a congruence relation on μ_p.

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Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)),$$

for all $a_1, \ldots, a_n \in A$ and the term-operations corresponding to terms u and v respectively.

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Proposition

If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.

Theorem

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies (all identities in) $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.

Proposition

The collection of cuts of E in an Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a closure system on A^2, a subposet of the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ of $\mathcal{A}$.

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Theorem (Representation)

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ such that

$$\text{if } a \neq b, \text{ then } (a,b) \notin \bigcap\{R \in \mathcal{R} \mid (a,a) \in R\} \quad \text{for all } a, b \in A.$$

Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E.

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Now we have that $E_R = R$ (the cut determined by R considered as an element of Ω, coincides with R as a weak congruence).

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Now we have that $E_R = R$ (the cut determined by R considered as an element of Ω, coincides with R as a weak congruence).

The structure $(\mathcal{A}, E)$ is then the required Ω-algebra, obtained by the *canonical construction*.

For a symmetric and transitive relation $R \subseteq A^2$, we denote by $\mathrm{dom}R$ the set $\{x \in A \mid (x,x) \in R\}$.

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Corollary

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ fulfilling condition:

$$\text{if } a \neq b, \text{ then } \ (a,b) \notin \bigcap\{R \in \mathcal{R} \mid (a,a) \in R\} \quad \text{for all } a,b \in A.$$

Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $R \in \mathcal{R}$, the algebra $\mathrm{dom}R/R$ fulfills these identities. Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.

Solving of equations in Ω-algebras.

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