# An introduction of a new operation on a free barycentric algebra

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# Barycentric algebra

### Definition (Barycentric Algebra)

We call set $A$ with family of bianry operation $I^0 = \{+_r\}_{r\in(0,1)}$ a *Barycentric algebra* if following conditions are fulfilled:

$\forall_{a\in A}\forall_r \quad a +_r a = a$

$\forall_{a,b\in A}\forall_r \quad a +_r b = b +_{1-r} a$

$\forall_{r,p<1} \quad (a +_p b) +_r c = a +_{pr} (b +_{\frac{r-pr}{1-pr}} c)$

# Free barycentric algebra

**Lemma**

*For an arbitrary set $G$, a set $A_G = \{a : G \longrightarrow \mathbb{R}_{\geqslant 0} \mid \sum_{g \in G} a(g) = 1 \ \wedge \ a^{-1}(\mathbb{R}_+) \text{ is finite}\}$ with a family $I^0 = \{+_r\}_{r \in (0,1)}$ of binary operations such that*

$$a +_r b = ra + (1-r)b,$$

*where $r \in (0,1)$, is a free barycentric algebra over a set $G$.*

# Extended operation

**Definition**

Let $\langle G, \circ \rangle$ be a magma. Let $\langle B(G), I^0 \rangle$ be a free barycentric algebra over the set $G$. We define the binary operation $\circledcirc$ on $B(G)$ by

$$\begin{aligned} \circledcirc : & B(G) \times B(G) \longrightarrow B(G); \\ & (p, q) \mapsto p \circledcirc q : B(G) \longrightarrow [0, 1]; \\ & g \mapsto (p \circledcirc q)(g) = \Sigma_{(x,y) \in G_g^{\circ}} p(x) \circ q(y), \end{aligned}$$

where $G_a^{\circledcirc} = \{(x, y) \in G \times G \mid x \circ y = a\}$. We will call it expanded operation.

# Intuition

Let $p$ be an element of $B(G)$. Then $p$ can be treated as the distribution of a random variable over $G$. Then the addition operations in the barycentric algebra $+_r$ can be interpreted as a conditional probability distribution.

Let $p, q \in B(G)$ be distributions of some random variables $X, Y$ over $G$. Let $R$ be Bernoulli distribution with $\mathbb{P}(X = 1) = r$. Then $\mathbb{P}(z = Z | Z = X \ \mathit{iff} \ R = 1, Z = Y \ \mathit{otherwise}) = \mathbb{P}(z = X | R = 1) + \mathbb{P}(z = Y | R = 0) = \frac{1}{r}\mathbb{P}(z = X) + \frac{1}{1-r}\mathbb{P}(z = Y) = (p +_r q)(z)$.

For our expanded operation we have following interpretation. Let $p, q \in B(G)$ be distributions of some independent random variables $X, Y$ over $G$. Then $(p \circledcirc q)(z) = \mathbb{P}(z = X \circ Y)$.

# Natural embedding

**Lemma**

*Assume that* $\langle G, \circ \rangle$ *is a magma. Let* $\langle B(G), I^0, \circledcirc \rangle$ *be a free barycentric algebra over the set* $G$ *with expanded operation and let* $\epsilon : G \longrightarrow B(G); g \mapsto \delta_g$ *be an embedding. Then the magma* $\langle G, \circ \rangle$ *is isomorphic with the magma* $\langle \epsilon(G), \circledcirc \rangle$*, and* $\epsilon$ *is an isomorphism.*

# Basic properties

**Theorem**

*Let $\langle G, \circ \rangle$ be a magma. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Then*

1. *if $\circ$ has associative property then $\circledcirc$ has associative property.*
2. *if $\circ$ has commutative property then $\circledcirc$ has commutative property.*
3. *if $\circ$ has identity element then $\circledcirc$ has identity element.*

# Interesting property

**Lemma**

*Let $\langle G, \circ \rangle$ be a finite group of size $n$. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Then $p \in B(G)$ such as $\forall_{g \in G} p(g) = \frac{1}{n}$, is a absorbing element $\mathbb{0}_{\circledcirc}$ in $B(G)$.*

**Corollary**

*If $\langle G, \circ \rangle$ is a group then $\langle B(G), \circledcirc \rangle$ is not a group.*

# Group of automorphisms

**Theorem**

*Let $\langle G, \circ \rangle$ be a finite group. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Then $Aut(G, \circ) \cong Aut(B(G), I^0, \circledcirc)$.*

# Group of automorphisms

**Theorem**

*Let $\langle G, \circ \rangle$ be a finite group. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Then $Aut(G, \circ) \cong Aut(B(G), I^0, \circledcirc)$.*

**Problem**

*Let $\langle G, \circ \rangle$ be a finite group. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Is $Aut(G, \circ) \cong Aut(B(G), \circledcirc)$?*

## Norm

We will now add infinity norm on $B(G)$. From definition if $p \in B(G)$ then $\|p\|_\infty = \|\Sigma_{g \in G} p_g g\|_\infty = max_{g \in G} p_g$.

### Lemma

*Let $\langle G, \circ \rangle$ be a finite group. Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. If $p, q \in B(G)$ then $\|p \circledcirc q\|_\infty \leqslant \|p\|_\infty$ and $\|p \circledcirc q\|_\infty \leqslant \|q\|_\infty$.*

# Entropy

### Definition

Let $A$ be a finite set, and let $p$ be distribution of some random variable $X$ on $A$, we will denote $p(x) = \mathbb{P}(x = X)$. *Shannon's entropy* will be a function $H : \mathcal{P} \longrightarrow \mathbb{R}$,
$H(p) = -\Sigma_{x \in A} p(x) \log(p(x))$.

Relation between Shannon's entropy and operation on $B(G)$.

### Theorem

*Let $\langle G, \circ, \backslash, / \rangle$ ($\langle G, \circ \rangle$) be a finite quasigroup (group). Let $\langle B(G), I^0, \circledcirc \rangle$ be a free barycentric algebra over set $G$ with expanded operation. Let $p, q \in B(G)$. Then $H(p \circledcirc q) \geqslant H(p)$ and $H(p \circledcirc q) \geqslant H(q)$.*

# Order

**Definition**

Let $\langle G, \circ \rangle$ be a finite magma with a partial order $\leqslant_G$ on the set $G$. Then we can define a partial order on the set $B(G)$ such as
$p \leqslant_{B(G)} q \Leftrightarrow \forall_{g \in G} \Sigma_{h \in \uparrow g} p(h) \leqslant \Sigma_{h \in \uparrow g} q(h)$.

# Thank you for your attention

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