T-based orthomodular dynamic algebras

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Motivation

In this lecture, we generalize the results of Kishida, Rafiee Rad, Sack, and Zhong by expanding the quantale $\mathcal{Q}(X)$ into a T-based orthomodular dynamic algebra. Here X is a complete orthomodular lattice (the category of complete orthomodular lattices and orthomodular lattice isomorphism between them shall be denoted as $\mathbb{COL}$) and $\mathcal{Q}(X) \subseteq \mathcal{P}(X^X)$ is generated by the Sasaki projections $\{\pi_a \mid a \in X\}$ and closed under the pointwise lifting of composition

$$A \cdot B = \{ f \circ g \mid f \in A, \ g \in B \}.$$

In other words, we work in a generalized setting that augments $\mathcal{Q}(X)$ with additional morphisms.

Standard Orthomodular dynamic algebras

Definition (Generalized dynamic algebra — signature)

A *GDA* is a tuple $(Q, \bigvee, \cdot, \sim)$ where

$$\bigvee : \mathcal{P}(Q) \to Q, \qquad \cdot : Q \times Q \to Q, \qquad \sim : Q \to Q.$$

No axioms are imposed here; any algebraic laws used later will be stated explicitly where needed.

Ordered set preliminary

Definition

An **ortho-lattice** is a tuple $\mathcal{M} = \left(M, \leq, -^{\perp}\right)$ satisfying the following conditions:

1. $(M, \leq)$ is a bounded lattice with least element 0 and greatest element 1.
2. The map $-^{\perp} : M \to M$ satisfies, for all $m, n \in M$,
    1. $m \wedge m^{\perp} = 0$ and $m \vee m^{\perp} = 1$
    2. $m \leq n \Rightarrow n^{\perp} \leq m^{\perp}$
    3. $(m^{\perp})^{\perp} = m$.

An ortho-lattice $\mathcal{M}$ is defined to be an **orthomodular lattice** if for all $m, n \in Y$ such that $m \leq n$, it holds that $n = m \vee \left(m^{\perp} \wedge n\right)$.

We write $x \perp y \iff x \leq y^{\perp}$.

Definition

A function $f : X \to Y$ is defined as a *linear map* (or *adjoint map*) from an orthomodular lattice X to an orthomodular lattice Y if there exists a function $g : Y \to X$, denoted as f^* and referred to as an *adjoint* of f, such that for all $x \in X$ and for all $y \in Y$, the following equivalence holds:

$$f(x) \perp_Y y \iff x \perp_X g(y).$$

The collection of all linear maps from an orthomodular lattice X to an orthomodular lattice Y is conventionally denoted by $\mathrm{Lin}(X, Y)$. Furthermore, if both X and Y are complete orthomodular lattices, then $\mathrm{Lin}(X, Y)$ forms a complete lattice. Additionally, when $X = Y$, $\mathrm{Lin}(X)$ (equivalent to $\mathrm{Lin}(X, X)$) is a Foulis quantale.

This subsection will eventually define the category of $\mathcal{T}$-based orthomodular dynamic algebras. A $\mathcal{T}$-based Orthomodular dynamic algebra fundamentally enriches a standard Orthomodular dynamic algebra. We will begin by defining the quantale, and then start by defining the standard Orthomodular dynamic algebra.

Notation

In the framework of GDAs, we introduce these definitions:

$$\begin{array}{ll}
\sqsubseteq & \stackrel{\text{def}}{=} \{(x,y) \in Q \times Q \mid \bigsqcup\{x,y\} = y\} \\
\mathcal{P}_{\mathfrak{Q}} & \stackrel{\text{def}}{=} \{\sim x \mid x \in Q\} \\
\bigvee X & \stackrel{\text{def}}{=} \sim\sim \bigsqcup X, \text{ for any } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\
\bigwedge X & \stackrel{\text{def}}{=} \sim \bigsqcup\{\sim x \mid x \in X\}, \text{ for any } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\
\preceq & \stackrel{\text{def}}{=} \{(p,q) \in \mathcal{P}_{\mathfrak{Q}} \times \mathcal{P}_{\mathfrak{Q}} \mid \bigvee\{p,q\} = q\} \\
\ulcorner x \urcorner & \stackrel{\text{def}}{=} \lambda y. \sim\sim (x \cdot y) \text{ or } \ulcorner x \urcorner : P_{\mathfrak{Q}} \to P_{\mathfrak{Q}} \ \ulcorner x \urcorner(p) \ = \sim\sim \ (x \cdot p). \\
\equiv & \stackrel{\text{def}}{=} \{(x,y) \in Q \times Q \mid \ulcorner x \urcorner(p) = \ulcorner y \urcorner(p), \text{ for each } p \in \mathcal{P}_{\mathfrak{Q}}\} \\
\mathcal{T}_{\mathfrak{Q}} & \stackrel{\text{def}}{=} \{x \in Q \mid x = p_1 \cdot \cdots \cdot p_n, \text{ for some } n \in \mathbb{N}^+ \text{ and } p_1, \ldots, p_n \in \mathcal{P}_{\mathfrak{Q}}\}
\end{array}$$

We define $\mathcal{T}_{\mathfrak{Q}}$ as the minimal set containing $\mathcal{P}_{\mathfrak{Q}}$ within Q that is closed under the operation denoted by "$\cdot$."
We may omit the subscript "$\mathfrak{Q}$" from $\mathcal{P}_{\mathfrak{Q}}$ and $\mathcal{T}_{\mathfrak{Q}}$ when the context is clear.

Quantales

Definition

A *quantale* is a pair $V = (V, \otimes)$, where V is a complete $\bigvee$-semilattice and $\otimes$ is a binary operation on V satisfying:

(V1) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all $a, b, c \in V$ (associativity).

(V2) $a \otimes (\bigvee S) = \bigvee_{s \in S}(a \otimes s)$ for every $S \subseteq V$ and every $a \in V$.

(V3) $(\bigvee S) \otimes a = \bigvee_{s \in S}(s \otimes a)$ for every $S \subseteq V$ and every $a \in V$.

A quantale $V = (V, \otimes)$ is called *unital* if there exists an element $e \in V$ such that for every $a \in V$ the equalities $a \otimes e = a$ and $e \otimes a = a$ hold.

By an *involutive quantale* we mean a quantale $\mathbf{V}$ equipped with a semigroup involution * satisfying

$$\left(\bigvee a_i\right)^* = \bigvee a_i^*$$

for all $a_i \in V, i \in I$.

Standard Orthomodular dynamic algebras

Definition

(Kishida, et al.) An *orthomodular dynamic algebra* is a generalized dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ such that

1. $(Q, \sqsubseteq, \cdot)$ is a quantale, and $\bigsqcup$ is the arbitrary join.
2. $(\mathcal{P}, \preceq, \sim)$ is a complete orthomodular lattice, where $\mathcal{P} \subseteq Q$.
3. If A is such that
    1. $\mathcal{P} \subseteq A \subseteq Q$
    2. A is closed under the operation $\cdot$, and
    3. A is closed under $\bigsqcup$, by which we mean, for any $Y \in \mathcal{P}(A)$, $\bigsqcup Y \in X$.

    Then $A = Q$ (minimality).
4. For any $A, B \subseteq \mathcal{T}$, $\bigsqcup A = \bigsqcup B$, if and only if $A = B$ (sets).
5. For any $x, y \in \mathcal{T}$, $x = y$ if and only if $x \equiv y$ (completeness).
6. For any $p, q \in \mathcal{P}$, $\ulcorner p \urcorner(q) = \pi_p(q)$, i.e., $\sim\sim (p \cdot q) = p \wedge (\sim p \vee q)$ (Sasaki projection).
7. $\ulcorner x \urcorner(y) = \ulcorner x \urcorner(\sim\sim y)$, for each $x, y \in Q$ (composition).

Standard Orthomodular dynamic algebras

Fix a complete orthomodular lattice X. Let

1. $\mathcal{F}_T(X)$ be the smallest set of X^X containing $\{\pi_a \mid a \in X\}$ and closed under function composition $\circ$; recall that $\pi_a \colon X \to X$ is the Sasaki projection to a;
2. $\mathcal{Q}(X) \overset{\text{def}}{=} \mathcal{P}(\mathcal{F}_T(X))$;
3. $\cdot \colon \mathcal{Q}(X) \times \mathcal{Q}(X) \to \mathcal{Q}(X) :: A \cdot B \mapsto \{a \circ b \in \mathcal{F}_T(X) \mid a \in A, b \in B\}$;
4. $\sim \colon \mathcal{Q}(X) \to \mathcal{Q}(X) :: A \mapsto \{\pi_{(\bigvee\{a(1) \mid a \in A\})^\perp}\}$.

It is proven that $\mathcal{Q}(X)$ is a an involutive quantale. With suitable modification, we are able to obtain even a so-called Foulis quantale.

Standard Orthomodular dynamic algebras

Given $\mathcal{O}_{\mathrm{GDA}}$ as the class of all generalized dynamic algebras and $\mathcal{O}_{\mathrm{SET}}$ as the class of all sets, we define a class function

$$\mathcal{T} : \mathcal{O}_{\mathrm{GDA}} \to \mathcal{O}_{\mathrm{SET}}$$

where for every $K \in \mathcal{O}_{\mathrm{GDA}}$, $\mathcal{T}(K)$ is a set such that

$$\widetilde{K} \subseteq \mathcal{T}(K) \subseteq K$$

This framework facilitates the introduction of the $\mathcal{T}$-based orthomodular dynamic algebra, which constitutes an expansion upon the extant structure of orthomodular dynamic algebras.

T-based ODAs (Part I)

A $\mathcal{T}$-based orthomodular dynamic algebra is a generalized dynamic algebra

$$\mathfrak{K} = (K, \bigsqcup, \odot, \sim)$$

extended with a unary operation $^{-*} : K \to K$. This extended structure must satisfy the following conditions:

- $(K, \bigsqcup, \odot, -^*)$ forms a unital involutive quantale, and $\bigsqcup$ is the arbitrary join.
- $(\widetilde{K}, \leq, \sim)$ is a complete orthomodular lattice. If $x \in \widetilde{K}$, then its conjugate $x^* = x \in \widetilde{K}$.
- If a set A meets these criteria:
  - $\mathcal{T}(\widetilde{K}) \subseteq A \subseteq K$,
  - A is closed under both the $\odot$ and $-^*$ operations,
  - A is closed under $\bigsqcup$ (for any $B \subseteq A$, $\bigsqcup B \in A$),

  then $A = K$. (Minimality)

T-based ODAs (Part II)

- (a) For any $S, T \subseteq \langle \widetilde{K} \rangle$, $\bigsqcup S = \bigsqcup T \;\Leftrightarrow\; S = T$. (Proper set equality)
- (b) For any $s, t \in \langle \widetilde{K} \rangle$, $s = t \;\Leftrightarrow\; s \equiv t$. (Completeness)
- (c) For any $v, w \in \widetilde{K}$, the unary operation $v'(w)$ is equivalent to the Sasaki projection:

$$\pi_v(w) = \sim\big(\sim(v \odot w)\big) = v \wedge (\sim v \vee w).$$

- (d) For each $k, l \in K$, the composition property holds:

$$k'(l) = k'\big(\sim(\sim l)\big).$$

The category of $\mathcal{T}$-based orthomodular dynamic algebra with structure preserving bijections shall be denoted as $\mathbb{T} - \mathbb{ODA}$.

The functor $\Gamma : \mathbf{COL} \to \mathcal{T}\text{-}\mathbf{ODA}$

Mapping of objects (I). Let $\mathcal{M} = (M, \leq, (-)^{\perp})$ be a complete orthomodular lattice. Define:

1. $L_M = \mathcal{T}(\mathrm{Lin}(M))$: a set of endomaps with

$$\{\pi_m : M \to M \mid m \in M\} \ \subseteq \ L_M \ \subseteq \ \mathrm{Lin}(M), \qquad \pi_m(b) = m \wedge (m^{\perp} \vee b).$$

2. $\langle L_M \rangle$: *involutive* subsemigroup of $\mathrm{Lin}(M)$ generated by L_M, which is a closure under composition $\circ$ and the involution $(\cdot)^*$ given on generators by $\pi_m^* = \pi_m$ and extended by

$$(f \circ g)^* = g^* \circ f^*.$$

3. $\odot$: a binary operation on $\mathcal{P}(\langle L_M \rangle)$ defined by

$$A \odot B = \{\, a \circ b \mid a \in A,\ b \in B \,\}.$$

The functor $\Gamma : \mathbf{COL} \to \mathcal{T}$-**ODA**

Mapping of objects (II).

5. $(\cdot)^*$: a unary operation on $\mathcal{P}(\langle L_M \rangle)$ defined by

$$A^* = \{\, a^* \mid a \in A \,\}.$$

6. $\sim$: a unary operation on $\mathcal{P}(\langle L_M \rangle)$ defined by

$$\sim A = \left\{\, \pi_{(\,\bigvee_{a \in A} a(1)\,)^{\perp}} \,\right\}.$$

Define

$$\Gamma(\mathcal{M}) := \big(\mathcal{P}(\langle L_M \rangle),\ \cup,\ \odot,\ \sim,\ ^*\big), \qquad \mathbf{1} = \{\mathrm{id}_M\}.$$

The functor $\Psi : \mathcal{T}$-**ODA** $\to$ **COL**

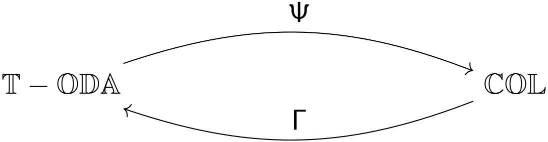
On objects. For a $\mathcal{T}$-based ODA K, let P_K be its set of *tests*. Define the OML operations for $X \subseteq P_K$ by

$$p^\perp :=\sim p, \qquad \bigwedge X :=\sim \bigvee \{\sim x \mid x \in X\}, \qquad \bigvee X :=\sim \bigwedge \{\sim x \mid x \in X\}.$$

Set $\Psi(K) := (P_K, \leq, {}^\perp)$ with $p \leq q \iff p \vee q = q$.

On morphisms. If $h : K \to L$ is a $\mathcal{T}$-ODA morphism (among other things, preserves joins and $\sim$), define

$$\Psi(h) : P_K \to P_L, \qquad \Psi(h)(p) := h(p).$$



- $\Gamma(M)$: pick $L_M = \mathcal{T}(\mathrm{Lin}(M))$, form $S = \langle L_M \rangle$, then $\mathcal{P}(S)$ with $\cup, \odot, ^*, \sim$.
- $\Psi(K)$: extract the test OML P_K (with $p^\perp = \sim p$, joins/meets via De Morgan).

For every $\mathcal{L} \in \mathbb{COL}$ and $\Omega \in \mathbb{T}-\mathbb{ODA}$, there is a natural bijection

$$\mathrm{Hom}_{\mathbb{T}-\mathbb{ODA}}\big(\Gamma(\mathcal{L}), \Omega\big) \;\cong\; \mathrm{Hom}_{\mathbb{COL}}\big(\mathcal{L}, \Psi(\Omega)\big),$$

exhibiting Ψ as left adjoint to Γ ($\Psi \dashv \Gamma$).

Thank you for your listening.

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