Closure operators on additively idempotent semirings

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Filters and ultrafilters

Definition

Given a set X and nonempty collection $\mathscr{F}$ of subsets of X, we say that $\mathscr{F}$ is *a filter* on X if the following properties hold:

(i) $\varnothing \notin \mathscr{F}$,

(ii) if $Y, Z \in \mathscr{F}$, then $Y \cap Z \in \mathscr{F}$,

(iii) if $Z \in \mathscr{F}$ and $Z \subseteq Y \subseteq X$, then $Y \in \mathscr{F}$.

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A filter $\mathscr{F}$ is *an ultrafilter* on X if $\mathscr{F}$ is a maximal element (under inclusion) in the set of all filters on X.

Equivalently, a filter $\mathscr{F}$ is an ultrafilter on X if for each $Y \subseteq X$, either $Y \in \mathscr{F}$ or $X \setminus Y \in \mathscr{F}$.

Spectral spaces

T_0-axiom: For any two points x, y in topological space X, there is an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

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Definition (Finocchiaro 2014)

A topological space X is *a spectral space* if X satisfies the T_0-axiom and there is a basis $\mathscr{B}$ of X such that

$$X_{\mathscr{B}}(\mathscr{U}) := \{x \in X \mid [\forall B \in \mathscr{B}, x \in B \iff B \in \mathscr{U}]\} \neq \varnothing$$

for any ultrafilter $\mathscr{U}$ on X.

Semirings and k-ideals

Definition

A semiring is a nonempty set S with two binary operations $+ : S \times S \to S$ and $\cdot : S \times S \to S$ such that:

(i) $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids,

(ii) $\forall a, b, c \in S \;\; (a + b)c = ac + bc$,

(iii) $\forall a \in S \;\; a \cdot 0 = 0$.

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An ideal of semiring S is a nonempty subset $I \subseteq S$ such that:

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Definition

A k-ideal of semiring S is an ideal I which satisfies the following condition:

$$\forall a, b \in S \quad a \in I, a + b \in I \implies b \in I.$$

Hull-kernel topology

Given a set X, the power set $\mathcal{P}X$ is a spectral space endowed with the *hull-kernel topology* whose open subbase is given by the sets of the form

$$D(F) := \{Y \in \mathcal{P}X \mid F \not\subseteq Y\},$$

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where $F \in \mathcal{P}X$, $|F| < \infty$.

Proposition (Jun, Ray and Tolliver 2022)

For a semiring S, the collections of all ideals (proper or not), all prime ideals and all k-ideals form spectral spaces with the hull-kernel topology.

Closure operators defined on the set of ideals

Some examples of closure operators defined on the set of ideals of an additively idempotent semiring S:

- radical closure

$$I \mapsto \sqrt{I} := \{a \in S \mid \exists n \in \mathbb{N} \text{ such that } a^n \in I\}$$

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- closure with respect to a congruence

$$I \mapsto I^C := \{a \in S \mid \exists b \in I \text{ such that } (a, b) \in C\}$$

where C is a given congruence on a semiring S.

Spectral spaces arising from semirings

Let S be a semiring and $\mathscr{I}$ be the poset of all ideals of S. A closure operation

$$cl : \mathscr{I} \longrightarrow \mathscr{I} \qquad I \mapsto I$$

is said to be *of finite type* if

$$cl(I) = \bigcup\{cl(J) \mid J \subseteq I,\ J \in \mathscr{I},\ J \text{ is finitely generated}\}.$$

All closure operators presented on the previous slide are closure operators of finite type.

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Proposition (Jun, Ray and Tolliver 2022)

Let S be a semiring and cl be a closure operator of finite type on the poset $\mathscr{I}$ of all ideals of S. Then the set

$$\{I \in \mathscr{I} \mid cl(I) = I\}$$

is a spectral space with the hull kernel topology.

Proposition (Jun, Ray and Tolliver 2022)

Let S be a semiring. The poset $(\mathscr{I}, \subseteq)$ of all ideals of S ordered by the inclusion relation is an algebraic lattice.

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- From the above statement we know that $(\mathscr{I}, \subseteq)$ is an algebraic lattice.
- Given a closure operator cl of finite type on the set $\mathscr{I}$, we can form the lattice $(cl(\mathscr{I}), \subseteq)$. This naturally raises the question about the properties of this lattice. In particular, is the lattice $(cl(\mathscr{I}), \subseteq)$ algebraic?

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- It turns out that this lattice must indeed be algebraic, and many authors establish this fact directly.
- In the remainder of this presentation, I will demonstrate an alternative proof, making use of a well-known continuous closure operator theorem.

Continuous closure operator theorem

Definition

Let $(P, \leq)$ be a poset. A subset $D \subseteq P$ is said to be *directed* if $\forall x, y \in D\ \exists z \in D \ \ x, y \leq z$.

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Definition

A closure operator $cl : P \to S$ between posets is *continous* if, for all directed subsets $D \subseteq P$, $cl\left(\bigvee D\right) = \bigvee cl(D)$ whenever $\bigvee D$ exists in P.

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Theorem (Rhodes and Steinberg 2009)

Let $(L, \vee_L, \wedge_L)$ be an algebraic lattice and suppose that $cl : L \to L$ is a continuous closure operator. Then $(cl(L), \vee_{cl(L)}, \wedge_{cl(L)})$ is an algebraic lattice where

$$\bigvee_{cl(L)} X := cl\Big(\bigvee_L X\Big), \qquad \bigwedge_{cl(L)} X := \bigwedge_L X$$

for every subset $X \subseteq cl(L)$.

Proposition

Let S be an additively idempotent semiring and cl be a closure operator of finite type on the poset $\mathscr{I}$. Then cl is continous.

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Corollary

Let S be an additively idempotent semiring. Then the set of all k-ideals of S, ordered by inclusion, forms an algebraic lattice.

Further directions of research

- **Spectral spaces.** Investigate order-theoretic properties of spectral spaces arising from closure operators on semirings. In particular, one could explore connections between the lattice-theoretic structure of ideals and the topology of the associated spectral spaces.

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- **Lattices of ideals.** Study the interplay between algebraic properties of semirings and the algebraicity of the lattices of certain ideals.

References

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Thank you for your attention!